计算传热学的近代进展

第二章　对流项离散格式研究进展

主讲　陶文铨

西安交通大学能源与动力工程学院
热流科学与工程教育部重点实验室
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第2 章 对流项离散格式研究进展

2.1 QUICK格式实施方式的优化及SGSD格式

2.1.1 对流项离散格式的重要性及QUICK格式的定义

2.1.2 二维问题五点格式界面插值通用形式

2.1.3 插值公式系数的取值及QUICK实施的优化

2.1.4 SGSD格式

2.1.5 关于格式稳定性与代数方程求解问题的说明

2.1 QUICK格式实施方式的优化及SGSD格式

2.1.1 对流项离散格式的重要性及QUICK格式的定义

1.对流项离散格式的重要性

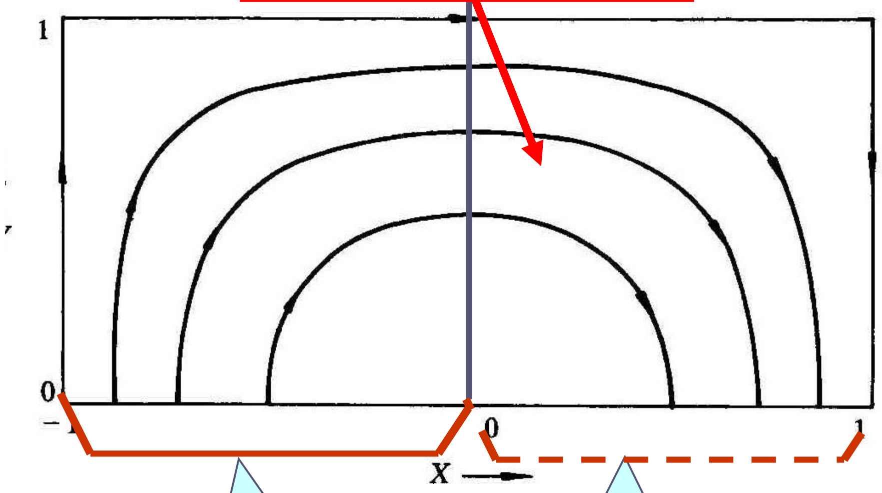
对流项离散格式影响到数值计算结**稳定性，经济性和准确性** (stability, economics and accuracy)。

以Smith-Hutton问题为例说明离散格式对准确性的影响。

Smith-Hutton 问题用于计算污染物的传递-**假定只有对流而无扩散**，则计算区域进口与出口污染物的分布应完全一样，但数值误差（假扩散）的存在导致进口与出口不同。

已知的流场

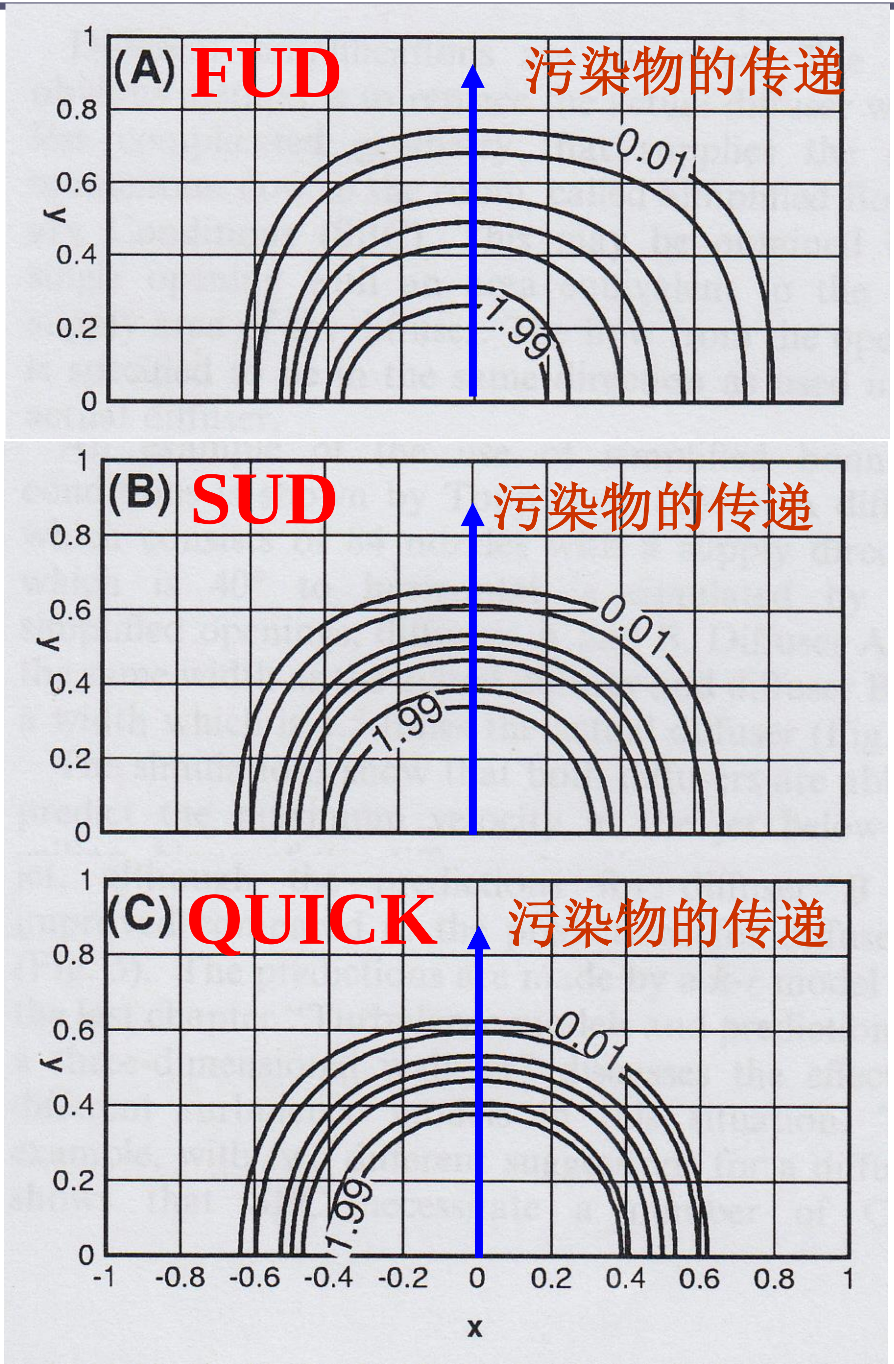
$$u = 2y(1-x^2),$$
$$v = -2x(1-y^3)$$



进口给定分布

求出的分布口

Smith-Hutton问题



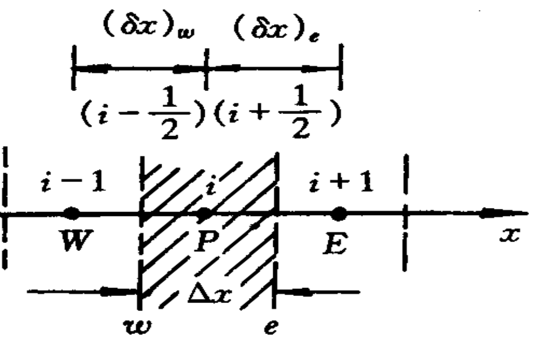
关于有限容积法（FVM）对流项格式的基本知识

通用控制方程

$$\frac{\partial(\rho\phi)}{\partial t}+div(\rho\phi\vec{U})=div(\Gamma_{\phi}^{*}grad(\phi))+S_{\phi}^{*}$$

Transient **Convection** **Diffusion** **Source**

讨论对流项离散格式采用1D稳态控制方程



$$\frac{\partial(\rho u\phi)}{\partial x}=\frac{\partial}{\partial x}(\Gamma\frac{\partial\phi}{\partial x})+S_{\phi}$$

FVM采用控制容积积分来离散对流项：

$$\rho\int_{w}^{e}\frac{\partial(u\phi)}{\partial x}dx=\rho[(u\phi)_e-(u\phi)_w]$$

采用交叉网格时界面速速是可获得的，但 ϕ 则存储在节点上

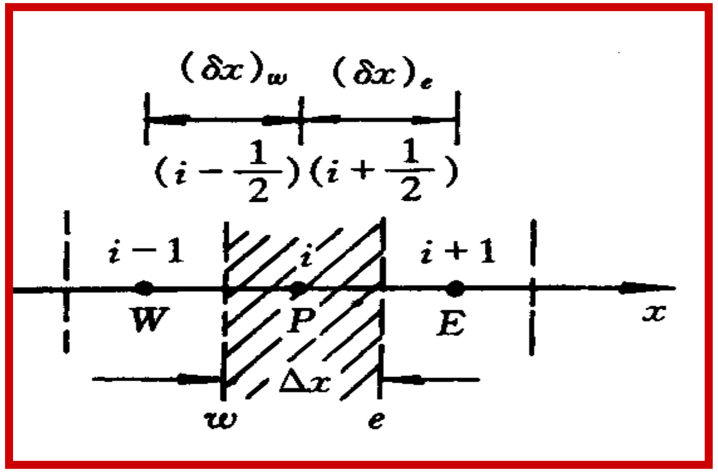
数值计算时所有的信息都存于节点上，要将对流项积分最终变为节点上的变量 ϕ 的值，**需要确定如何利用节点的变量值来得到界面值的插值方法，这就是FVM中对流项的格式。**

中心差分
CD,均分网格

$$\phi_e = \frac{\phi_E + \phi_P}{2}$$

一阶迎风
FUD

$$\phi_e = \phi_P, u > 0$$



QUICK定义为在中心差分基础上加型线的曲率修正。

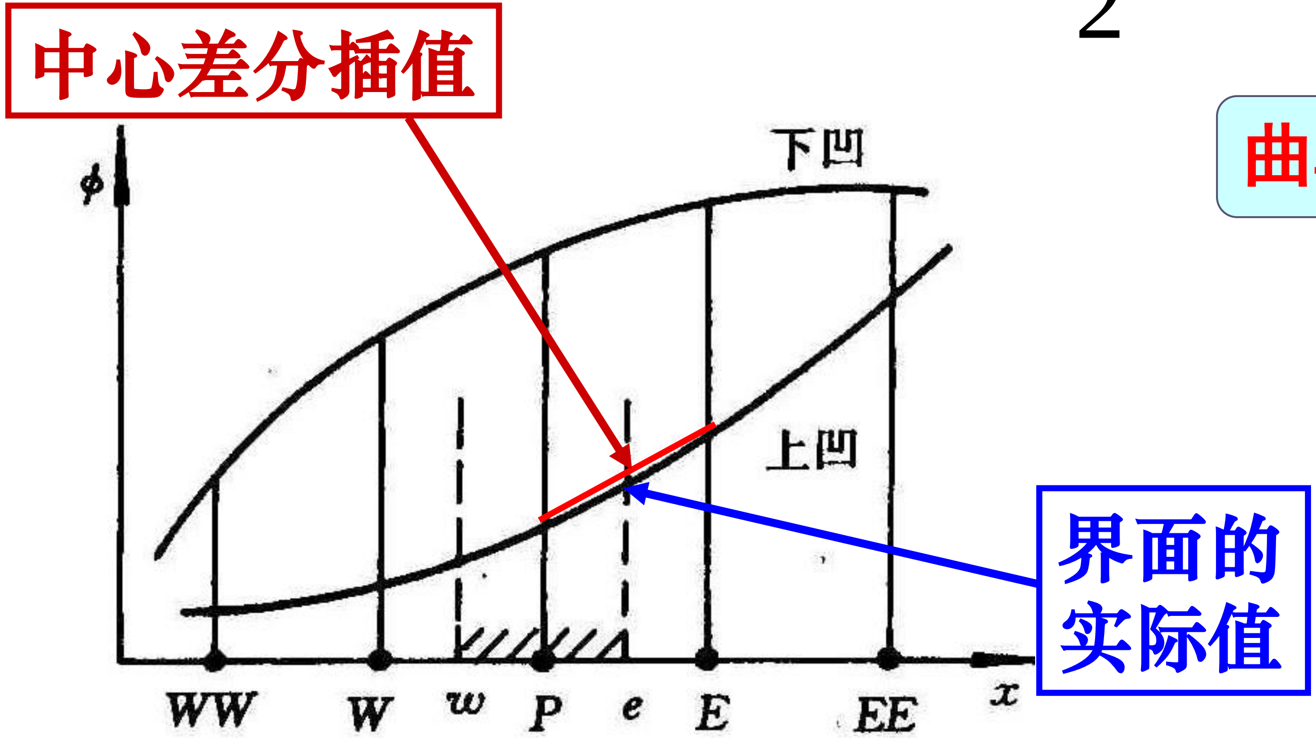
注意：插值各项系数之和需为**1**！以满足均匀场的需要。

2. QUICK格式的定义

FVM定义－界面的插值：在中心差分基础上考虑曲率修正：

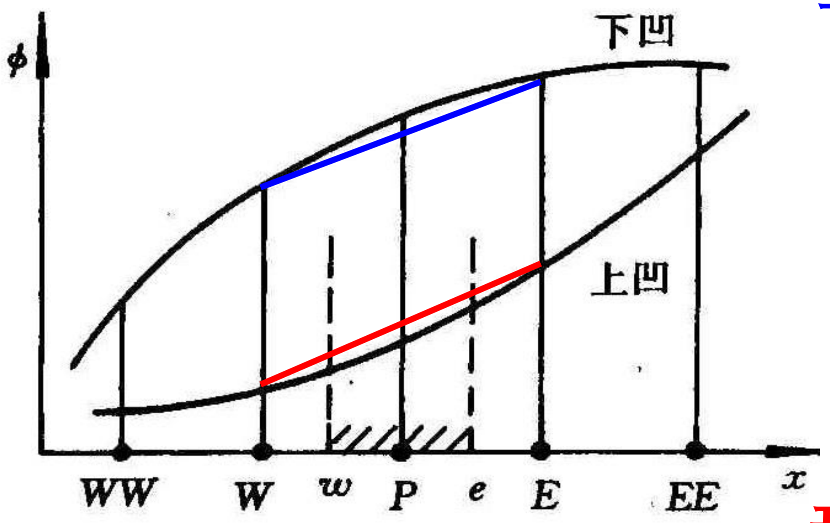
$$\phi_e = \frac{\phi_E + \phi_P}{2} - \frac{1}{8} Cur$$

曲率修正



如何确定曲率修正*Cur*？需要满足两个条件：

(1) 能自动反应型线凹向对CD插值的正确修正：相邻三点间二阶导数中心差分的结构可以反应型线的凹向



型线下凹 $\boxed{(\phi_W - 2\phi_P + \phi_E) < 0}$

二阶导数中心差分的结构

$$(\phi_W - 2\phi_P + \phi_E)$$

可以自动反应型线的凹向

型线上凹 $\boxed{(\phi_W - 2\phi_P + \phi_E) > 0}$

如何选定与界面有关的相邻三点？

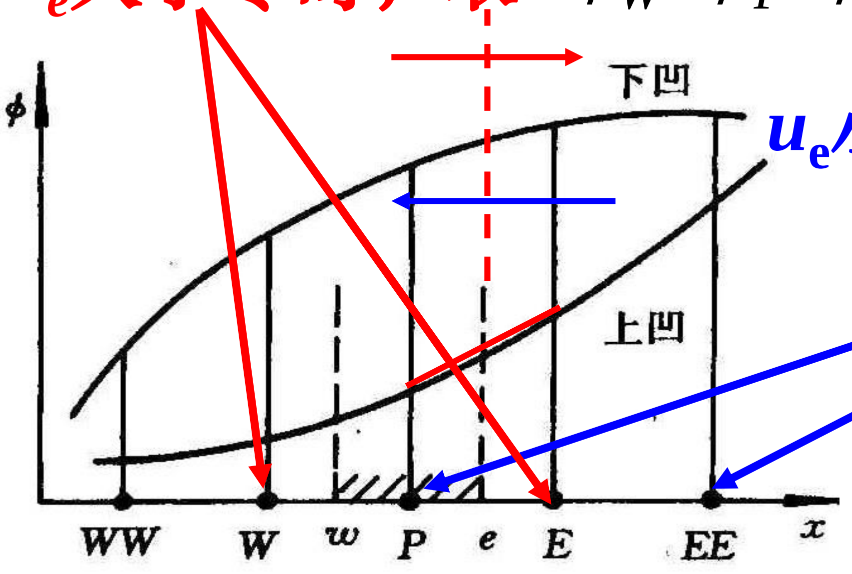
(2) 一般认为，为增加格式的稳定性要引入迎风思想，

对e-界面：

当界面流速u_e大于零时，取 ϕ_W,ϕ_P,ϕ_E **上游节点W**

当界面流速u_e小于零时，取 ϕ_P,ϕ_E,ϕ_{EE} **上游节点EE**

u_e大于零时，取 ϕ_W,ϕ_P,ϕ_E



u_e小于零时，取 ϕ_P,ϕ_E,ϕ_{EE}

e-界面QUICK
格式的曲率修正：

$$Cur = \begin{cases} \phi_W - 2\phi_P + \phi_E, \; u > 0 \\ \phi_P - 2\phi_E + \phi_{EE}, \; u < 0 \end{cases}$$

QUICK格式展开定义（节点位置用 *i* 表示），对于

$$u > 0 \quad \phi_e = \phi_{i+1/2} = \frac{\phi_i + \phi_{i+1}}{2} - \frac{1}{8}(\phi_{i+1} - 2\phi_i + \phi_{i-1})$$

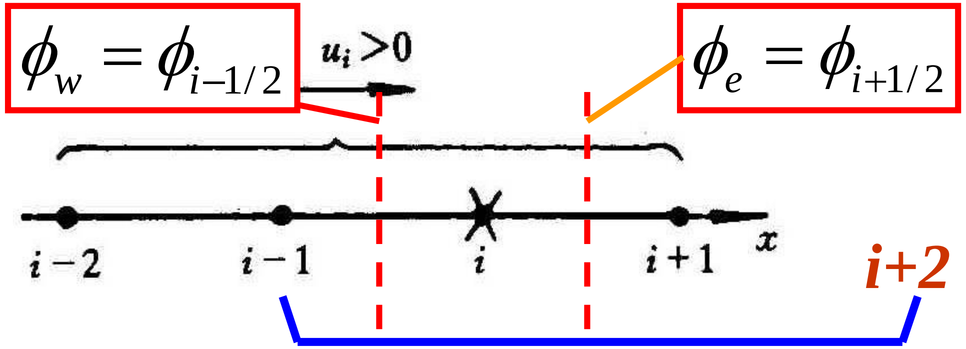
$$\phi_e = \phi_{i+1/2} = \frac{1}{8}(3\phi_{i+1} + 6\phi_i - \phi_{i-1})$$

$$\phi_w = \phi_{i-1/2} = \frac{1}{8}(3\phi_i + 6\phi_{i-1} - \underline{\phi_{i-2}})$$

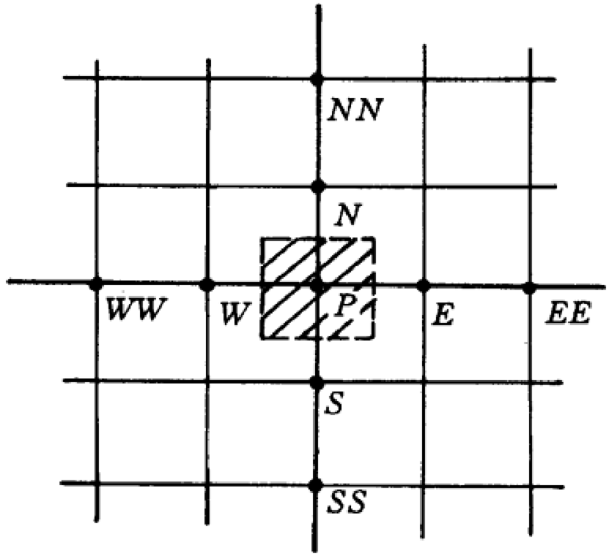
u>0，e与w界面的插值引入了节点：
i -2 , *i*-1, *i*, *i*+1

u<0， e与w界面的插值引入节点：
i-1, *i*, *i*+1, *i*+2 。

5个节点的位置都需要

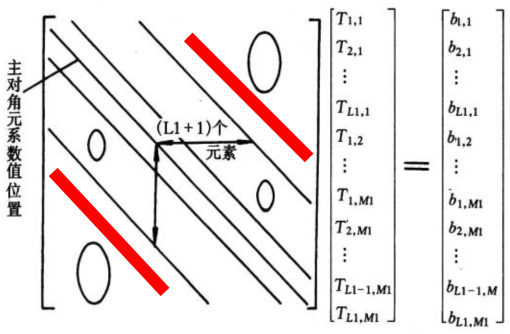


二维问题QUICK格式离散方程系数形成 9 对角矩阵



对于P点计算前无法知道其 *x* 方向流速的正负，因此WW点和EE点的空间须同时保留；对y方向也如此；总体上就形成了9对角阵代数方程组。

$$\vec{A}\vec{T}=\vec{b}$$



如何将远邻点引入到源项，对2D问题避免求解9对角阵的代数方程，同时保证代数方程求解过程的稳定性一直是自1979年QUICK格式提出后引起研究的问题，到1992年获得圆满解决。

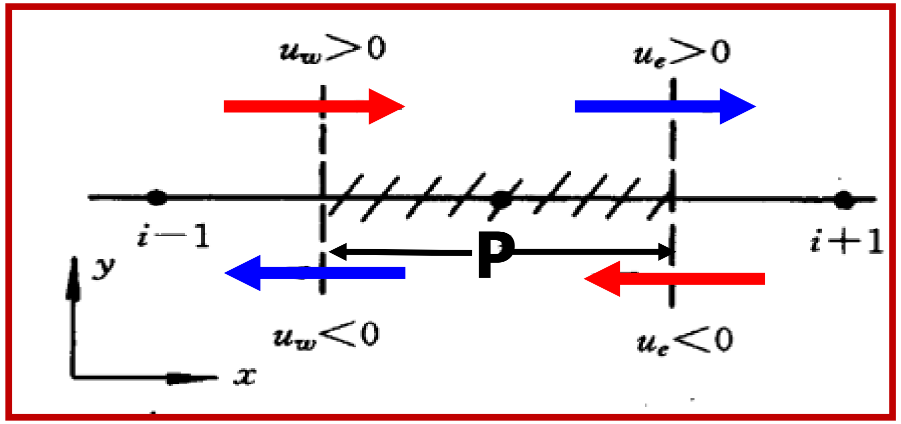
Hayase T, Humphery J A C, Grief A R. J Comput Physics, 1992,93:108-119

推荐阅读(4)

2.1.2 二维问题五点格式界面插值通用形式

1.为得出2维问题5点格式建立界面插值公式的基本原则

（1）为得出2维问题5点格式的离散方程，某个坐标方向界面插值只能由该方向控制容积的两相邻界面两侧各一个节点（主节点及两邻点共3点）来显式地表示，其余节点必须进入源项；



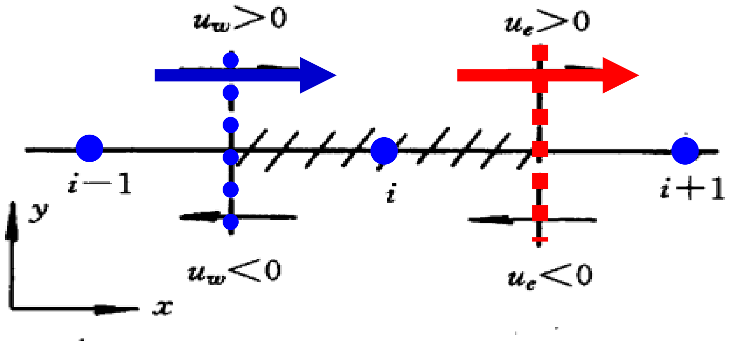
（2）从对流通量对控制容积*i*(P) 的作用而言，$u_e>0$与 $u_w<0$ 等价；$u_e<0$与 $u_w>0$等价。

根据此两原则来建立界面变量插值的通用公式，并依据一些基本要求决定插值公式中的系数之值。

2. 界面插值的通用形式

第一种情形流速大于0

$u_e>0, u_w>0$



$$\phi_e = a_1\phi_{i-1} + a_2\phi_i + a_3\phi_{i+1} + S_e^+; \qquad (1)$$

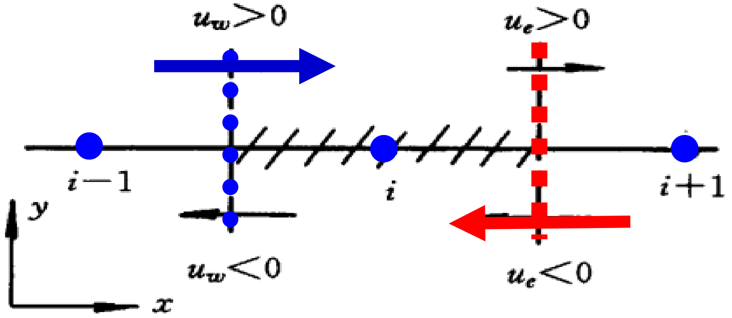
$$\phi_w = b_1\phi_{i-1} + b_2\phi_i + b_3\phi_{i+1} + S_w^+ \qquad (2)$$

东、西界面插值的显式表示式中节点不变，系数不同（显式表示，节点不变）！

注意：物理过程要求，上述系数必须大于等于0!

第二种情形流速小于0

（a） $u_e<0$



$$\phi_e = b_3\phi_{i-1} + b_2\phi_i + b_1\phi_{i+1} + S_e^-;$$

$$\boxed{\phi_w = b_1\phi_{i-1} + b_2\phi_i + b_3\phi_{i+1} + S_w^+}$$

$u_e<0$时的 ϕ_{i-1} 与$u_w>0$时的 ϕ_{i+1} 作用等价

显式表示,节点不变！（首先判定是b,再定顺序）

（b） $u_w<0$

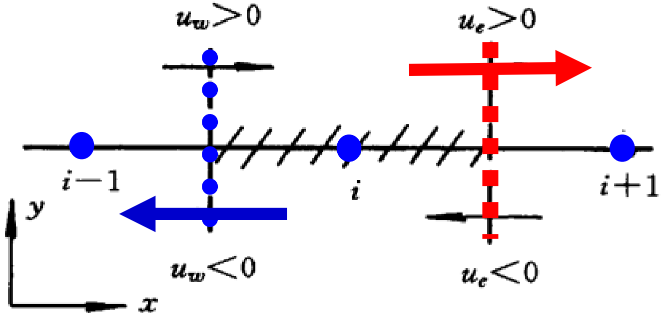
$$\phi_w = a_3\phi_{i-1} + a_2\phi_i + a_1\phi_{i+1} + S_w^-$$

$u_w<0$时的 ϕ_{i-1} 与$u_e>0$时的 ϕ_{i+1}作用等价

$$\boxed{\phi_e = a_1\phi_{i-1} + a_2\phi_i + a_3\phi_{i+1} + S_e^+}$$

显式表示,节点不变！

（首先判定是a,再定顺序）



3. 源项S 的表示式

将上述定义与格式原始定义对照，可以得出各个格式源项的表达式，对QUICK有：

$$S_e^+ = (-\frac{1}{8} - a_1)\phi_{i-1} + (\frac{3}{4} - a_2)\phi_i + (\frac{3}{8} - a_3)\phi_{i+1}$$

$$S_e^- = (-\frac{1}{8})\phi_{i+2} + (\frac{3}{4} - b_1)\phi_{i+1} + (\frac{3}{8} - b_2)\phi_i - b_3\phi_{i-1}$$

$$S_w^- = (-\frac{1}{8} - a_1)\phi_{i+1} + (\frac{3}{4} - a_2)\phi_i + (\frac{3}{8} - a_3)\phi_{i-1}$$

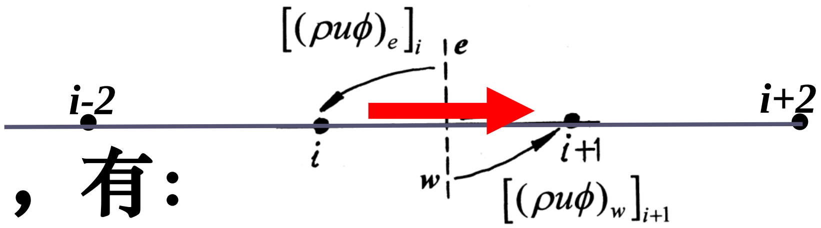
$$S_w^+ = (-\frac{1}{8})\phi_{i-2} + (\frac{3}{4} - b_1)\phi_{i-1} + (\frac{3}{8} - b_2)\phi_i - b_3\phi_{i+1}$$

这样定义的源项不仅有远邻点，而且有近邻点！

2.1.3 插值公式系数的取值及QUICK格式实施的优化

1. 决定界面插值系数的四个物理与数学原则

1) 界面上对流通量连续原则



对共享一个界面的相邻两节点，有：

$$[(\rho u\phi)_e]_i = [(\rho u\phi)_w]_{i+1} \qquad [(\phi)_e]_i = [(\phi)_w]_{i+1}$$

对共享一个界面的相邻两节点，当 $(u_e)_i>0$, 及 $(u_w)_{i+1}>0$, 由式（1），（2）可有：

$$\phi_e = \underline{a_1\phi_{i-1} + a_2\phi_i + a_3\phi_{i+1} + (S_e^+)}_i =$$

$$\phi_w = \underline{b_1\phi_i + b_2\phi_{i+1} + b_3\phi_{i+2} + (S_w^+)}_{i+1}$$

$(\phi_e)_i$

第1项是 ϕ_i

$(\phi_w)_{i+1}$

先假设 $(S_e^+)_i = (S_w^+)_{i+1}$ ，然后再验证； 则显然有：

$$\phi_e = \overset{0}{\cancel{a_1}}\phi_{i-1} + a_2\phi_i + a_3\phi_{i+1} + \cancel{(S_e^+)_i} =$$

$$\phi_w = b_1\phi_i + b_2\phi_{i+1} + \overset{0}{\cancel{b_3}}\phi_{i+2} + \cancel{(S_w^+)_{i+1}}$$

$$\underline{a_1 = 0;\ b_3 = 0;\ a_2 = b_1;\ a_3 = b_2}$$

容易证明，此时有： $(S_e^+)_i = (S_w^+)_{i+1}$

2) 离散方程正系数原则

将QUICK用于一维稳态对流扩散方程，

$$\frac{d(\rho u\phi)}{dx} = \frac{d}{dx}(\Gamma\frac{d\phi}{dx}) \xrightarrow[\text{积分}]{} (\rho u\phi)_e - (\rho u\phi)_w = (\Gamma\frac{d\phi}{dx})_e - (\Gamma\frac{d\phi}{dx})_w$$

扩散项取分段线性（中心差分），将QUICK代入对流项的界面插值，经整理有：

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + b$$

$$\underline{a_E = -a_3F_e^+ - b_1F_e^- + b_3F_w^+ + a_1F_w^- + D_e}$$

$$D_e = \frac{\Gamma_e}{(\delta x)_e}$$

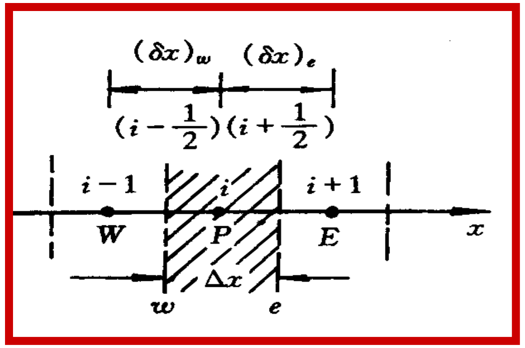
$$a_W = -a_1F_e^+ - b_3F_e^- + b_1F_w^+ + a_3F_w^- + D_w$$

$$a_P = a_2F_e^+ + b_2F_e^- - b_2F_w^+ - a_2F_w^- + (D_e + D_w)$$

$$b = -S_e^+F_e^+ - S_e^-F_e^- + S_w^+F_w^+ + S_w^-F_w^-$$

$$F_e^+ = (\rho u)_e, if\ u_e > 0;\ \ F_e^+ = 0,\ if\ u_e < 0$$

$$F_e^- = -\left|(\rho u)_e\right|_e, if\ u_e < 0; F_e^- = 0, if\ u_e > 0$$



系数为正要求： $a_E \ge 0;\ a_W \ge 0;\ a_P \ge 0$

对常物性一维问题： $F_e^+ = F_w^+,\ F_e^- = F_w^-,\ D_e = D_w$

对 $a_E = -\underline{a_3} F_e^+ - \underline{b_1} F_e^- + \underline{b_3} F_w^+ + \underline{a_1} F_w^- + D_e$

据 $a_E \ge 0$ **有：** $(b_3 - a_3)F_e^+ + (a_1 - b_1)\cancelto{0}{F_e^-} + D_w \ge 0$

对*u*>0： $(b_3 - a_3)F_e^+ + D_w \ge 0$

此条件应对任何*u* 成立，包括速度极大，扩散的影响可以不计时的情形，于是：

$$b_3 - a_3 \ge 0 \quad \longrightarrow \quad \boxed{a_3 \le b_3}$$

类似地根据 $a_W \geq 0, a_P \geq 0$ 得出： $\boxed{a_1 \leq b_1, a_2 \geq b_2}$

3) 邻点之值对界面插值应具有正影响的原则

这要求在式（1），（2）中所有的系数均应大于零：

$$\boxed{a_i \geq 0; b_i \geq 0}$$

4) 界面插值公式中系数之和为1的原则

$$\boxed{a_1 + a_2 + a_3 = 1,\ b_1 + b_2 + b_3 = 1}$$

可以证明当上述条件成立时，对于均匀场源项 S 均为零。

对上述四个要求的综合分析：

1) 界面连续原则 $\boxed{a_1=0;\,a_2=b_1;\;a_3=b_2;\,b_3=0}$

2) 离散方程正系数原则 $\boxed{a_1\le b_1}$ $\boxed{a_2\ge b_2}$ $\boxed{a_3\le b_3}$

3) 插值系数正影响原则 $\boxed{a_i\ge 0;\,b_i\ge 0}$

4) 适用均匀场原则 $\boxed{a_1+a_2+a_3=1,\;b_1+b_2+b_3=1}$

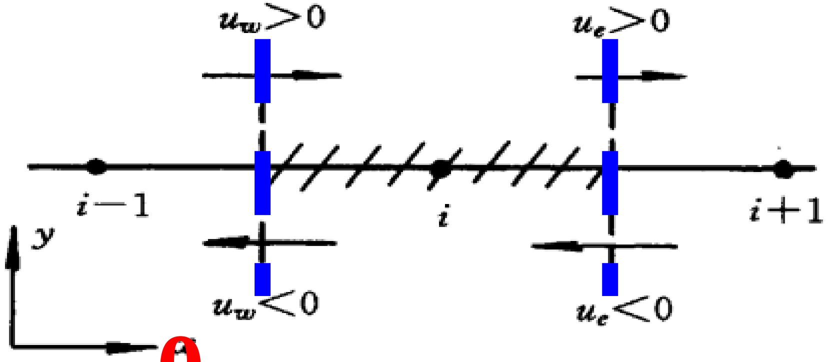
导致: $a_3=0$ **进而:** $b_2=0$ **再据** $\sum a_i=1$ **及** $a_1=0$,

$\boxed{a_2=1!}$ $\boxed{a_2=b_1=1!}$ **最终得出十分简单的结果!**

$$a_1 = a_3 = b_2 = b_3 = 0 \qquad a_2 = b_1 = 1!$$

$$\phi_e = \overset{0}{\cancel{a_1}}\phi_{i-1} + \overset{=1}{a_2}\phi_i + \overset{0}{\cancel{a_3}}\phi_{i+1} + S_e^+;$$

$$\underline{\phi_e = \phi_i + S_e^+};$$



$$\phi_w = \overset{=1}{b_1}\phi_{i-1} + \overset{0}{\cancel{b_2}}\phi_i + \overset{0}{\cancel{b_3}}\phi_{i+1} + S_w^+$$

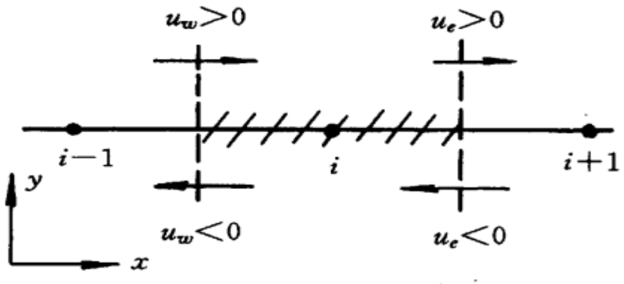
$$\underline{\phi_w = \phi_{i-1} + S_w^+}$$

均为一阶迎风+源项的表示方式！

2. QUICK 格式界面插值公式的优化表示

（1） $u_e>0, u_w>0$

$$\left.\begin{aligned}
\phi_e &= \phi_{i+1/2} = \phi_i + \underline{\frac{1}{8}(3\phi_{i+1} - 2\phi_i - \phi_{i-1})} \\
\phi_w &= \phi_{i-1/2} = \phi_{i-1} + \underline{\frac{1}{8}(3\phi_i - 2\phi_{i-1} - \phi_{i-2})}
\end{aligned}\right\}\text{源项}$$



（2） $u_e<0, u_w<0$

$$\left.\begin{aligned}
\phi_e &= \phi_{i+1/2} = \phi_{i+1} + \underline{\frac{1}{8}(3\phi_i - 2\phi_{i+1} - \phi_{i+2})} \\
\phi_w &= \phi_{i-1/2} = \phi_i + \underline{\frac{1}{8}(3\phi_{i-1} - 2\phi_i - \phi_{i+1})}
\end{aligned}\right\}\text{源项}$$

统一的表示模式：$\boxed{\phi_f^{QUICK} = \phi_f^{FUD} + S_f^{QUICK}}$

1974年Kholsa-Rubin提出了实施高阶格式的延迟修正(**Deferred correction**)：

$$\boxed{\phi_e^{H} = \phi_e^{L} + (\phi_e^{H} - \phi_e^{L})^{old}}$$

因此QUICK的优化表达即为延迟修正方式：

$$\boxed{\phi_f^{QUICK} = \phi_f^{FUD} + (\phi_f^{QUICK} - \phi_f^{FUD})^{old}}$$

对2D问题要使QUICK格式形成的代数方程为5对角阵且迭代求解不发散**应采用延迟修正**；对13年中（1979－1992）国际CHT界的各种尝试划上了句号。

2.1.4 SGSD格式

1.SCSD格式(1998)

1) 均分网格上的CD与SUD

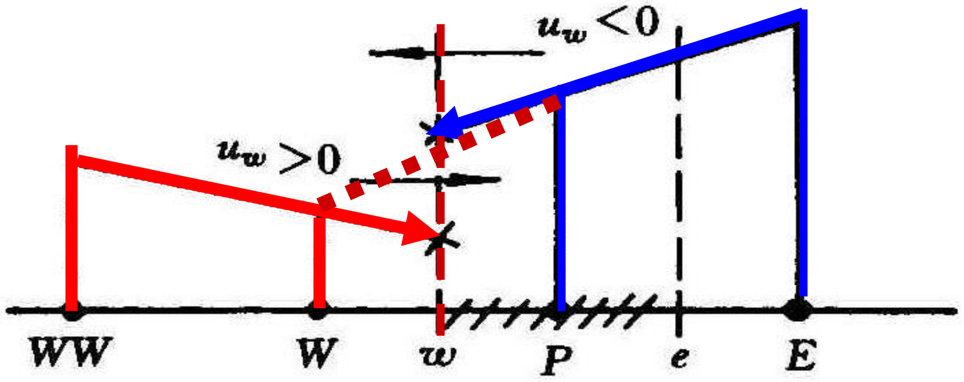
CD： $\phi_w = 0.5(\phi_P + \phi_W)$ 无二阶假扩散，但条件稳定!

SUD：

$$\phi_w = 1.5\phi_P - 0.5\phi_E, \quad u_w < 0$$

$$\phi_w = 1.5\phi_W - 0.5\phi_{WW}, \quad u_w > 0$$

绝对稳定，但仍然有一定的二阶假扩散计算误差!



2) 发挥CD与SUD各自优点的SCSD

将CD与SUD组合起来，Pe数小时CD占优，Pe大时SUD为主：

$$\phi_e^{SCSD} = \beta\phi_e^{CD} + (1-\beta)\phi_e^{SUD},\ 0 \le \beta \le 1$$

$$\beta = 1,\ \phi^{SCSD} \equiv \phi^{CD};\quad \beta = 0, \phi^{SCSD} \equiv \phi^{SUD};$$

可以证明，其临界网格Peclet数为： $\dfrac{\rho u \delta x}{\Gamma} = P_{\Delta,cr} = \dfrac{2}{\beta}$

通过调节Beta其临界Peclet数可在0～无穷之间变化，

故称为：Stability-controllable second-order difference —SCSD。

Ni M J，Tao WQ. Journal Thermal Science, 1998,7(2):119-130

可以证明 当 $\beta = 3/4, \phi^{SCSD} \equiv \phi^{QUICK}$

$$\phi_e = \frac{3}{4}\frac{\phi_E + \phi_P}{2} + \frac{1}{4}(1.5\phi_P - 0.5\phi_W) = \frac{1}{8}(3\phi_E + 6\phi_P - \phi_W)$$

$$= \phi_e^{QUICK}$$

从这一角度看QUICK也只有二阶精度（文献中有争议的问题）。

SCSD格式的用途：可用于多重网格的计算中，通过调节Beta使同一个格式可以用于不同疏密网格上的离散。20240423

但是：如何确定Beta值，特别是如何由计算结果来自动决定Beta之值？

2. SGSD格式(2002)

由 $P_{\Delta,cr}=\dfrac{2}{\beta}$ 可得 $\beta=\dfrac{2}{P_{\Delta,cr}}$，将分母中的 $P_{\Delta,cr}$ 用 $2+P_{\Delta}$ 来代替：

$$\beta=\frac{2}{2+P_{\Delta}}\begin{cases}P_{\Delta}\rightarrow 0,\beta\rightarrow 1, & \textbf{CD占优;}\\ P_{\Delta}\rightarrow \infty,\beta\rightarrow 0, & \textbf{SUD占优}\end{cases}$$

代表扩散作用

代表对流作用

可由计算得到的流场自动考虑扩散与对流的影响！还可区别不同的方向，*x, y, z* 可用各自方向流速。

显然这样确定的Beta值，格式一定是稳定的：

因为SCSD的 $P_{\Delta,cr}=\dfrac{2}{\beta}$

由 $\beta=\dfrac{2}{2+P_{\Delta}}\rightarrow\beta(2+P_{\Delta})=2;$

$$\longrightarrow P_{\Delta}=\frac{2}{\beta}-2<P_{\Delta,cr}=\frac{2}{\beta}$$

李增耀

Li Z Y, Tao WQ. A new stability-guaranteed second-order difference scheme. Numerical Heat Transfer-Part B, 2002， 42 (4): 349-365

推荐阅读(5)

SGSD格式的特点：

（1） 绝对稳定；

（2）具有某种自适应性，通过Peclet数将对流与扩散的相对重要性反映到格式中；

（3）至少具有二阶精度；

（4）计算工作量增加不大；

（5）当网格Peclet较大时，其特性很快接近SUD，计算精度有所下降。

总体上较优，建议采用，特别对于多重网格。

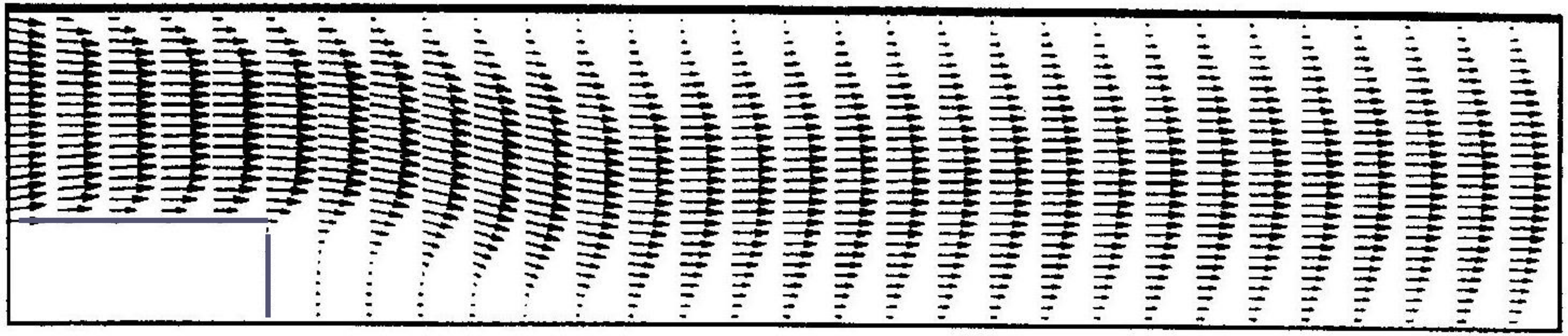


Figure Velocity vector using SGSD (Re=300, Er=1.5, 62 × 32 uniform grid)

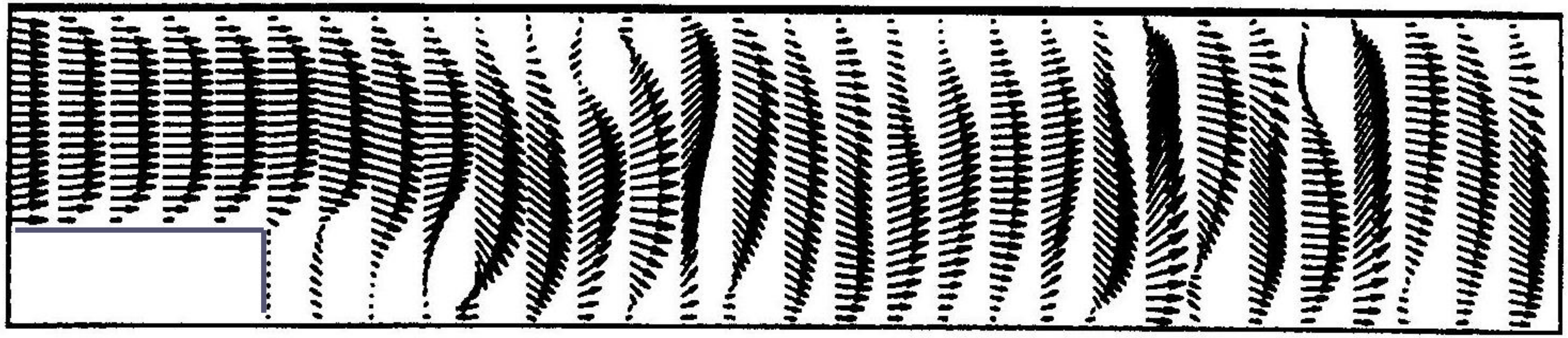
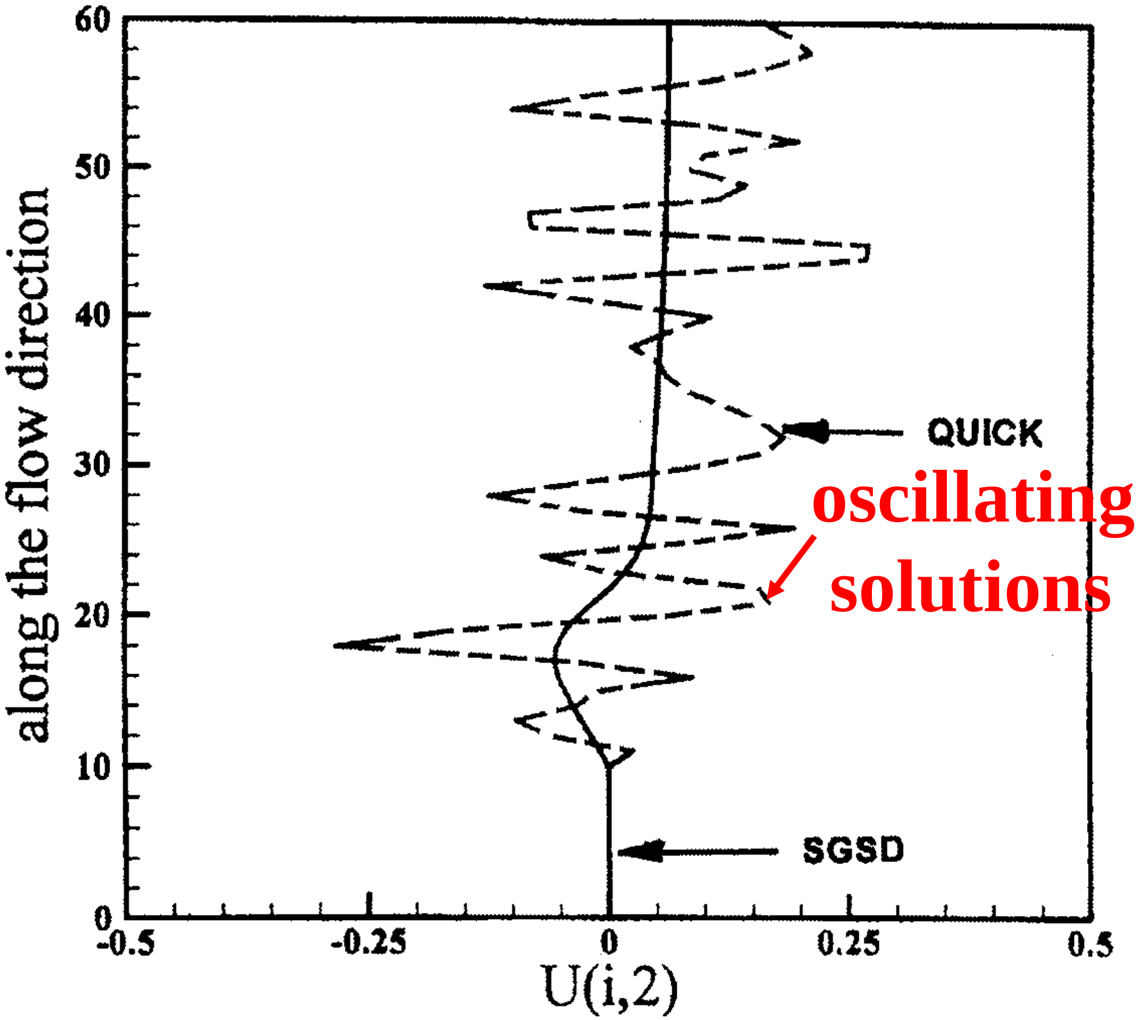


Figure Velocity vector using QUICK (Re=300, Er=1.5, 62 × 32 uniform grid)



Velocity distribution along the grid line parallel to the bottom

Example—Lid-driven cavity flow

Relative error

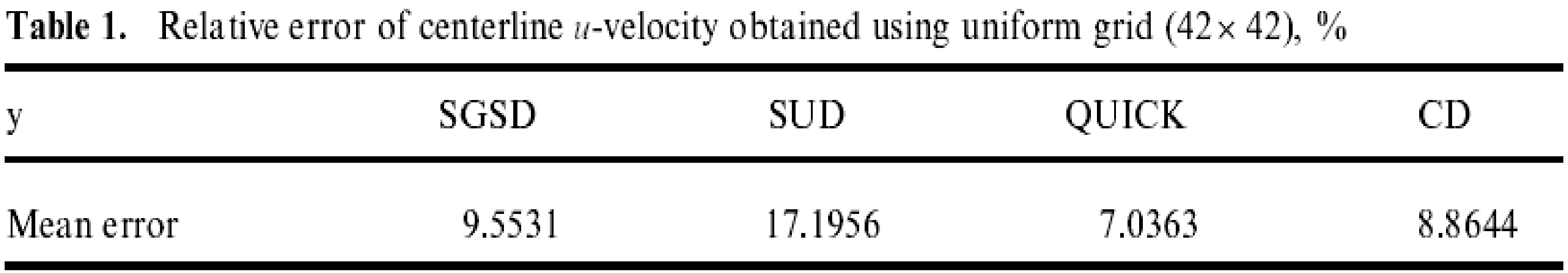
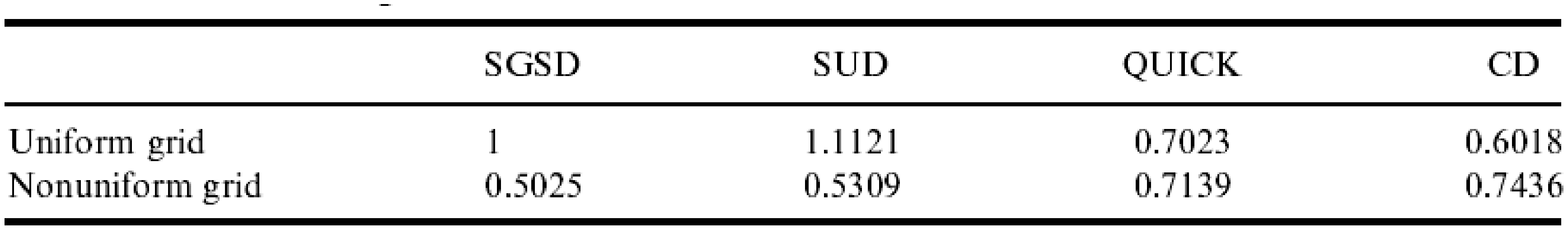


Table 1. Relative error of centerline *u*-velocity obtained using uniform grid (42× 42), %

y	SGSD	SUD	QUICK	CD
Mean error	9.5531	17.1956	7.0363	8.8644

Computational effort

CPU time



	SGSD	SUD	QUICK	CD
Uniform grid	1	1.1121	0.7023	0.6018
Nonuniform grid	0.5025	0.5309	0.7139	0.7436

2.1.5 关于格式稳定性与代数方程求解问题的说明

1. 现有分析格式稳定性的方法均基于5个假设：（1）一维问题；（2）线性问题（流速已知）；（3）无源项；（4）均分网格；（5）第一类边界条件。

任何一个假设的偏离均导致稳定性增加。

2. 现有分析对流格式稳定性的方法中扩散项均取二阶中心差分格式，因为扩散起增加稳定性的作用，扩散项格式变化导致对流项临界Peclet数的变化。

3. 延迟修正在高阶格式实施中得到广泛的采用：可以通过对采用FUD格式而编制的程序来实施高阶格式；

Yu Bo et al. Num. Heat Transfer, B, 2001, 40(4):343-365

4. 要区分代数方程迭代式求解过程的稳定性与格式的稳定性：代数方程迭代的稳定性可以保证得到解，但所得之解是否是振荡则取决于格式的稳定性。**格式的稳定性是其固有的属性，不能通过延迟修正来改进**。

Versteeg ,Malalasekera编著的 An Introduction to Computational Fluid Dynamics 对此解释有误。代数方程求解稳定性的分析可用von Neumann 方法，参见：

倪明玖

Ni M J et al. Numer Heat Transfer , B, 1999, 35(3):369-388

2.2 格式的有界性及规整变量图

2.2.1 格式有界性的定义

2.2.2 格式有界性与稳定性的联系与区别

2.2.3 研究有界性的重要工具---规整变量

2.2.4 采用规整变量的现有格式的界面插值定义

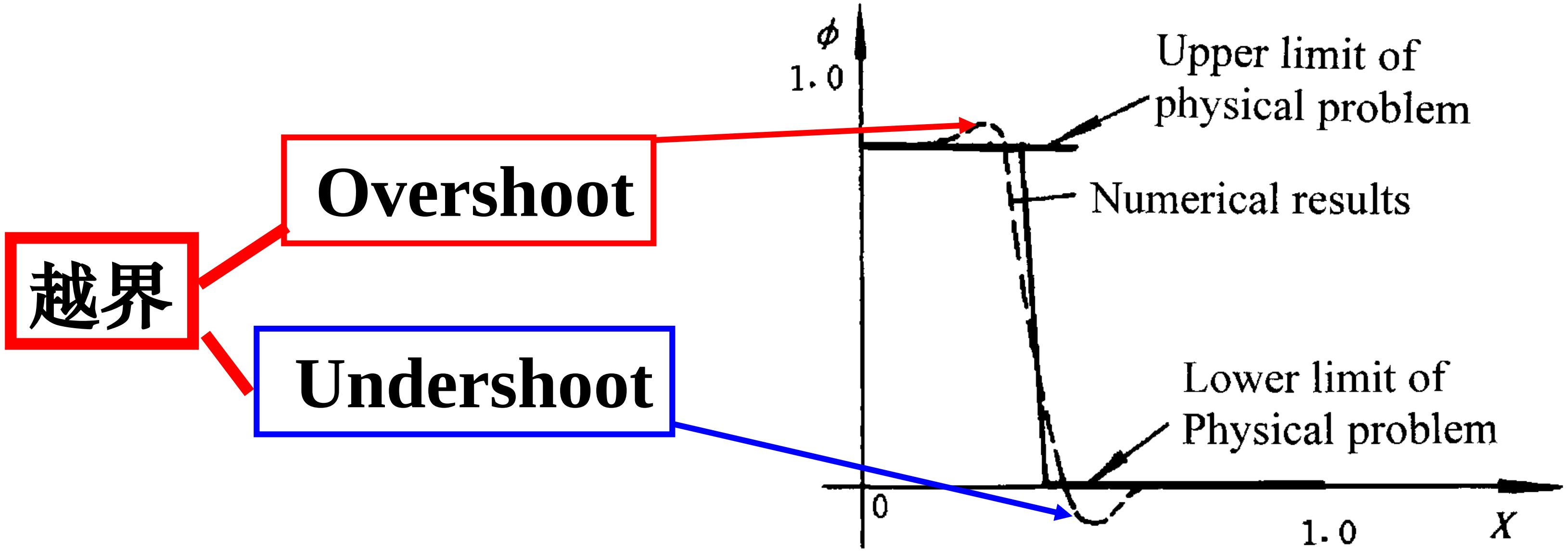
2.2.5 规整变量图

2.2 格式的有界性及规整变量图

2.2.1 格式有界性的定义

进行对流问题的数值计算时，如果计算结果不会超出物理问题本身所规定的上、下限的，称所采用的对流项格式具有有界性（Boundedness)。

采用不具有有界性的格式来离散时，如问题中物理场发生剧烈变化，会出现越界现象。



越界

Overshoot

Undershoot

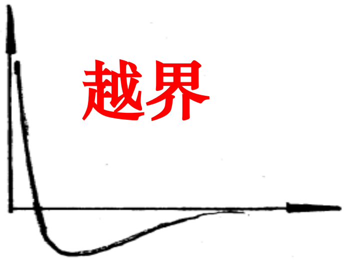
2.2.2 格式有界性与稳定性的联系与区别

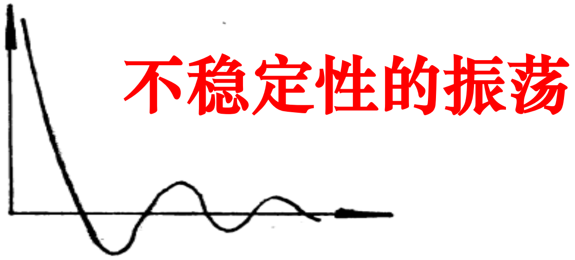
1. 区别

（1）有界性的判断仅取决与对流项离散方式，稳定性的判断取决于对流与扩散的联合作用结果；

（2）当物理量有剧烈变化时可能发生越界，而当 Peclet 数大时可能发生不稳定的振荡；

（3）越界使物理量一次过冲，但不稳定性则表现为多次的振荡；





2. 联系

凡有界的对流项格式必然绝对稳定；但绝对稳定的格式未必有界，如SUD。

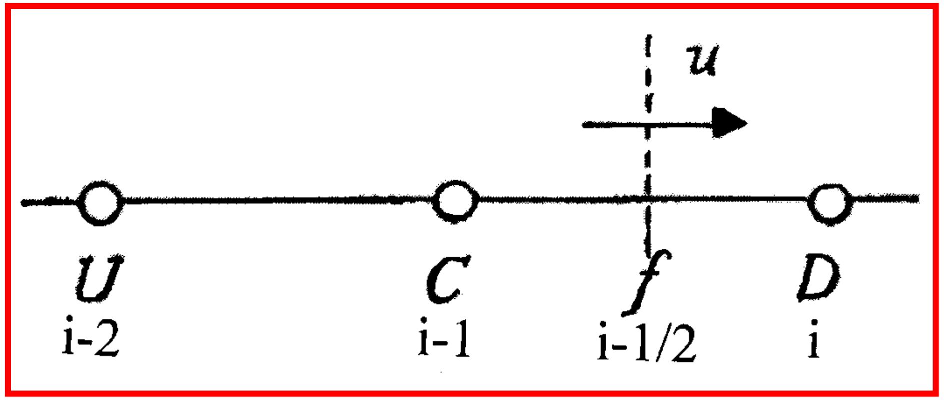
2.2.3 规整变量的定义

FVM中格式的定义为界面的插值： $\phi_f = f(\phi_U, \phi_C, \phi_D)$

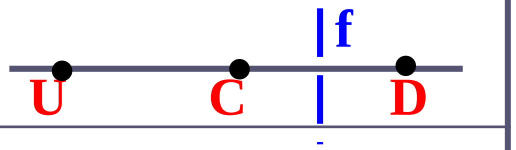
定义： $\tilde{\phi} = \dfrac{\phi - \phi_U}{\phi_D - \phi_U}$； $\tilde{\phi}_U = 0$; $\tilde{\phi}_D = 1$

称为规整变量 (Normalized variable)

则格式的定义简化为： $\tilde{\phi}_f = \tilde{\phi}_{i-\frac{1}{2}} = f(\tilde{\phi}_C)$



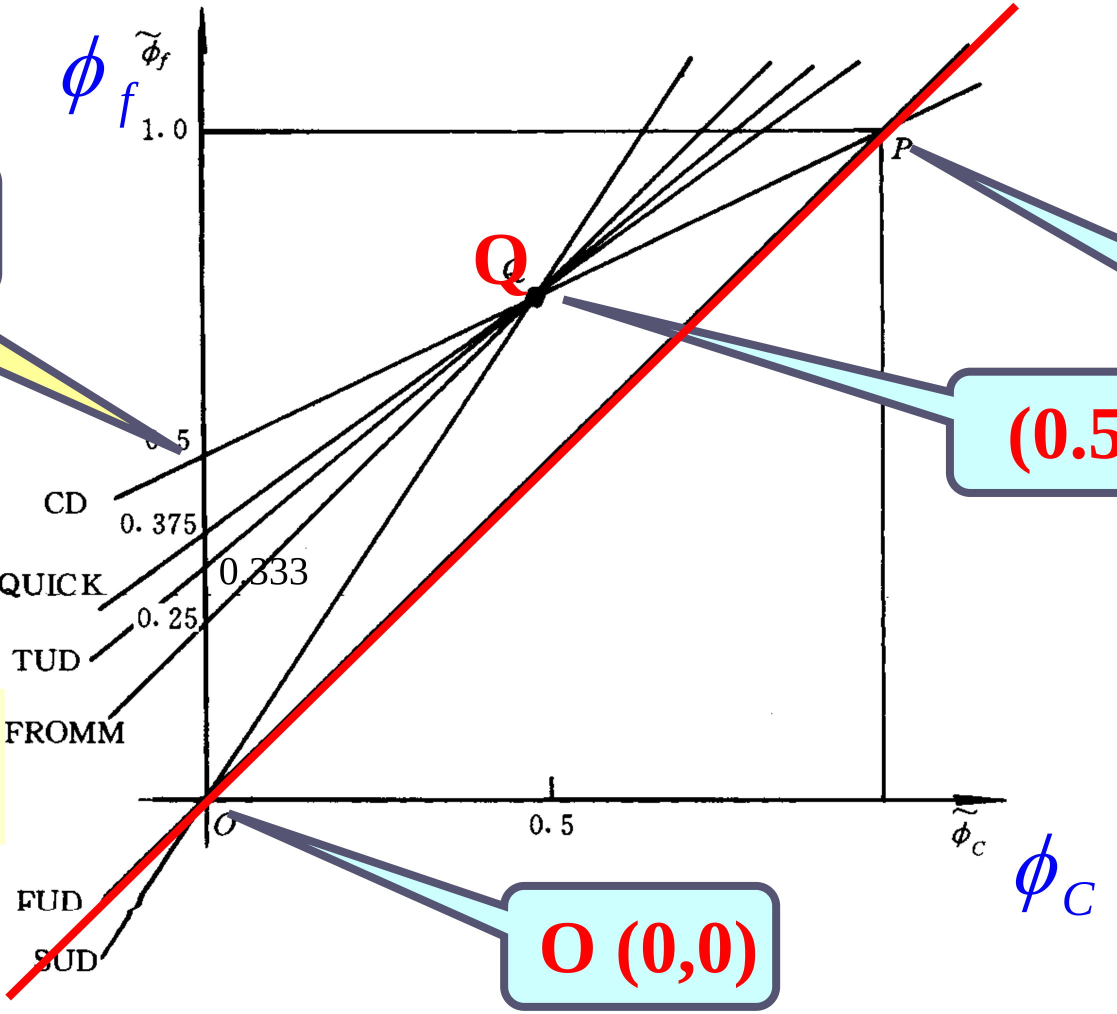
2.2.4 采用规整变量时现有格式的界面插值定义 (u>0)

No	格式	常规定义	规整变量定义
1	FUD	$\phi_f=\phi_C$	$\phi_f=\phi_C$ 
2	CD	$\phi_f=(\phi_C+\phi_D)/2$	$\phi_f=0.5(\phi_C+1)=\underline{0.75}+0.5(\phi_C-\underline{0.5})$
3	SUD	$\phi_f=1.5\phi_C-0.5\phi_U$	$\phi_f=1.5\phi_C-0=\underline{0.75}+1.5(\phi_C-\underline{0.5})$
4	QUICK	$\phi_f=\frac{1}{8}(6\phi_C+3\phi_D-\phi_U)$	$\phi_f=\frac{1}{8}(6\phi_C+3-0)=\underline{0.75}+0.75(\phi_C-\underline{0.5})$
5	TUD	$\phi_f=\frac{1}{6}(5\phi_C+2\phi_D-\phi_U)$	$\phi_f=\frac{1}{6}(5\phi_C+2-0)=\underline{0.75}+\frac{5}{6}(\phi_C-\underline{0.5})$

从2−5均为2阶及以上的格式： $\boxed{\phi_f=0.75+m(\phi_C-0.5)}$

2.2.5 规整变量图 (NVD)及其用途

以 ϕ_C 为横坐标，ϕ_f 为纵坐标，形成规整变量图，现有格式在该图上均为直线线（**特征线**）。



特征线截距

P (1,1)

(0.5,0.75)

O (0,0)

$$\phi_f = \frac{\phi_D + \phi_C - \phi_U}{4}$$

规整变量图的用途:

1. 判断格式的截差范围

凡是二阶及以上的格式特征线一定通过Q 点（0.5,0.75）。

2. 判断格式的稳定性

格式	规整图上特征线的截距	格式的 $P_{\Delta,cr}$
CD	0.5	2.0
QUICK	0.375	8/3
TUD	0.333	3
FROMM	0.25	4

截距的倒数等于格式的临界 Peclet 数。

凡是特征线过原点（0,0) 的格式绝对稳定。

3. 判断格式假扩散严重程度

特征线越接近对角线（FUD），假扩散越严重。

4. 判断格式的有界性

5. 构建高阶对流有界格式 (构建通过O, Q, P的折线)

FVM中对流项离散格式发展的两个里程碑

第一个里程碑是Patankar教授关于五种三点格式特性的总结（1980）。

第二个里程碑是Leonard提出的规整变量及规整变量图的分析方法（1988）。

2.3 格式有界性的判别准则

2.3.1 格式有界性对变量型线的要求

2.3.2 Gaskell/Lau提出的CBC (Convective boundedness criterion)

2.3.3 高阶组合格式

2.3.4 格式有界性G-L判据的改进与发展

2.3.5 Wei J J （魏进家）的进一步分析

2.3.6 不均匀及非结构化网格上高阶格式的实施

2.3 格式有界性的判别准则

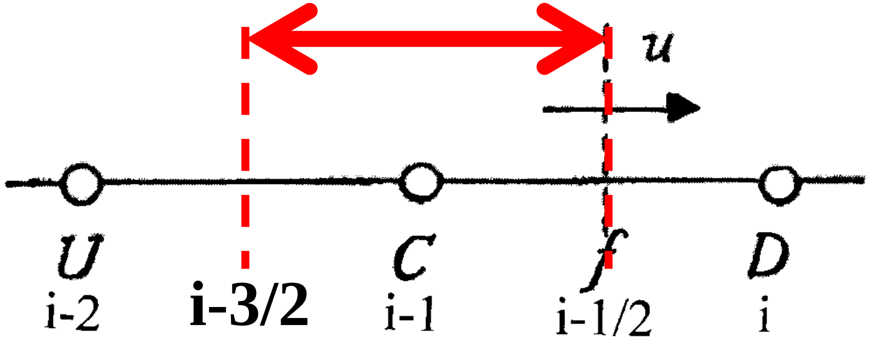
2.3.1 格式有界性对变量型线的要求

1. 1D模型方程的规整变量分析

将带源项的1D稳态模型方程

$$\frac{d(\rho u\phi)}{dx}=\frac{d}{dx}(\Gamma\frac{d\phi}{dx})+S_{\phi}$$

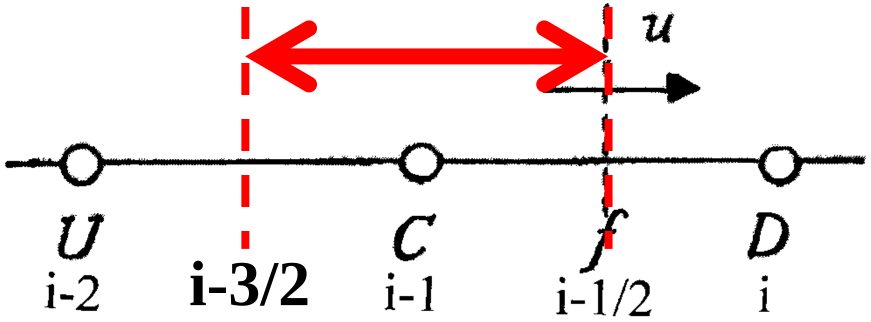
对控制容积 (i-1)做积分：



$$\rho u(\phi_{i-1/2}-\phi_{i-3/2})=(\Gamma\frac{d\phi}{dx})_{i-1/2}-(\Gamma\frac{d\phi}{dx})_{i-3/2}+\int_{i-3/2}^{i-1/2}S_\phi dx$$

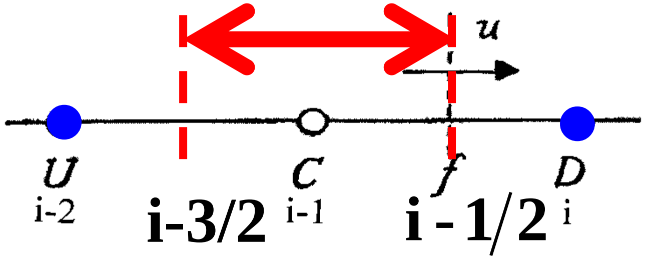
以 $\rho u(\phi_D-\phi_U)=\rho u(\phi_i-\phi_{i-2})$ 来规整上式

$$(\phi_{i-\frac{1}{2}}-\phi_{i-\frac{3}{2}})=\frac{(\Gamma\frac{d\phi}{dx})_{i-1/2}-(\Gamma\frac{d\phi}{dx})_{i-3/2}+\int_{i-3/2}^{i-1/2}S_\phi dx}{\underline{\rho u(\phi_i-\phi_{i-2})}}=S_\phi^*$$



于是对于规整变量沿着x 轴的变化，有以下关系：

$$\left\{\begin{array}{l}\phi_{i-1/2}-\phi_{i-3/2}=S_{\phi}^{*}\\ \phi_{i-2}=0;\ \phi_{i}=1.0\end{array}\right.$$



这一关系式规定了两个界面规整值应该满足的条件；

现要寻找界面值 ϕ_f 与C 点值 ϕ_C 之间的关系。

以下分三种可能的情形讨论界面的取值特性。

2. 三种可能情形

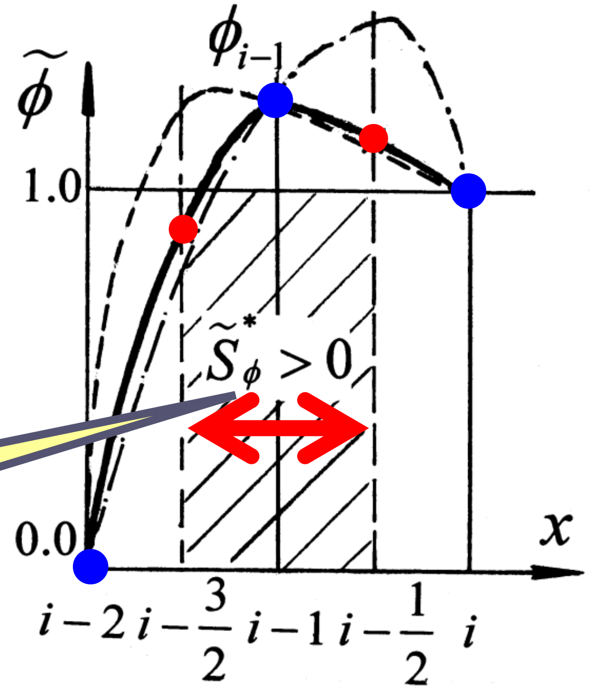
1) $S_{\phi}^{*}>0,\phi_{i-1}>1.0$

黑实线为合理型线

$$0\le\phi_{i-\frac{3}{2}}\le\phi_{i-\frac{1}{2}}$$

$$1\le\phi_{i-\frac{1}{2}}\le\phi_{i-1}$$

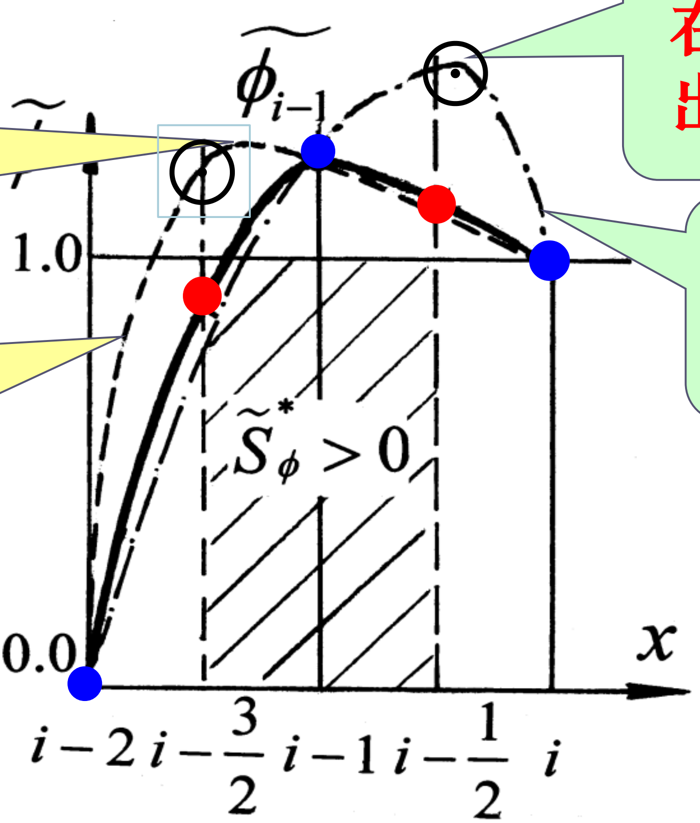
关注界面值应满足的条件



在此区域内不应出现局部最大值

因为：在[i-3/2, i-1] 内源项>0

$[\cancel{\phi_{i-3/2} > \phi_{i-1/2}}]$



在此区域内不应出现局部最大值

因为：在[i-1/2, 1] 之内源项 =0

$[\cancel{\phi_{i-1/2} > \phi_{i-1}}]$

$$\boxed{0 \le \phi_{i-\frac{3}{2}} \le \phi_{i-\frac{1}{2}}} \qquad \boxed{1 \le \phi_{i-\frac{1}{2}} \le \phi_{i-1}}$$

是合理的选择。

沿曲线右行

沿曲线左行

其它两种型线的讨论

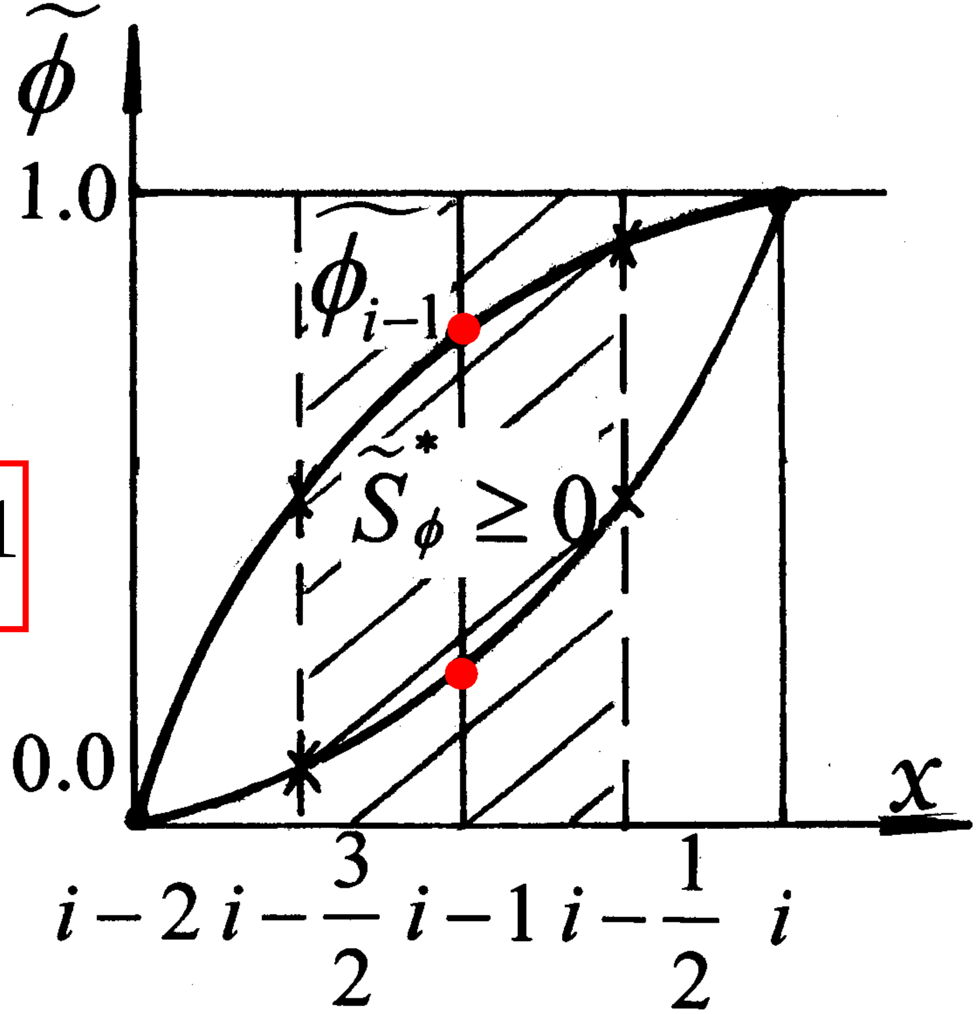
2) $S_{\phi}^{*} \geq 0,\ 0 \leq \phi_{i-1} \leq 1$

任何在[0,1]之间单调上升的曲线均满足要求：

$$\boxed{0 \leq \phi_{i-\frac{3}{2}} \leq \phi_{i-1} \leq \phi_{i-\frac{1}{2}} \leq 1}$$

$$\phi_{i-1} \leq \phi_{i-\frac{1}{2}} \leq 1,$$

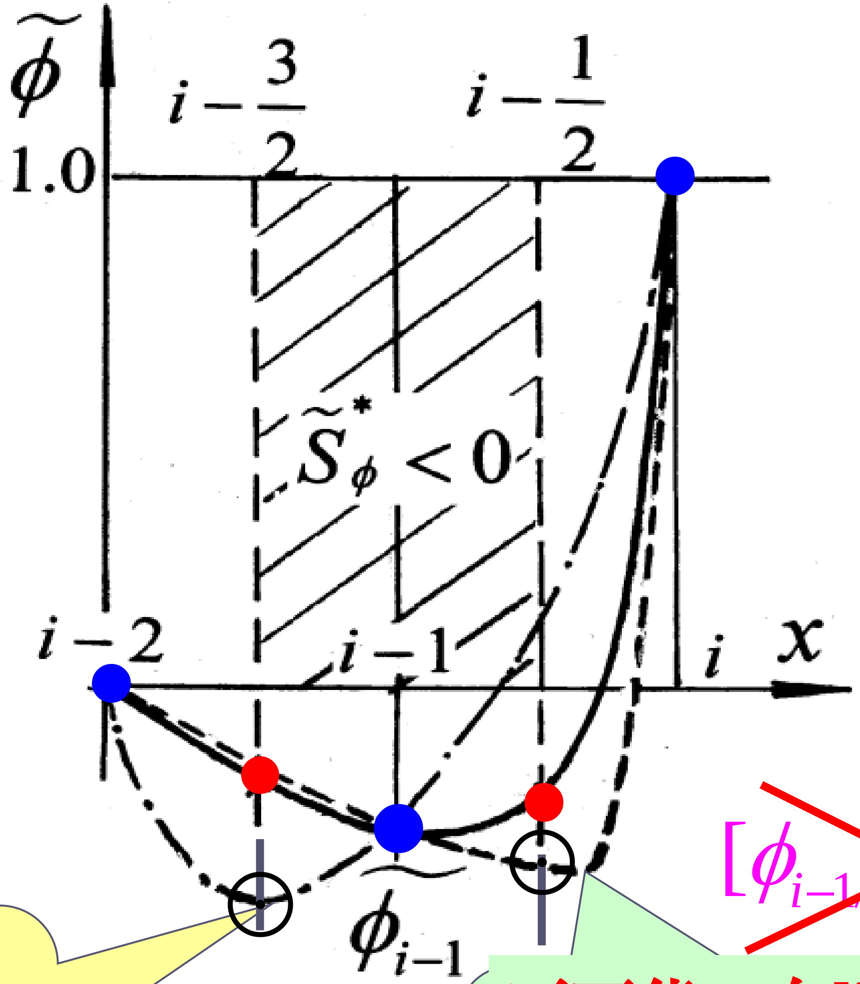
$$\phi_{C} \leq \phi_{f} \leq 1$$



3) $S_\phi^* \le 0, \phi_{i-1} < 0$

黑实线为合理型线

$$\boxed{0 \ge \phi_{i-\frac{3}{2}} \ge \phi_{i-\frac{1}{2}} \ge \phi_{i-1}}$$



~~$[\phi_{i-3/2} < \phi_{i-1/2}]$~~

~~$[\phi_{i-1/2} < \phi_{i-1}]$~~

不可能：在[i-3/2, i-1] 内源项<0，故在此区域不应有局部最小值

不可能：在[i-1/2，1]之内源项 =0,故在此区域不应出现局部最小值

2.3.2 Gaskell/Lau 的CBC

1988 Gaskell/Lau 根据上述分析提出为使格式具有有界性，界面插值 $\phi_f = f(\phi_C)$ 应满足：

---

1. $f(\phi_C)$ **是** ϕ_C **的连续的或分段连续的递增函数（正影响的原则）；**

2. 当 $0 \le \phi_C \le 1$ **时** $\phi_C \le \phi_f \le 1$ （据型线2）

3. 当 $\phi_C > 1$ **或** $\phi_C < 0$ **时**

$$\boxed{1 \le \phi_{i-\frac{1}{2}} \le \phi_{i-1}} \rightarrow \phi_f \le \phi_C$$

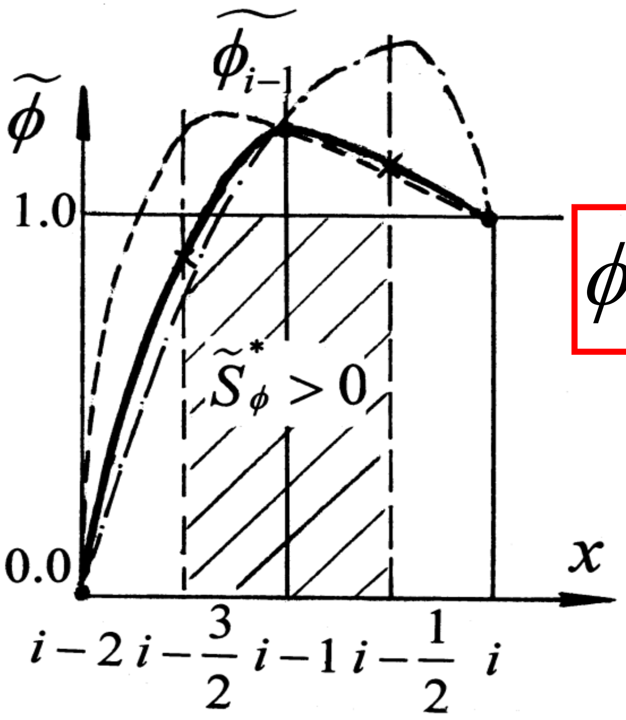
$$\boxed{0 \ge \phi_{i-\frac{3}{2}} \ge \phi_{i-\frac{1}{2}} \ge \phi_{i-1}} \rightarrow \phi_f \ge \phi_C$$

$$\phi_f = \phi_C$$

（据型线1, 3）

型线1 $\phi_{i-1} > 1 : \phi_{i-1/2} \le \phi_{i-1}$

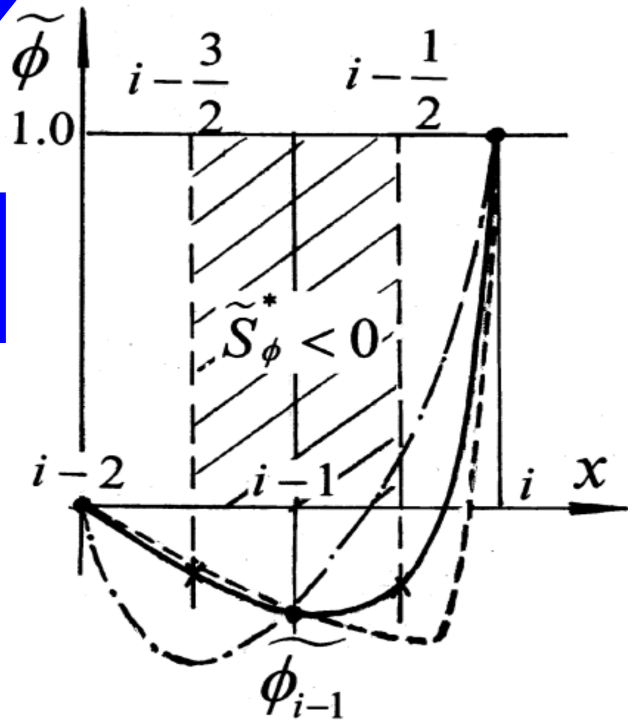
$\phi_{i-1} < 0 : \phi_{i-1/2} \ge \phi_{i-1}$ 型线3



$\phi_f \le \phi_C$

$\phi_f \ge \phi_C$

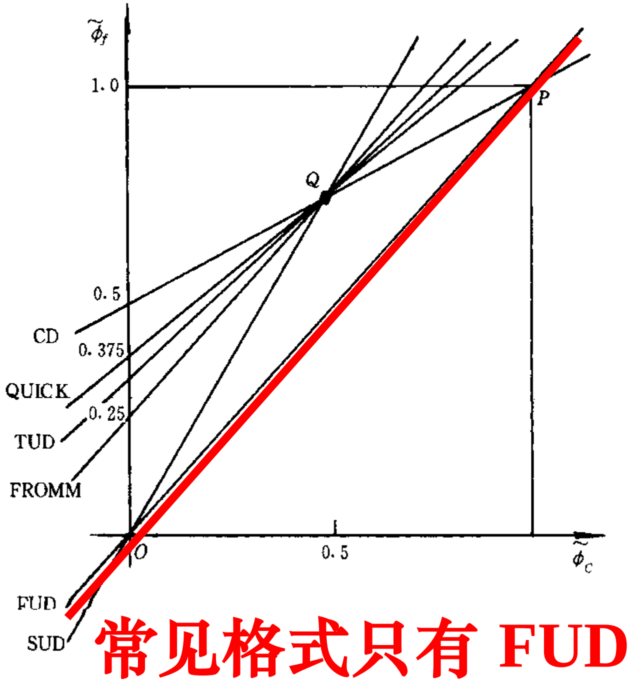
$\phi_f = \phi_C$



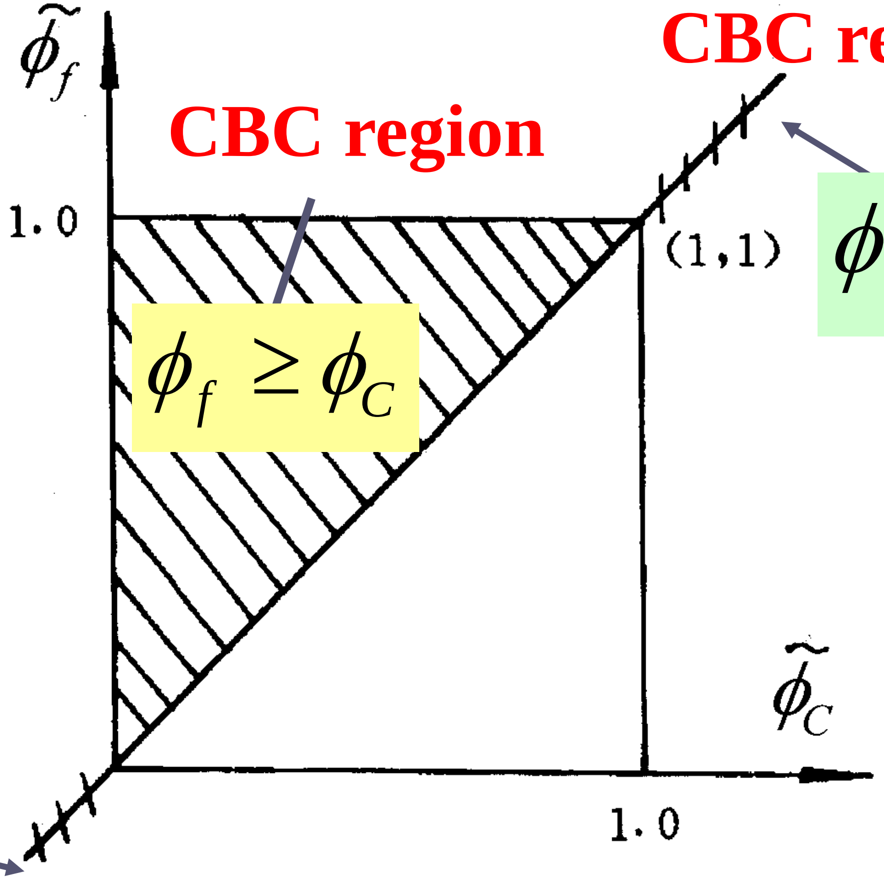
满足这一CBC的有界格式的特征线位于NVD图的上三角区域。

Gaskell P H, Lau A K C. Int J Numer Methods in Fluids, 1988,8:617-641

在NVD图上的G/L 的满足有界性条件的区域(1988)



常见格式只有 FUD
具有有界性！



CBC region

CBC region

$\phi_f \ge \phi_C$

$\phi_f = \phi_C$

CBC region

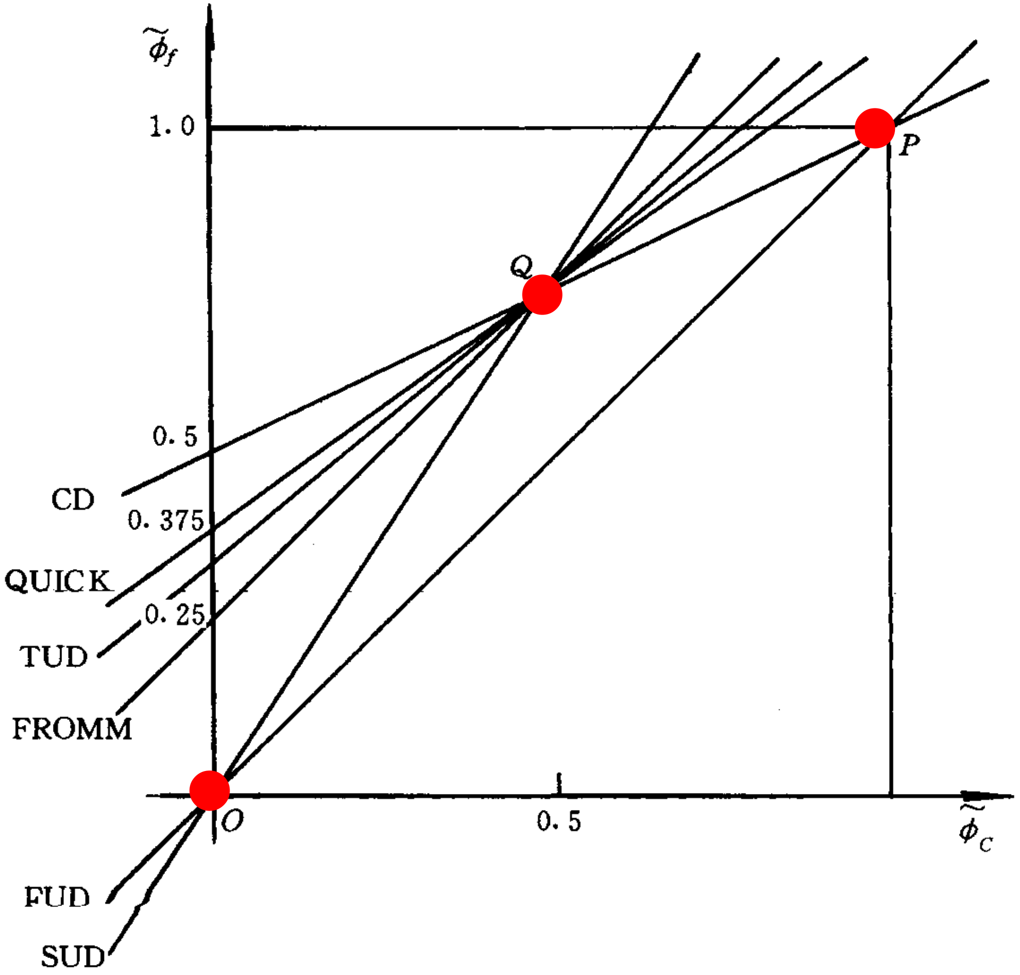
$\phi_f = \phi_C$

但如果格式的特征线不是直线，则过P,Q,O三点的任何连线都有界而且至少二阶精度！

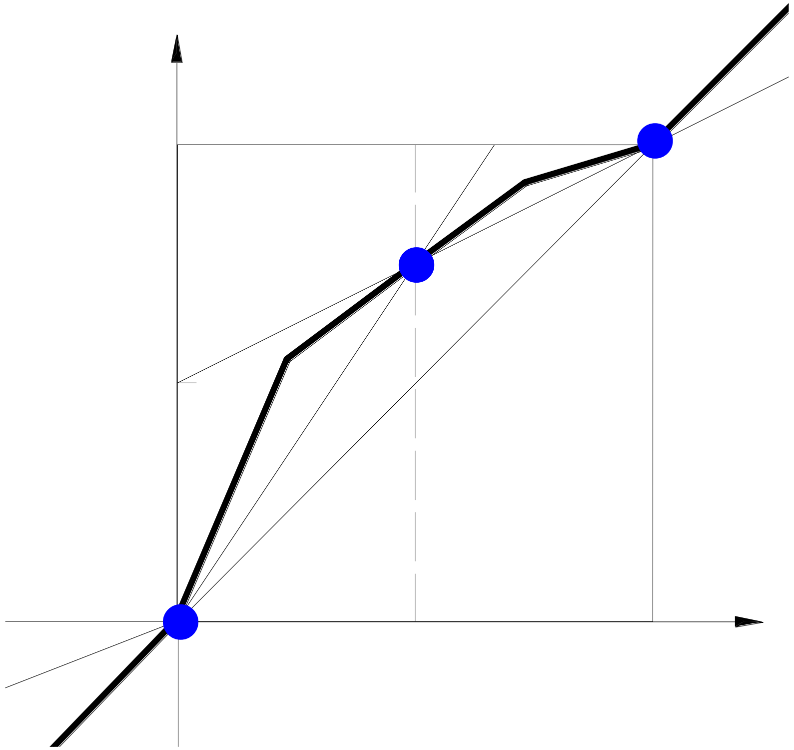
2.3.3 高阶组合格式

1.高阶有界格式的特征线在NVD图上不是直线

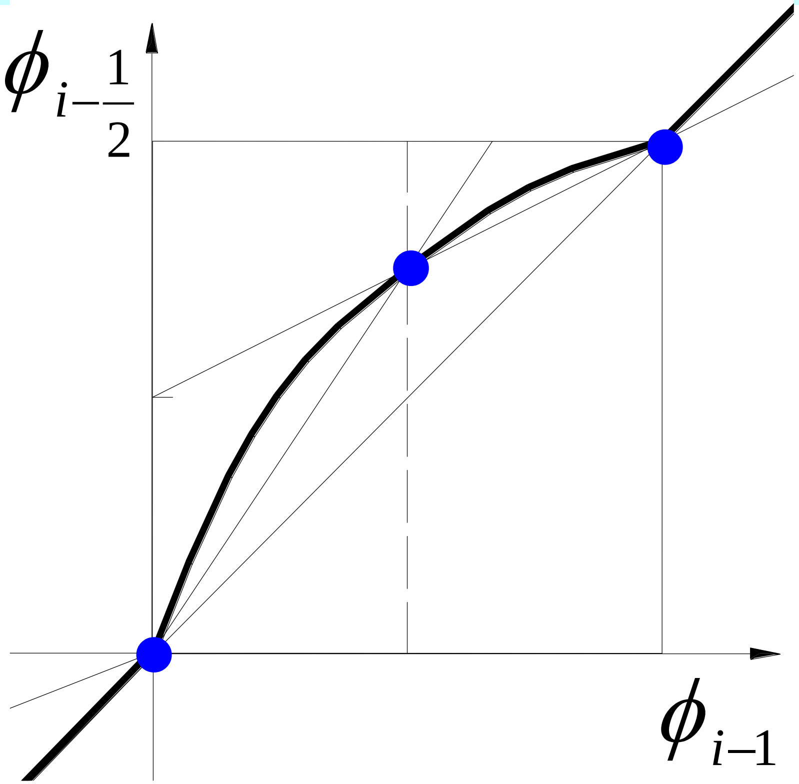
高阶有解格式的**特征线必须通过P，Q，O三个点**，因此其特征线不可能是直线；唯一有界的低阶格式是FUD，其特征线是直线。



文献中已经提出了十余种将该三点连接起来的方案，称为组合格式 (Composite scheme)。例如：



COPLA (combination of piecewise linear approximation)



HLPA (hybrid linear/ parabolic approximation) (CLAM)

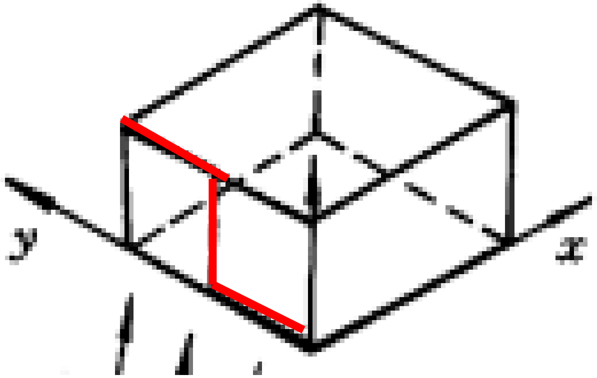
2. 考核高阶组合格式的常见问题

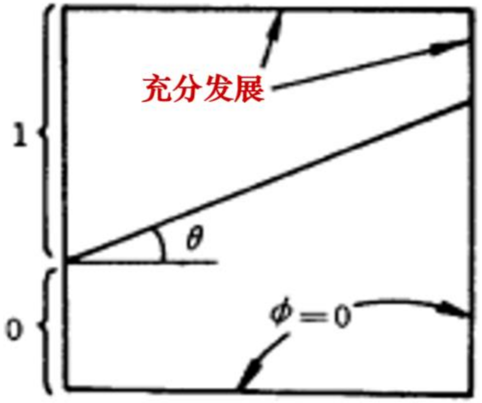
格式的有界性只取决于对流项，因此均用纯对流问题来考核：在给定的流场下利用所考核的格式研究物理量被传递的情形。对2D问题，控制方程为：

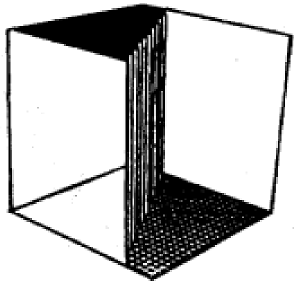
$$\begin{cases} u\dfrac{\partial\phi}{\partial x}+v\dfrac{\partial\phi}{\partial y}=0,\ u,v\ \text{给定} \\ \text{上游边界}\ \phi\ \text{给定，下游按充分发展处理。} \end{cases}$$

常用问题举例如下：

1) 阶梯型标量场在倾斜均匀流场中的传递

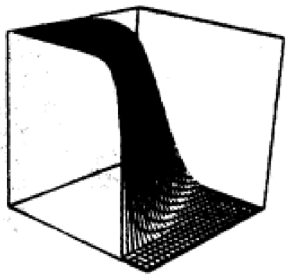






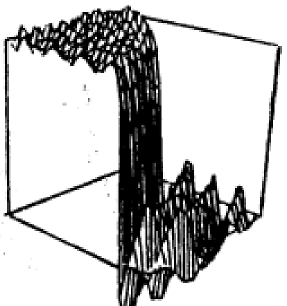
(a) 精确解

Exact solution



(b) FUD

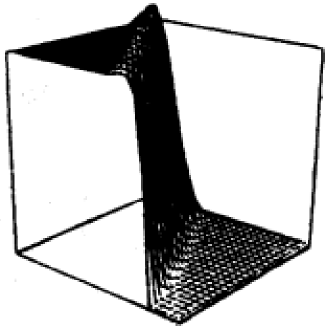
FUD



(c) CD

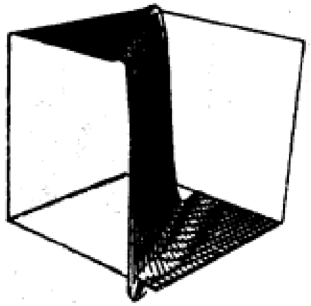
CD

CLAM



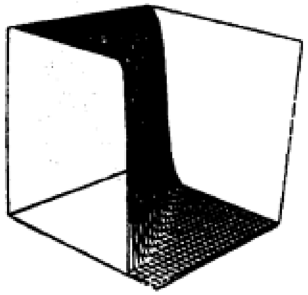
(d) SUD

SUD



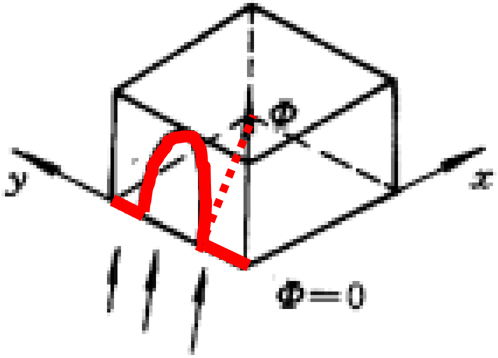
(e) QUICK

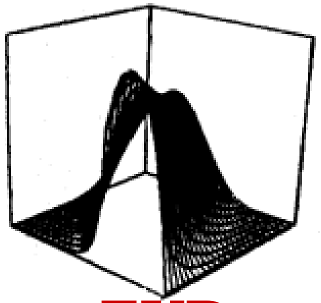
QUICK

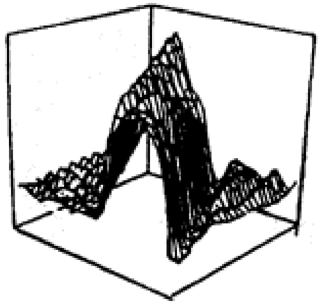


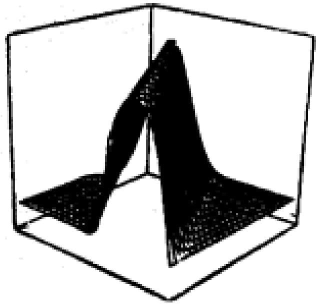
(f) CLAM

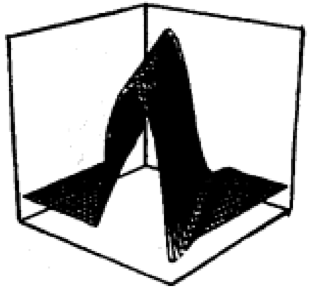
2) 抛物型标量场在倾斜均匀流场中的传递

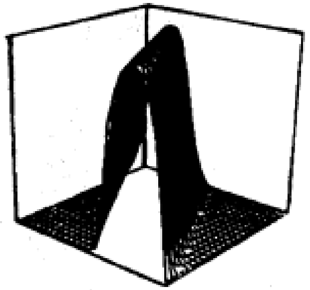


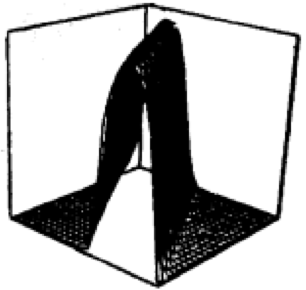

FUD

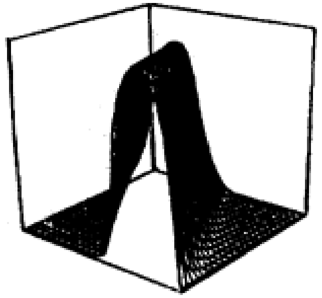

CD

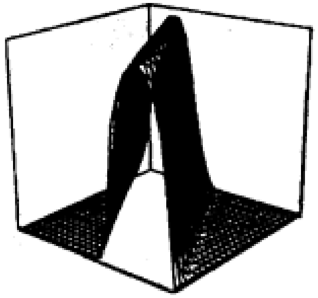

SUD

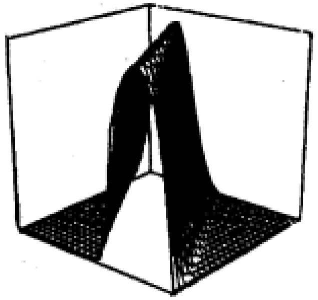

QUICK


CLAM


EULER

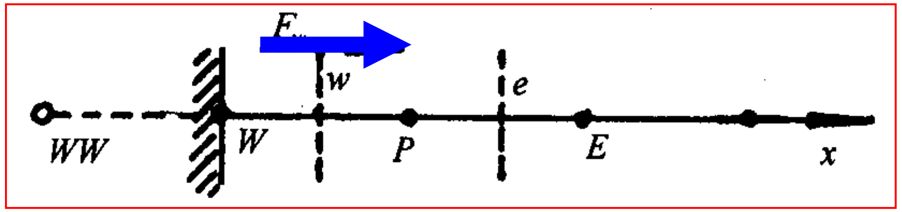

MINMOD


MUSCL


OSHER

3. 高阶组合格式实施中的一些处理

1) 近边界点的界面的处理



对邻近边界的内节点的界面可能找不到构建高阶格式所需的远邻点，建议采用降阶方法处理：

$$\text{当 } F_w > 0 \text{ 时}\begin{cases}\phi_w = \phi_W & \text{降阶处理}\\ \text{不降阶,虚拟点法确定} & \phi_{WW} = 2\phi_W - \phi_P\end{cases}$$

界面之值 ϕ_w 按照格式的定义由 ϕ_W, ϕ_{WW} 而定。

2) 代数方程求解方法 建议采用延迟修正方法。

2.3.4 格式有界性G-L判据的改进与发展

1. Gaskell/Lau的CBC只是充分条件而不是充要条件（Yu B的工作）

Gaskell/Lau 的CBC提出后文献中普遍认为（包括作者本人）这是格式有界的充要条件。

宇波(Yu B.)在其博士论文（1998）中第一个指出Gaskell / Lau的CBC只是充分条件而不是充要条件，并提出了另外一个CBC的判据，称为Extended CBC (ECBC)。

宇波的ECBC图为：

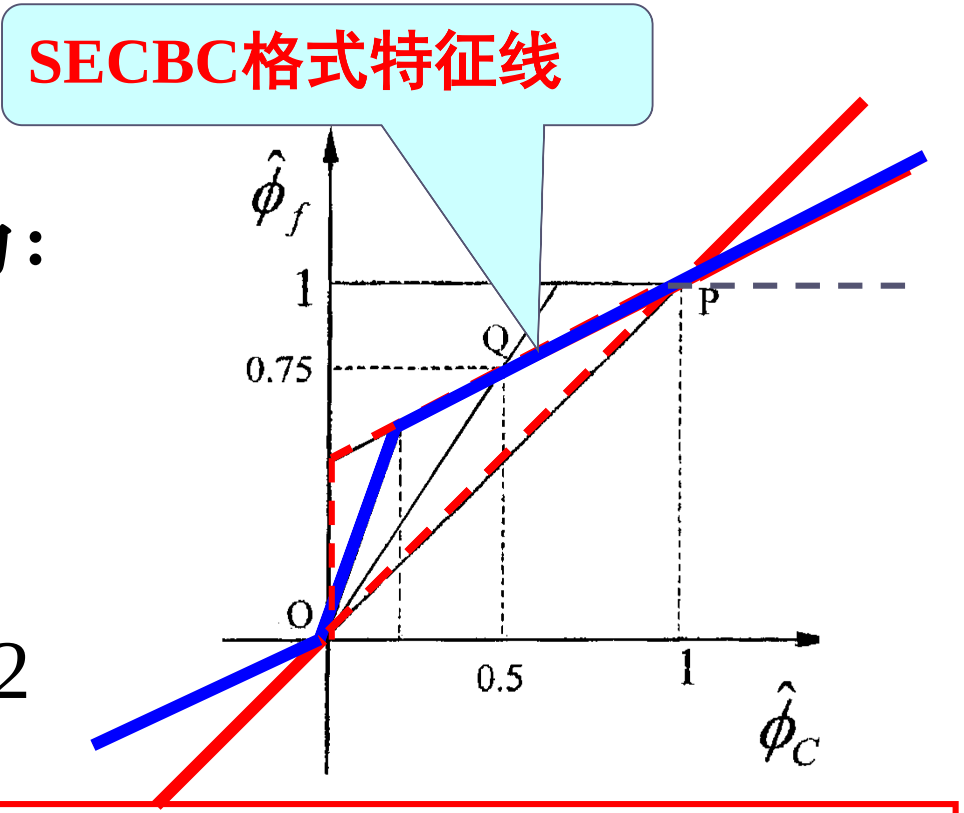
其上的一个有界格式

（称为SECBC）的取值为：

$$\phi_C \le 0, \phi_f = 0.5\phi_C;$$

$$0 < \phi_C < 0.2, \phi_f = 1.5\phi_C;$$

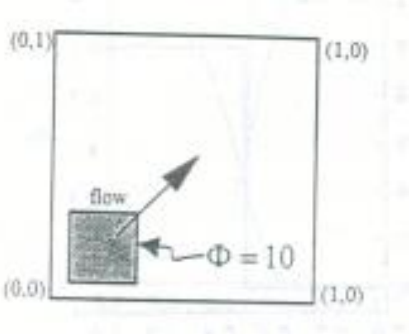
$$\phi_C \ge 0.2, \phi_f = (\phi_C + 1)/2$$

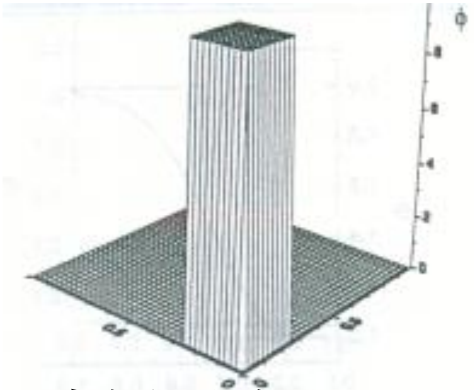


对于柱体标量场在倾斜均匀流场中的传递，数值计算证明该格式具有有界性。

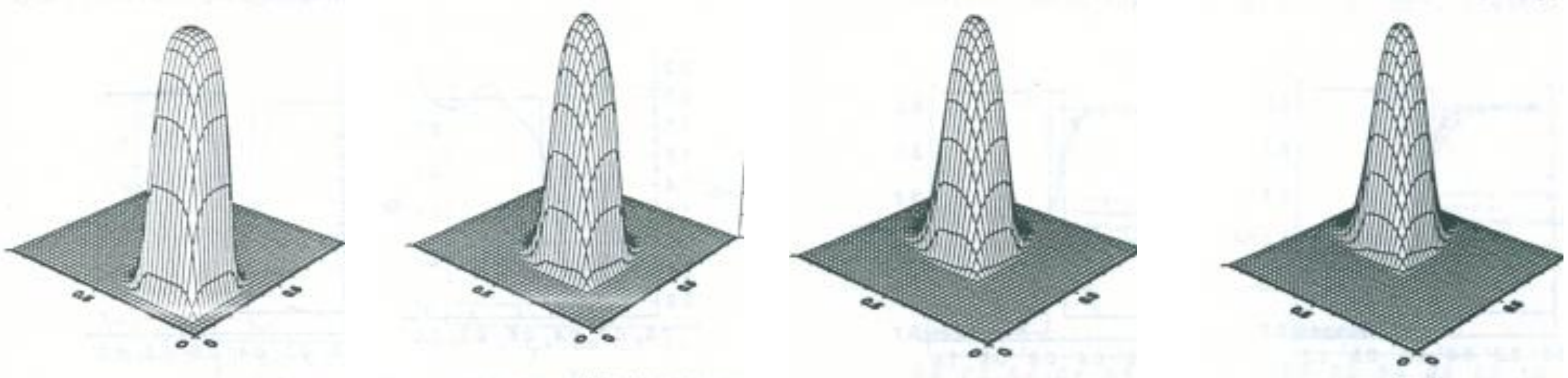
Yu B, Tao WQ, Wei JJ, et al. Discussion on momentum interpolation method for collocated grids of incompressible flow. Numerical Heat Transfer, Part B,2002,42 (2): 141-166

该问题的控制方程为： $\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}=0$





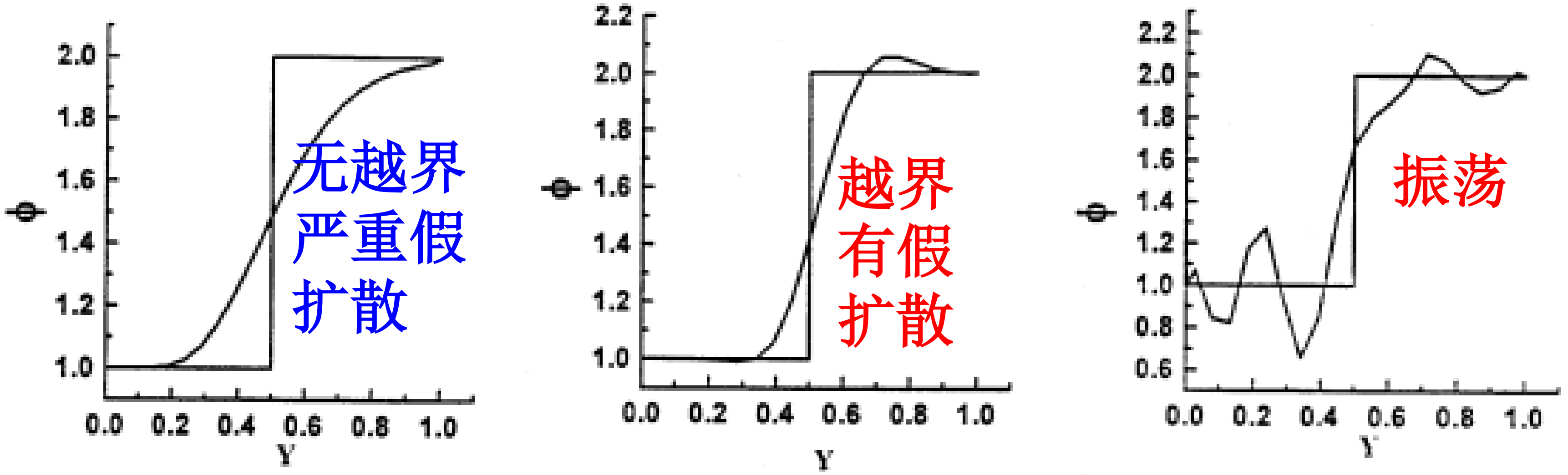
用SECBC计算得到不同时刻的标量场：



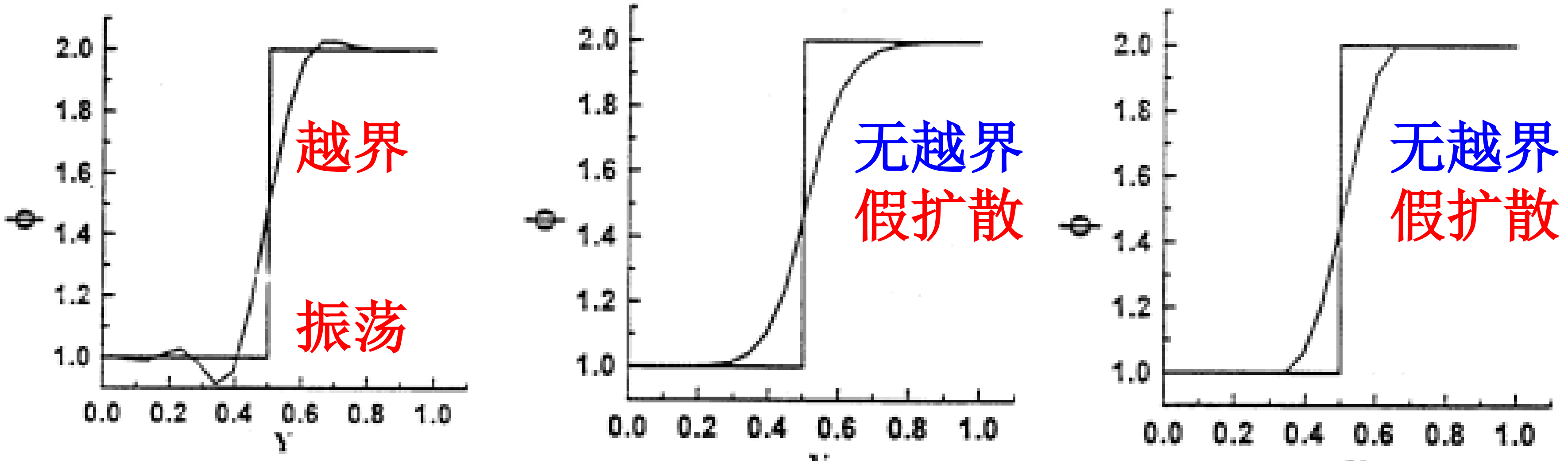
20 时间步计算标量场　50 时间步计算标　80 时间步计算标量场　100 时间步计算标

用SECBC计算得到的结果没有出现越界,有假扩散。

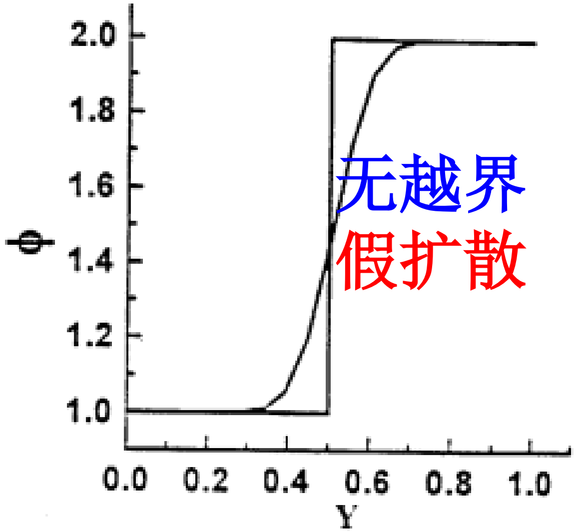
用不同格式计算阶梯型标量场在倾斜均匀流场中传递的结果：



(a)一阶迎风格式与精确解的比 (b) 二阶迎风格式与精确解 中心差分格式与精确解的比较



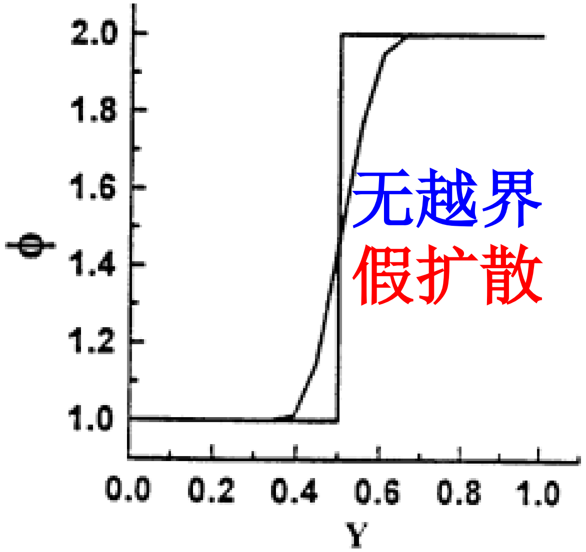
(d) QUICK 格式与精确解的比 (e)MINMOD（SOUCOU (f) OSHER（BDBD）



无越界

假扩散

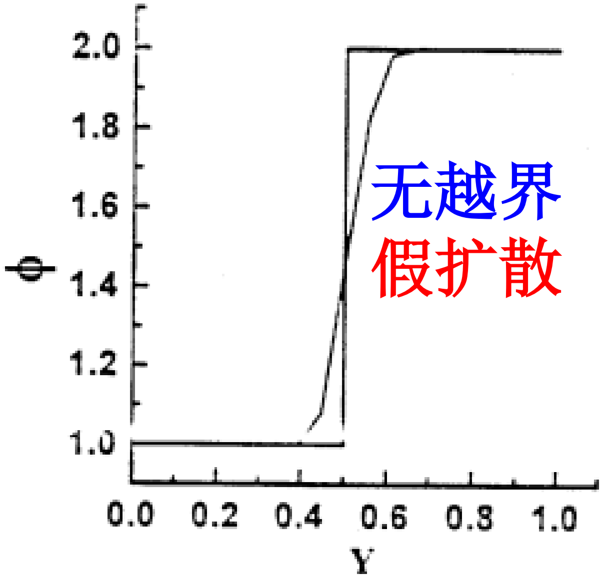
(g) CLAM（HLPA）



无越界

假扩散

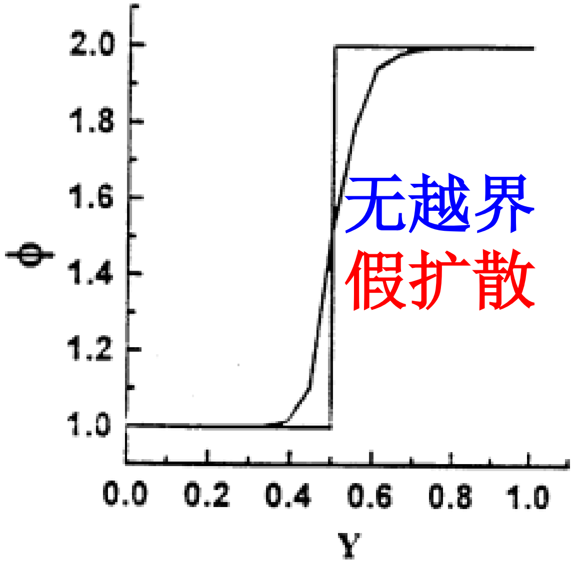
(h) SMART



无越界

假扩散

(i) STOIC 格式与精确解



无越界

假扩散

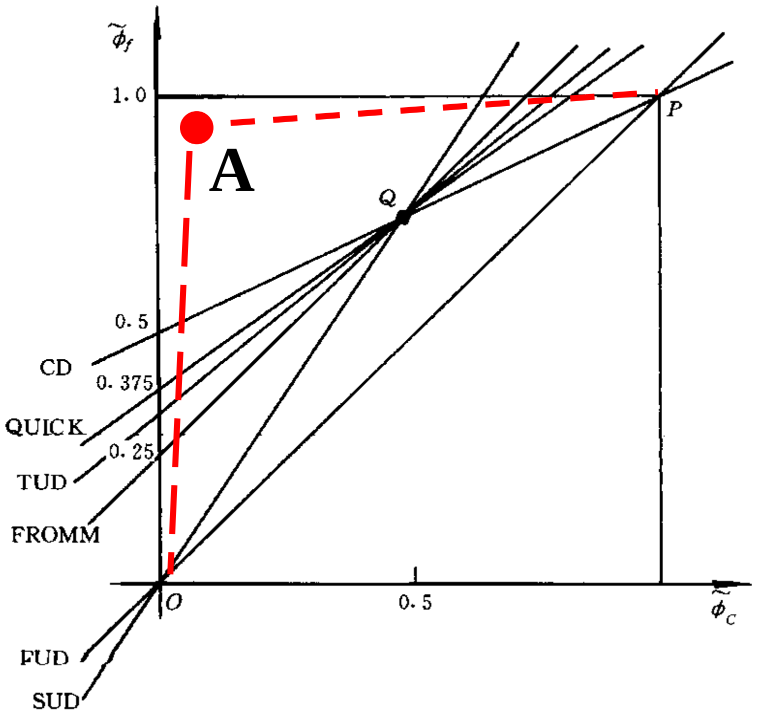
(j) SBECBC 格式与精确解的比

2. Hou P L 的改进 (2003)

在Gaskell/Lau 的CBC区域中取一点A:

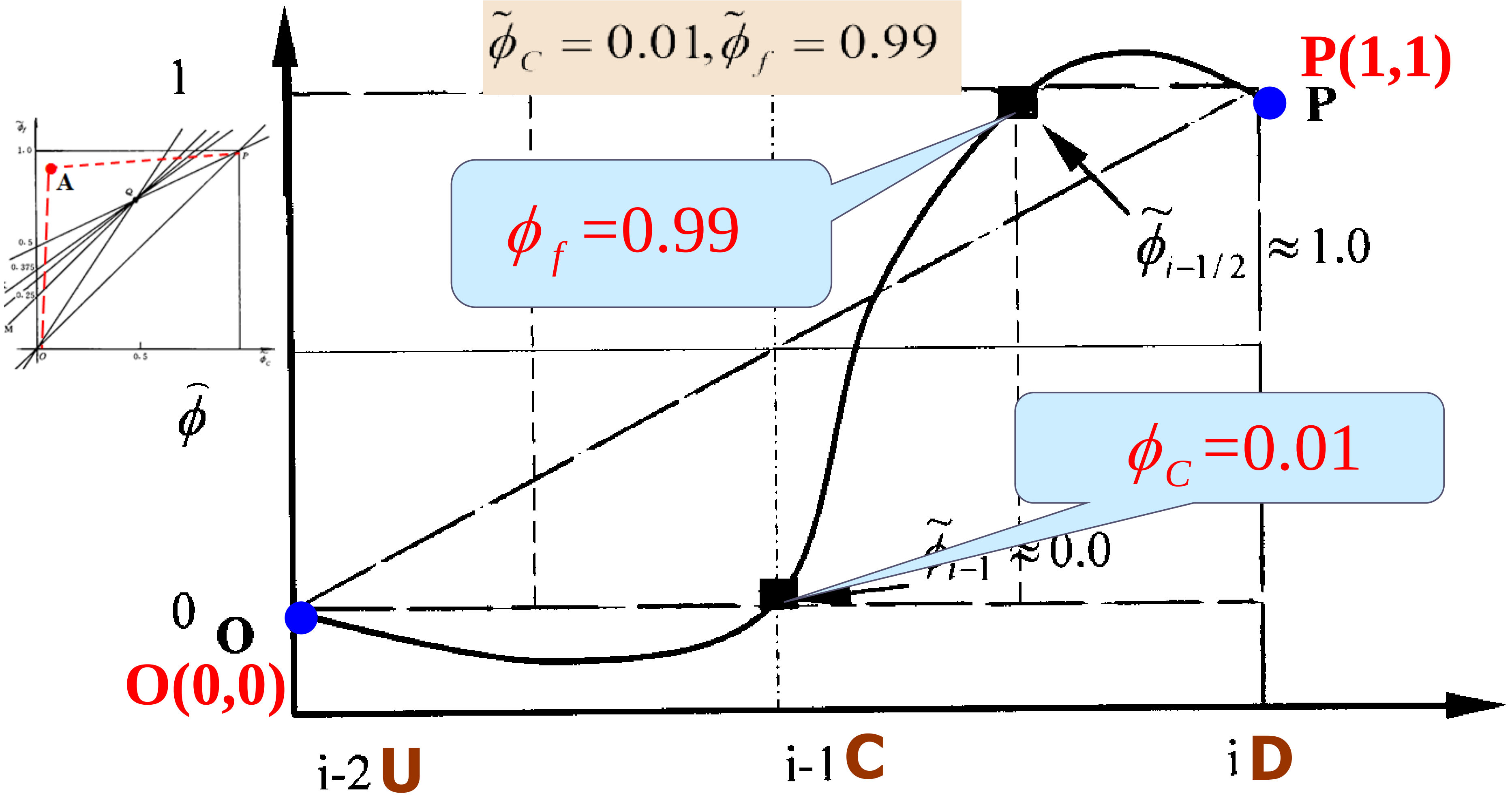
$$\boxed{\tilde{\phi}_C = 0.01, \tilde{\phi}_f = 0.99}$$

以位置为横坐标，规整变量为纵坐标，作图，会得出很不合理的型线:



Hou P L, Tao W Q, Yu M Z., Refinement of the convective boundedness criterion of Gaskell and Lau, Engineering Computations, 20(2003) 1023-1043

推荐阅读(6)



$\tilde{\phi}_C = 0.01, \tilde{\phi}_f = 0.99$

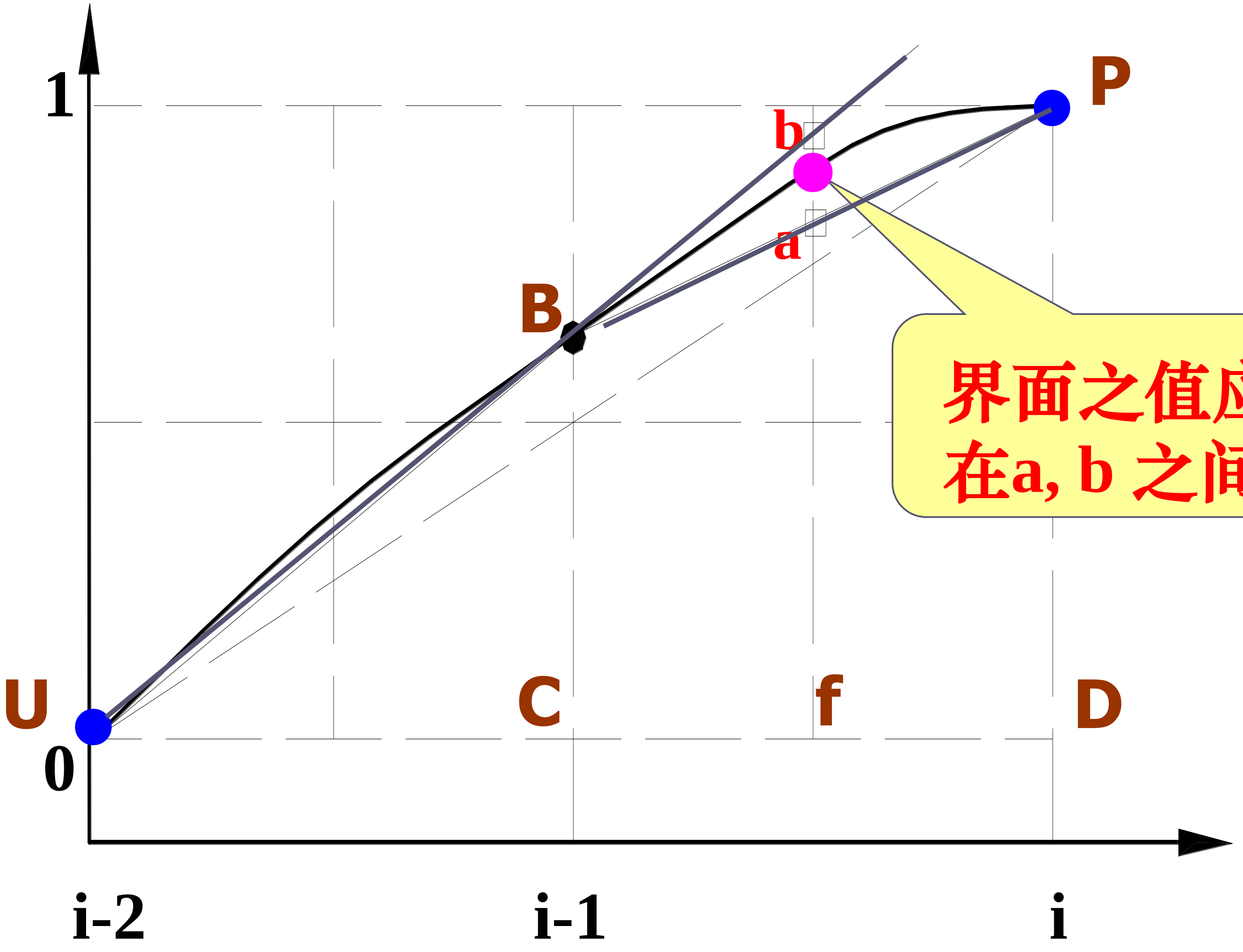
ϕ_f =0.99

ϕ_C =0.01

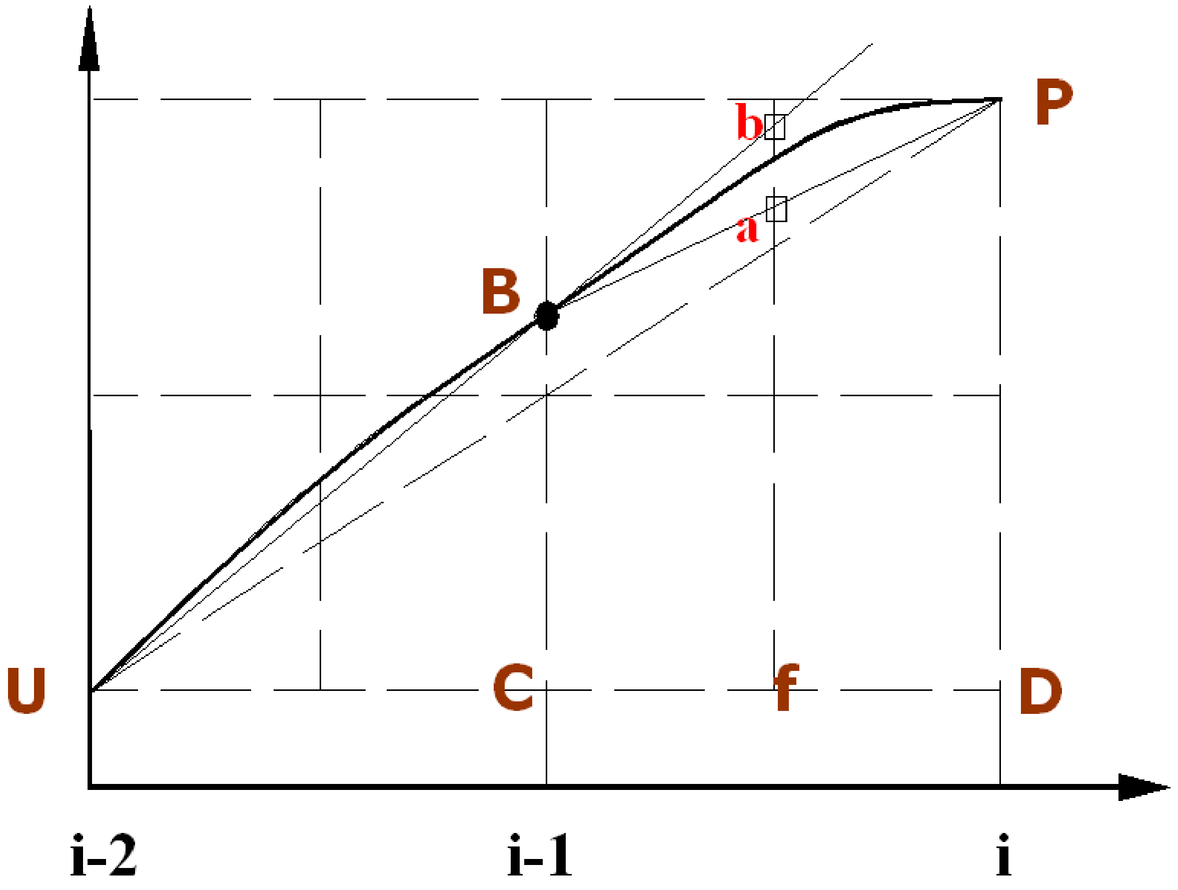
这样的型线显然不合理!

以 $0.5 \le \phi_{i-1} \le 0.75$时为例讨论合理的型线应是什么?

当 $0.5 \le \phi_{i-1} \le 0.75$ 时，合理的型线应该是



界面之值应在a, b 之间



在三角形Uf b中

f b=1.5CB

f b=1.5 ϕ_C

在四边形BCDP中

f a=0.5(CB+DP)

f a=0.5(ϕ_C+1)

当 $0.5 < \phi_C < 0.75$ 时 ：

$$\boxed{\phi_a = (\phi_C + 1)/2} < \phi_f < \boxed{\phi_b = 1.5\phi_C}$$

详细地分析了 ϕ_C 的不同取值范围内的合理型线后，侯平利得出以下改进的CBC：

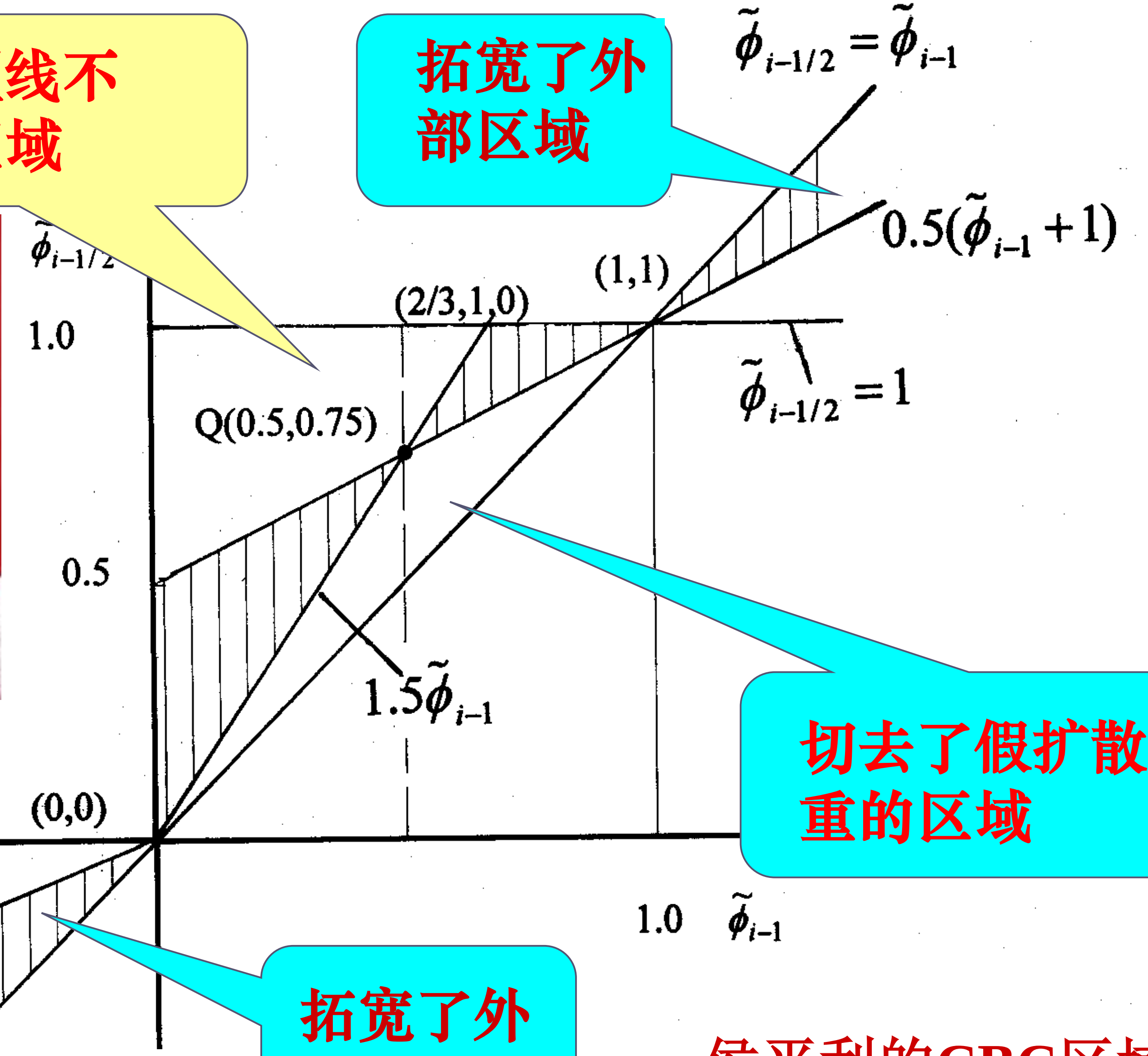
$$\left\{\begin{array}{ll}
\phi_C \in (1,\infty), \phi_f \in (0.5(1+\phi_C), \phi_C]; & \phi_C > 1 \\
\phi_C \in (-\infty,0),\ \phi_f \in [\phi_C, 0.5\phi_C); & \phi_C < 0 \\
\phi_C \in [0,0.5),\ \phi_f \in (1.5\phi_C, 0.5(1+\phi_C)]; & \\
\phi_C \in [0.5,0.75),\ \phi_f \in [0.5(1+\phi_C), 1.5\phi_C); & \\
\phi_C \in [0.75,1],\ \phi_f \in (0.5(1+\phi_C), 1.0]; &
\end{array}\right.$$

与宇波一致

$0 < \phi_C < 1$

与宇波不同

侯平利改进的CBC图示如下：



切去了型线不合理的区域

拓宽了外部区域

切去了假扩散严重的区域

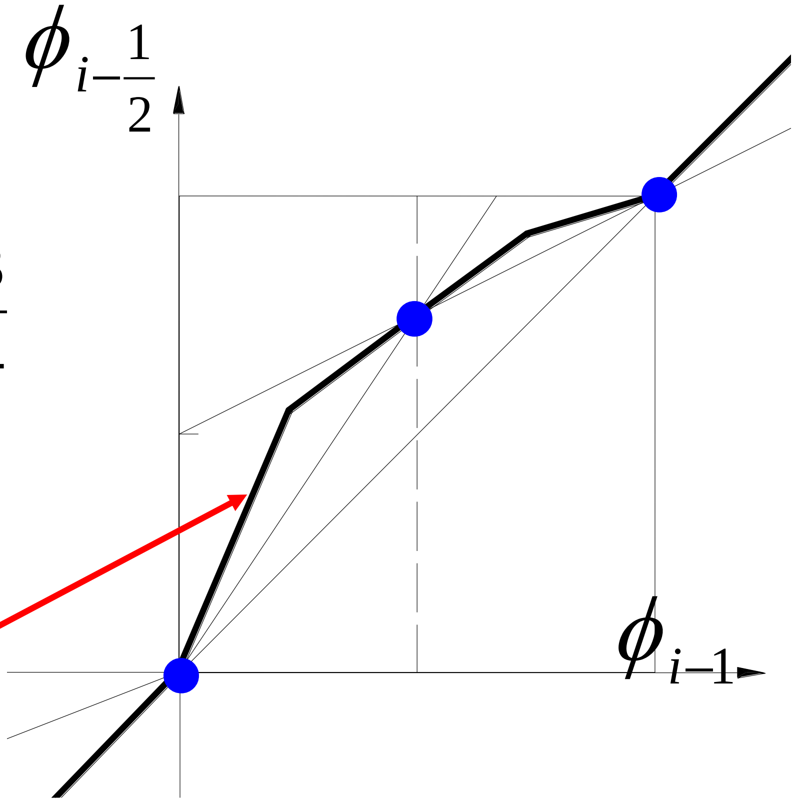
拓宽了外部区域

侯平利的CBC区域

对现有所有高阶有界格式检验的结果全部符合侯平利改进的CBC：

1 COPLA (combination of piecewise linear approximation)

$$\tilde{\phi}_{i-1/2}\begin{cases} = 2.25\tilde{\phi}_{i-1} & 0 \le \tilde{\phi}_{i-1} \le 1/4 \\ = \dfrac{3}{8} + \dfrac{3}{4}\tilde{\phi}_{i-1} & \dfrac{1}{4} \le \tilde{\phi}_{i-1} \le \dfrac{3}{4} \\ = \dfrac{3}{4} + \dfrac{1}{4}\tilde{\phi}_{i-1} & \dfrac{3}{4} \le \tilde{\phi}_{i-1} \le 1 \\ = \tilde{\phi}_{i-1} & else \end{cases}$$

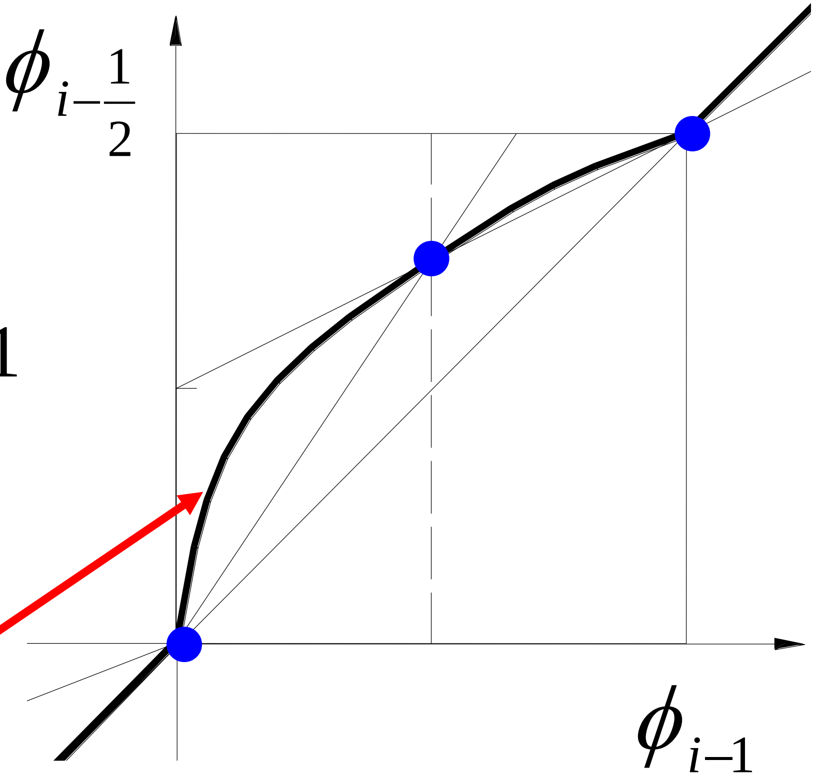


Defining curve of COPLA

2 EULER

$$\tilde{\phi}_{i-1/2}$$

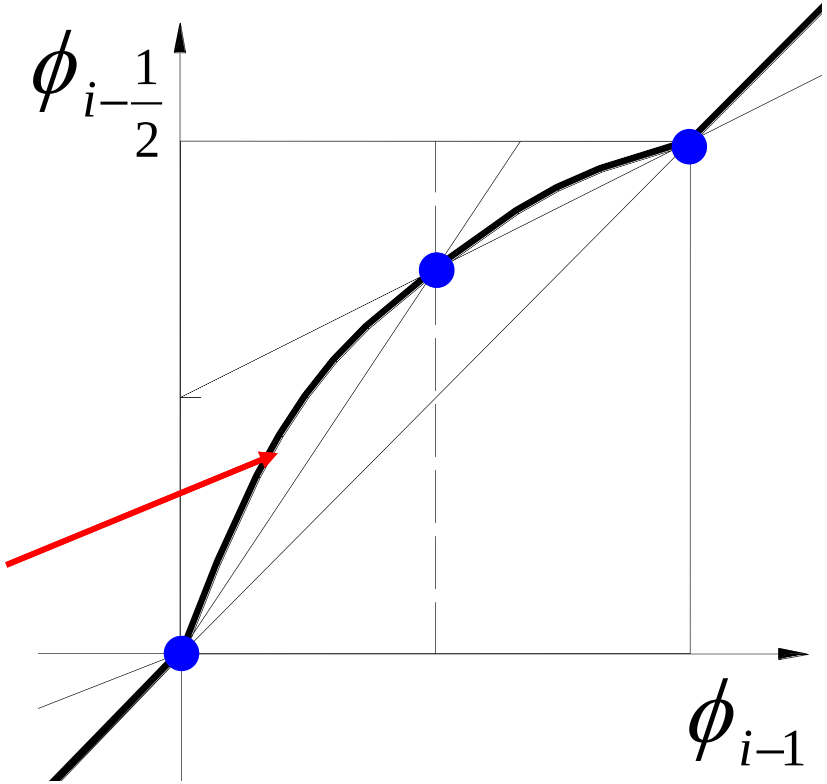
$$\begin{cases} = \dfrac{\sqrt{\tilde{\phi}_{i-1}(1-\tilde{\phi}_{i-1})^3} - \tilde{\phi}_{i-1}^2}{1-2\tilde{\phi}_{i-1}}, 0 \le \tilde{\phi}_{i-1} \le 1 \\ = 3/4 \qquad \tilde{\phi}_{i-1} = 0.5 \\ = \tilde{\phi}_{i-1} \qquad else \end{cases}$$



Defining curve of EULER

3 CLAM (hybrid linear/parabolic approximation)

$$\tilde{\phi}_{i-1/2}\begin{cases}=\tilde{\phi}_{i-1}(2-\tilde{\phi}_{i-1}) & 0\le\tilde{\phi}_{i-1}\le 1\\ =\tilde{\phi}_{i-1} & else\end{cases}$$

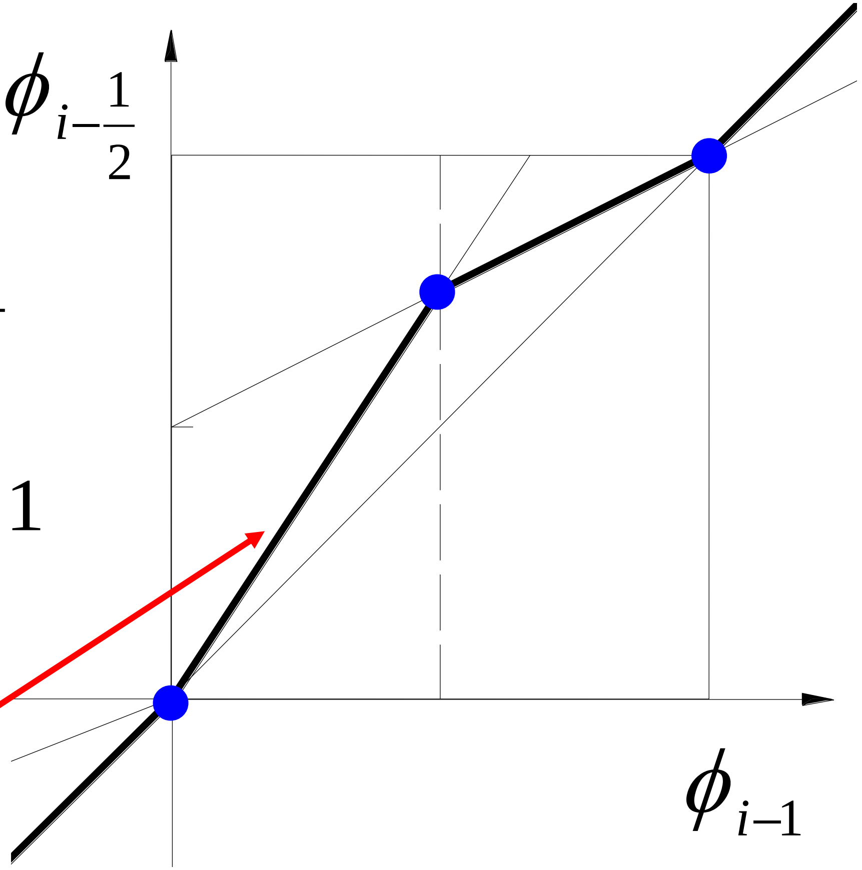


Defining curve of CLAM

4 MINMOD (minimum modulus)

$$\tilde{\phi}_{i-1/2} =$$

$$\begin{cases} = \dfrac{3}{2}\tilde{\phi}_{i-1} & 0 \le \tilde{\phi}_{i-1} \le \dfrac{1}{2} \\ = \dfrac{1}{2}(1+\tilde{\phi}_{i-1}) & 0.5 \le \tilde{\phi}_{i-1} \le 1 \\ = \tilde{\phi}_{i-1} & else \end{cases}$$

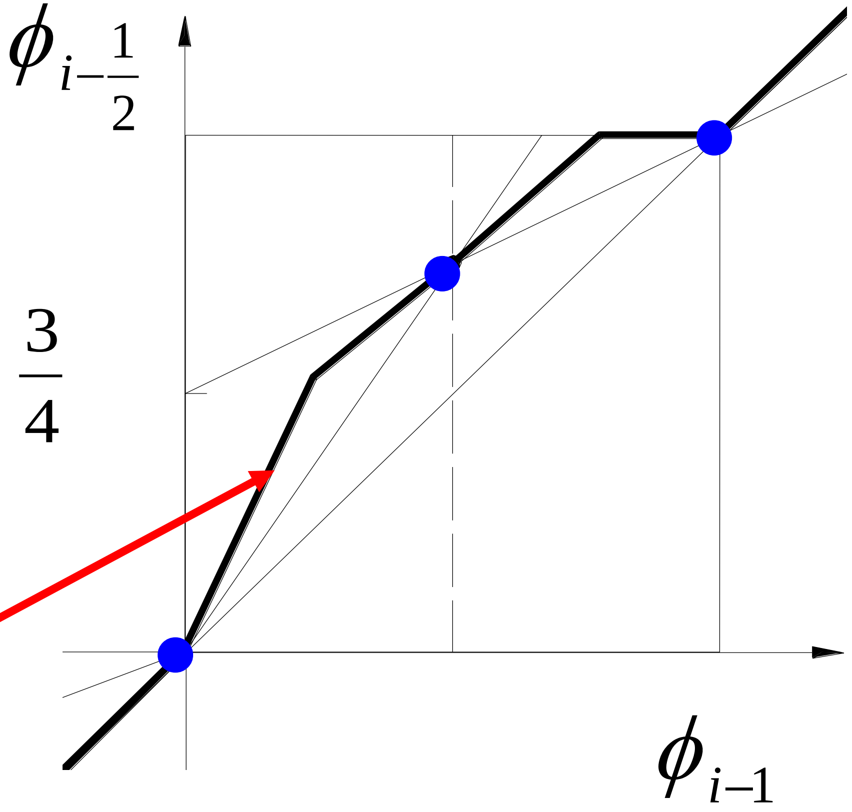


Defining curve of MINMOD

5 MUSCL (monotonic upwind scheme for conservation law)

$$\tilde{\phi}_{i-1/2} =$$

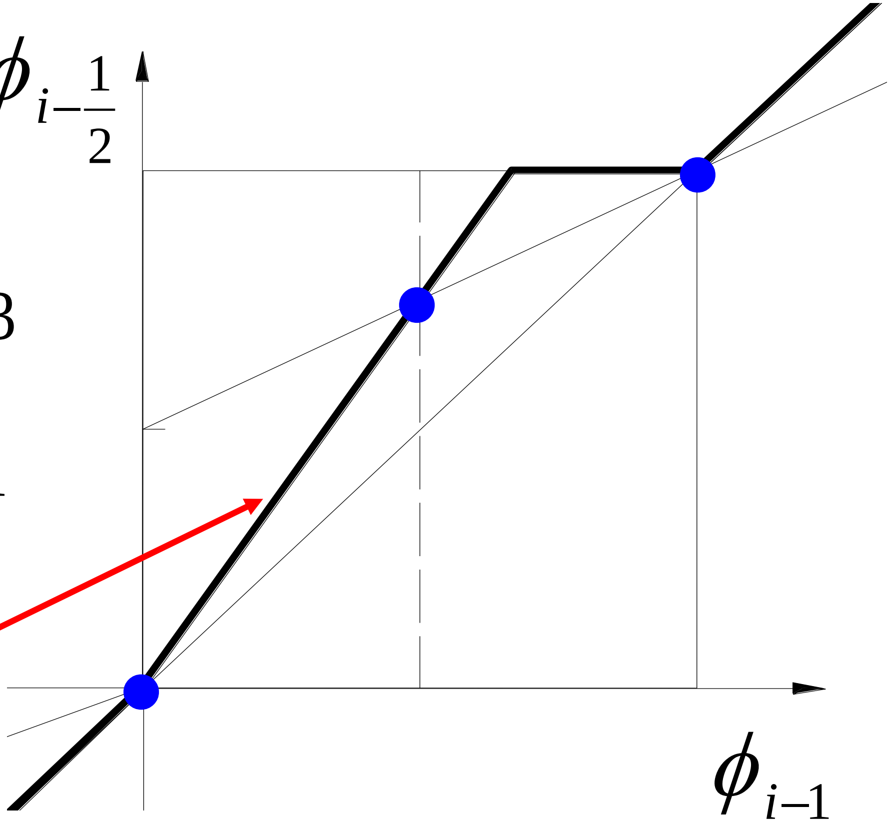
$$\begin{cases} = 2\tilde{\phi}_{i-1} & 0 \le \tilde{\phi}_{i-1} \le 1/4 \\ = 1/4 + \tilde{\phi}_{i-1} & \frac{1}{4} \le \tilde{\phi}_{i-1} \le \frac{3}{4} \\ = 1 & \frac{3}{4} \le \tilde{\phi}_{i-1} \le 1 \\ = \tilde{\phi}_{i-1} & else \end{cases}$$



Defining curve of MUSCL

6 OSHER

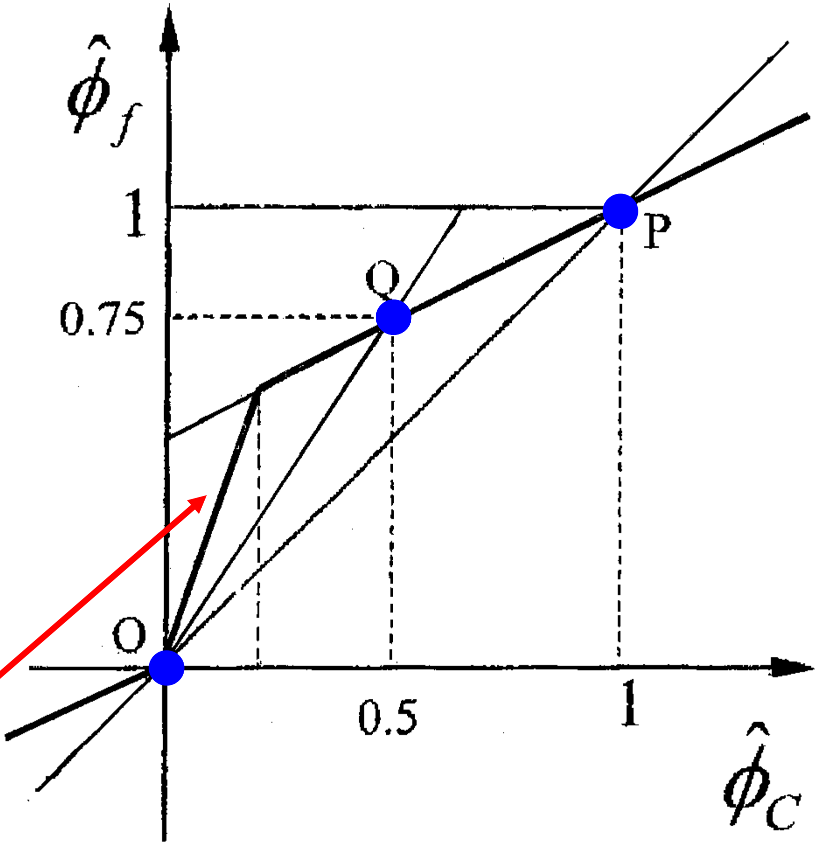
$$\tilde{\phi}_{i-1/2} =
\begin{cases}
= \frac{3}{2}\tilde{\phi}_{i-1} & 0 \le \tilde{\phi}_{i-1} \le 2/3 \\
= 1 & 2/3 \le \tilde{\phi}_{i-1} \le 1 \\
= \tilde{\phi}_{i-1} & else
\end{cases}$$



Defining curve of OSHER

7 SECBC (scheme based on extended CBC)

$$\tilde{\phi}_{i-1/2} = \begin{cases} = \dfrac{1}{2}\tilde{\phi}_{i-1} & \tilde{\phi}_{i-1} \le 0 \\ = 3\tilde{\phi}_{i-1} & 0 < \tilde{\phi}_{i-1} < 0.2 \\ = \dfrac{\tilde{\phi}_{i-1}+1}{2} & \tilde{\phi}_{i-1} \ge 0.2 \end{cases}$$

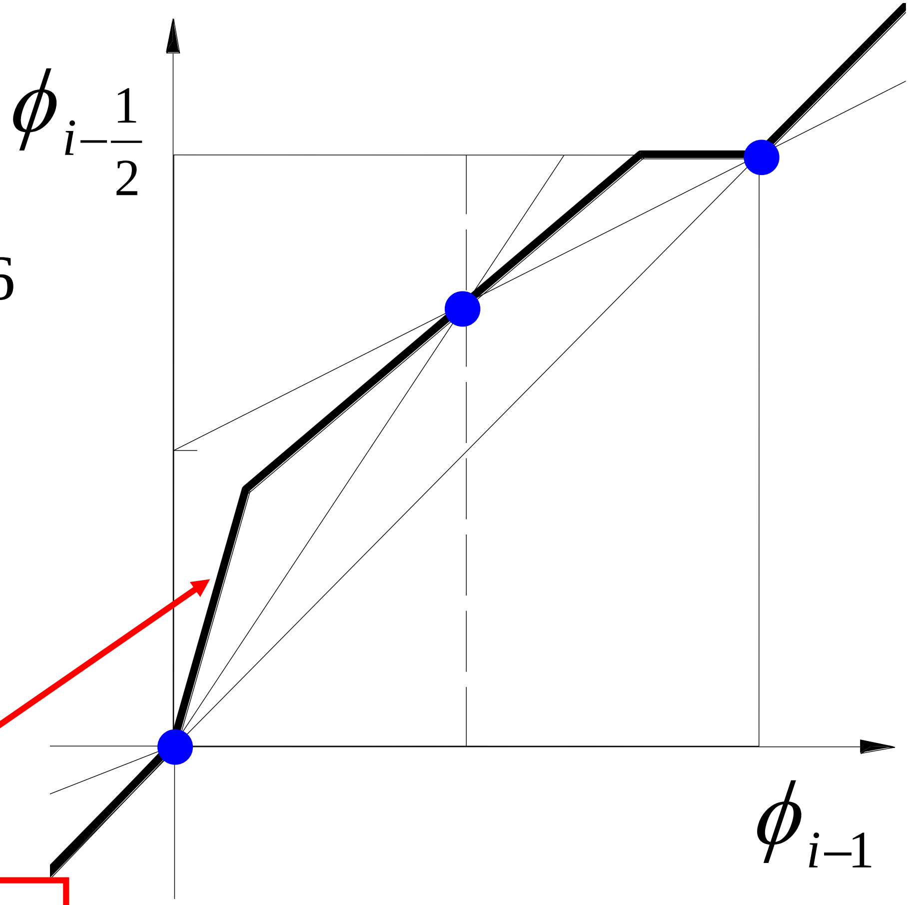


Defining curve of SECBC

8 SMART (sharp and monotonic algorithm for realistic transport)

$$\tilde{\phi}_{i-1/2} =$$

$$\begin{cases} = 3\tilde{\phi}_{i-1} & 0 \le \tilde{\phi}_{i-1} \le 1/6 \\ = 3/8 + 3/4\tilde{\phi}_{i-1} & \dfrac{1}{6} \le \tilde{\phi}_{i-1} \le \dfrac{5}{6} \\ = 1 & \dfrac{5}{6} \le \tilde{\phi}_{i-1} \le 1 \\ = \tilde{\phi}_{i-1} & else \end{cases}$$

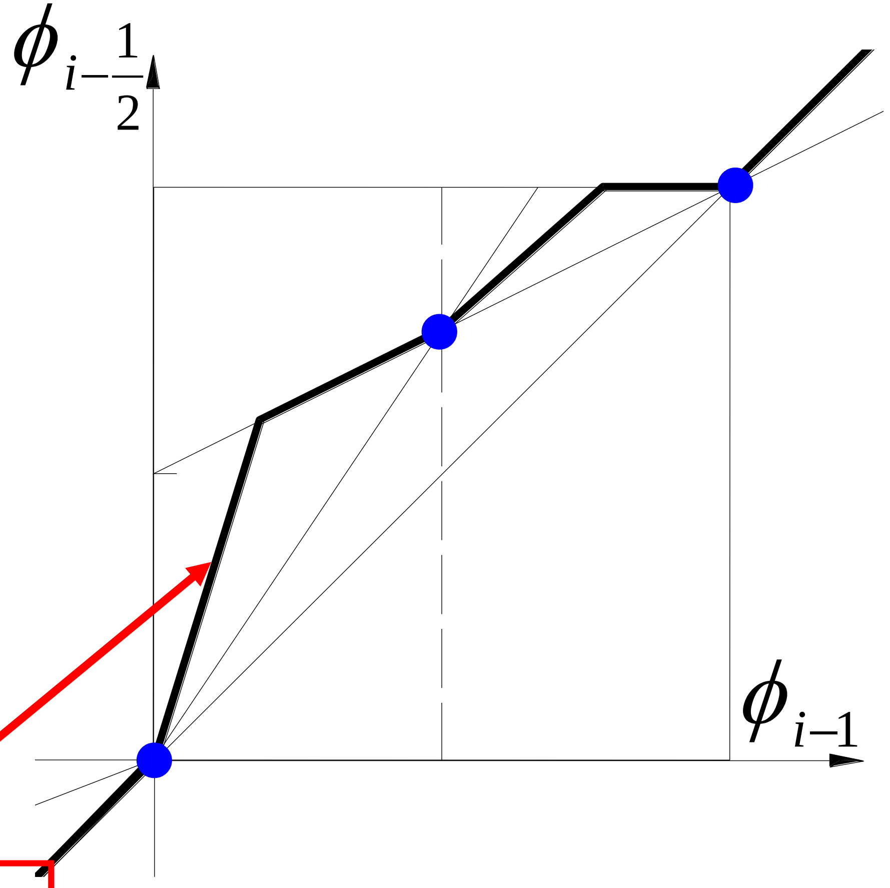


Defining curve of SMART

9 STOIC (second and third order interpolation for convection)

$$\tilde{\phi}_{i-1/2} =$$

$$\begin{cases} = 3\tilde{\phi}_{i-1} & 0 \le \tilde{\phi}_{i-1} \le 1/5 \\ = 1/2(1+\tilde{\phi}_{i-1}) & \dfrac{1}{5} \le \tilde{\phi}_{i-1} \le \dfrac{1}{2} \\ = \dfrac{3}{8}+\dfrac{3}{4}\tilde{\phi}_{i-1} & \dfrac{1}{2} \le \tilde{\phi}_{i-1} \le \dfrac{5}{6} \\ = 1 & \dfrac{5}{6} \le \tilde{\phi}_{i-1} \le 1 \\ = \tilde{\phi}_{i-1} & else \end{cases}$$

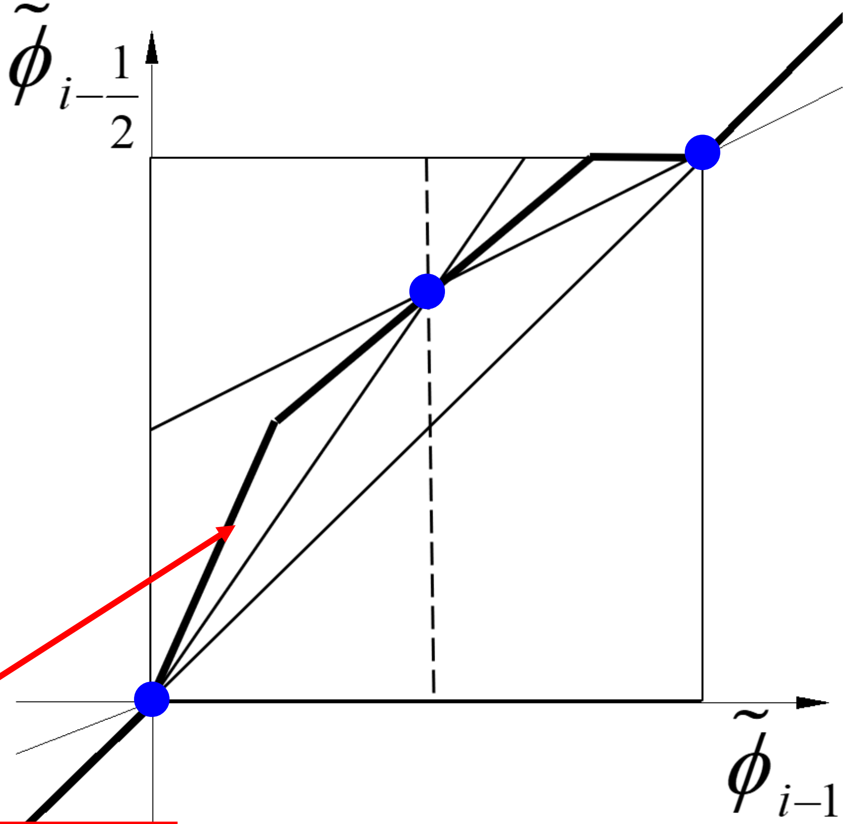


Defining curve of STOIC

10 WACEB (weighted-average coefficient ensuring boundedness)

$$\tilde{\phi}_{i-1/2} =$$

$$\begin{cases} = 2\tilde{\phi}_{i-1} & 0 \le \tilde{\phi}_{i-1} \le 0.3 \\ = 3/8 + 3/4\tilde{\phi}_{i-1} & 0.3 \le \tilde{\phi}_{i-1} \le \dfrac{5}{6} \\ = 1 & \dfrac{5}{6} \le \tilde{\phi}_{i-1} \le 1 \\ = \tilde{\phi}_{i-1} & else \end{cases}$$

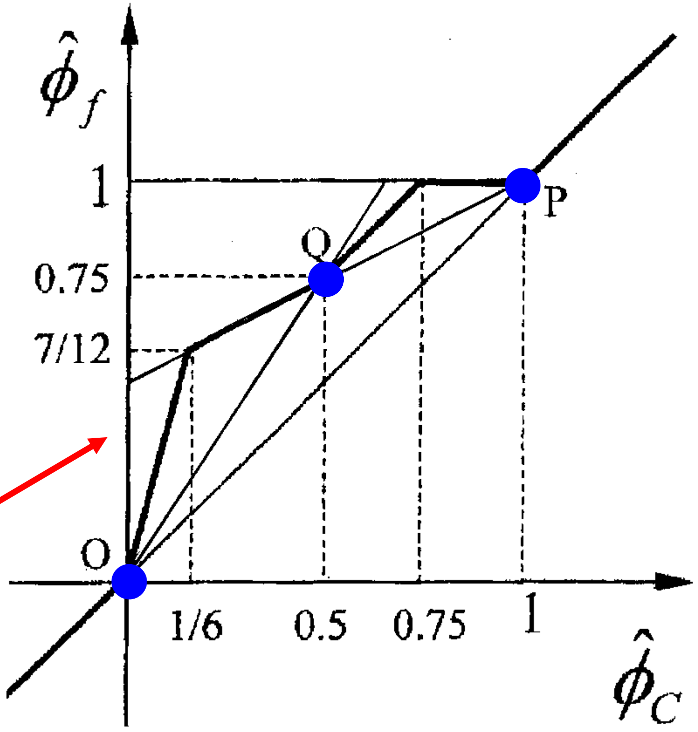


Defining curve of WACEB

11 HOAB (high-order-accurate bounded scheme)

$$\tilde{\phi}_{i-1/2} =$$

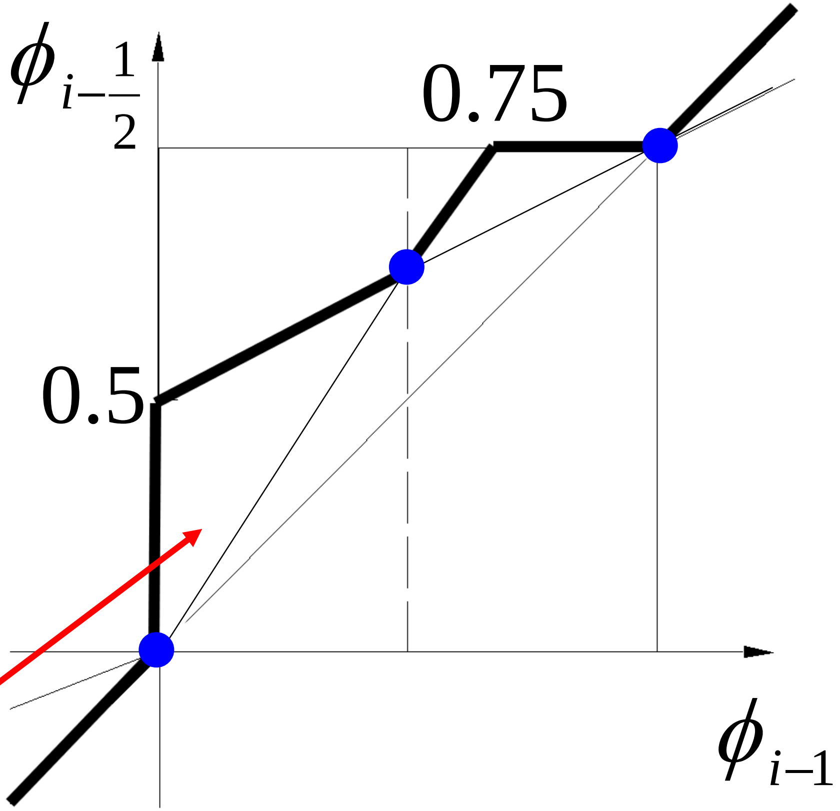
$$\begin{cases} \phi_{i-1/2} = 3.5\phi_i & 0 < \phi_i \le 1/6 \\ \phi_{i-1/2} = 0.5\phi_i + 0.5 & 1/6 < \phi_i \le 0.5 \\ \phi_{i-1/2} = \phi_i + 0.25 & 0.5 < \phi_i \le 0.75 \\ \phi_{i-1/2} = 1_i & 0.75 < \phi_i \le 1 \\ \phi_{i-1/2} = \phi_i & \text{elsewhere} \end{cases}$$



Defining curve of HOAB

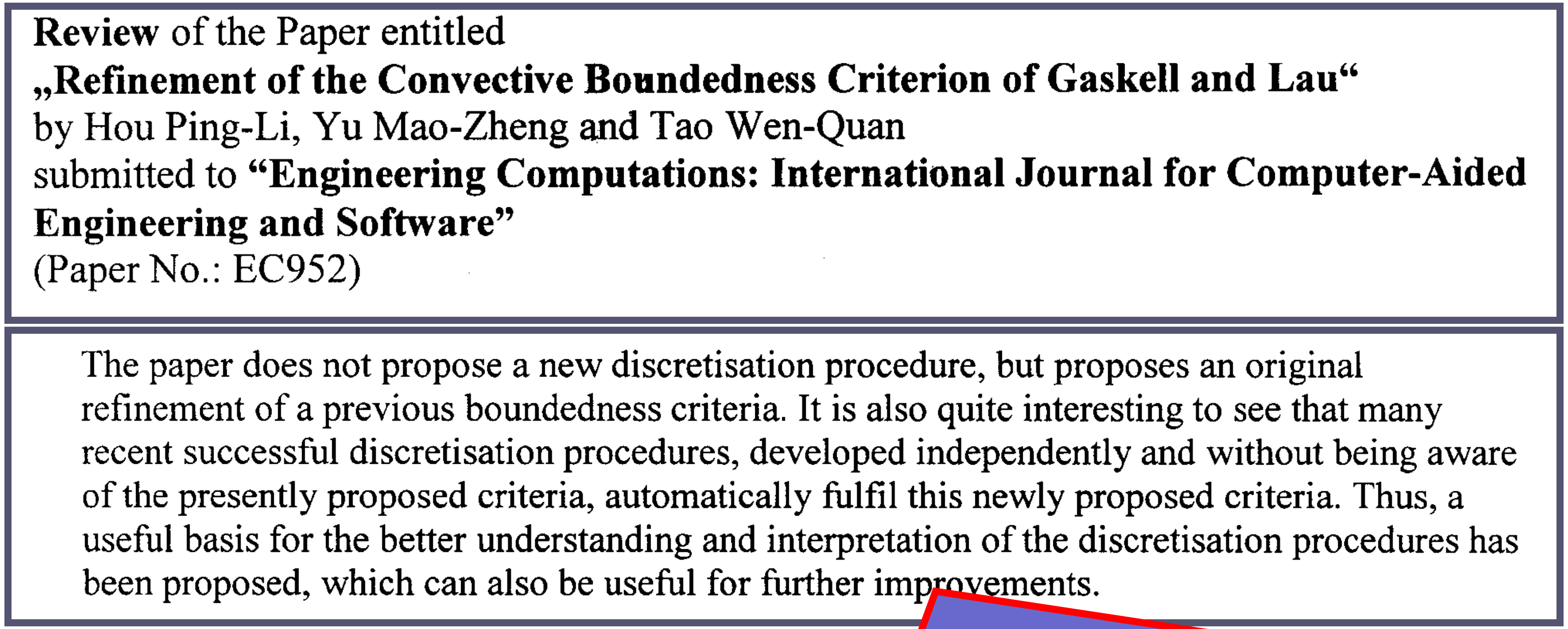
12 SUPERBEE

$$\tilde{\phi}_{i-1/2} = \begin{cases} \phi_{i-1/2} = 0.5 + 0.5\phi_i & 0 < \phi_i \le 1/2 \\ \phi_{i-1/2} = 1.5\phi_i & 1/2 < \phi_i \le 2/3 \\ \phi_{i-1/2} = 1 & 2/3 < \phi_i \le 1 \\ \phi_{i-1/2} = \phi_i & elsewhere \end{cases}$$



Defining curve of SUPERBEE

国际杂志Engineering Computations评审人对该论文的评价



Review of the Paper entitled
„Refinement of the Convective Boundedness Criterion of Gaskell and Lau“
by Hou Ping-Li, Yu Mao-Zheng and Tao Wen-Quan
submitted to **“Engineering Computations: International Journal for Computer-Aided Engineering and Software”**
(Paper No.: EC952)

The paper does not propose a new discretisation procedure, but proposes an original refinement of a previous boundedness criteria. It is also quite interesting to see that many recent successful discretisation procedures, developed independently and without being aware of the presently proposed criteria, automatically fulfil this newly proposed criteria. Thus, a useful basis for the better understanding and interpretation of the discretisation procedures has been proposed, which can also be useful for further improvements.

本文并未提出一个新的离散格式，但对已有格式有界性准则提出一个**原创性**的改进。特别有意义的是许多近来独立地提出来的离散格式，作者们并不知道本文提出的准则，但这些格式都自动地满足本文提出的条件。因此**本文提出了一个能更好的理解与解释离散格式的理论基础，对今后的格式的改进也都颇有价值。**

2.3.5 Wei J J （魏进家）的进一步分析

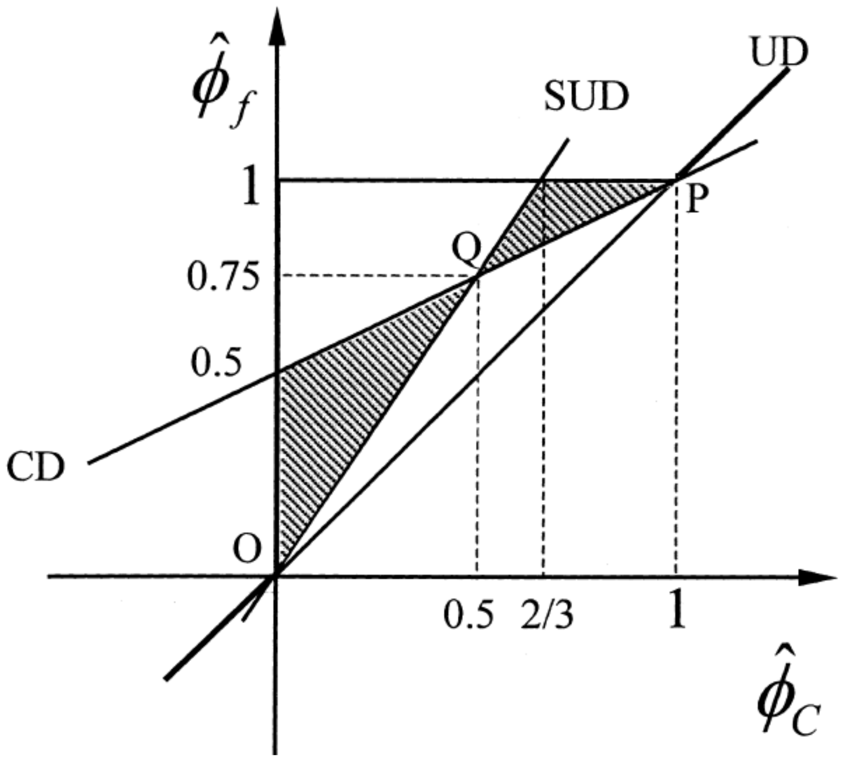
1.对界面插值提出的要求获得与侯平利相似的结果

(1）正响应要求：界面插值公式对节点上物理量扰动的响应该是正的；

(2）迁移特性要求：要求插值公式中对流的扰动只能向下游传播而不能向上游传播

据此导出在$0 \le \phi_C \le 1$范围内与侯平利完全一致的结果。

Wei J J, Yu B, Tao W Q, Kawaguchi Y, Wang H S. A new high-order accurate and bounded scheme for incompressible flow. Numerical Heat Transfer, Part B, 2003, 43:19-41



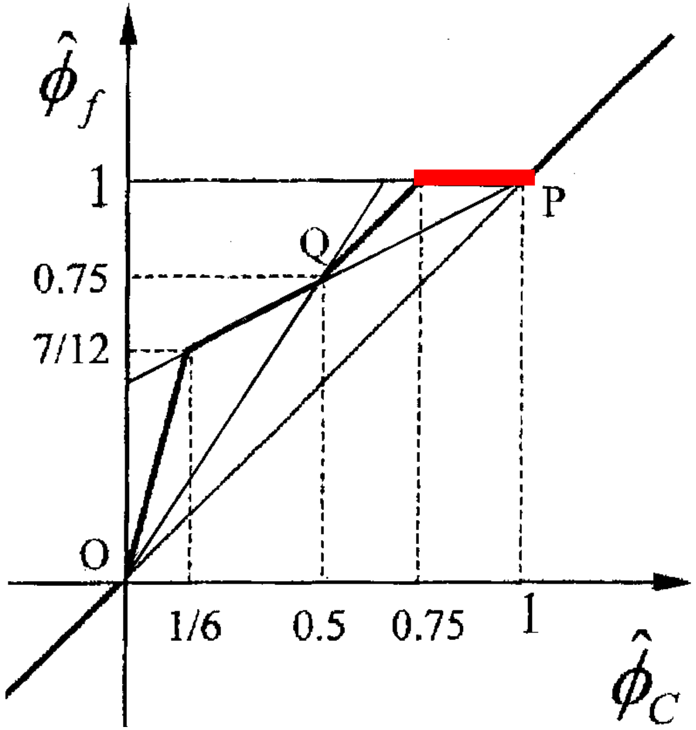
根据此图魏进家提出了：
HOAB (high-order-accurate bounded scheme)

2. 界面插值定义中应去掉 $\phi_f = 1$ 的部分

当格式定义式中包含有 $\phi_C = 1$ 时，数值计算结果在个别情况下仍然可能会出现越界现象，因此高阶有界组合格式的定义中应该永远使 $\phi_C < 1$

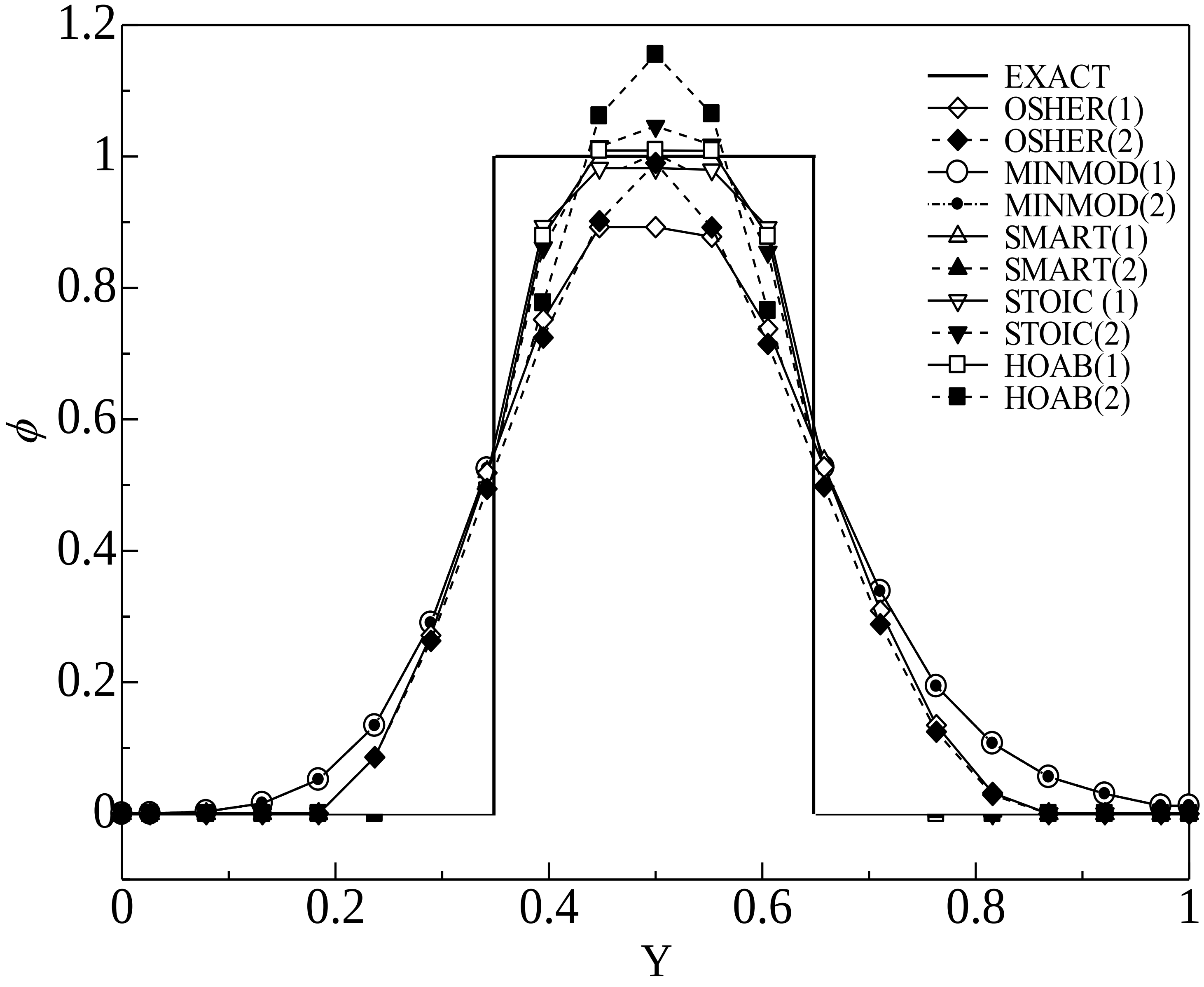
HOAB (high-order-accurate bounded scheme)

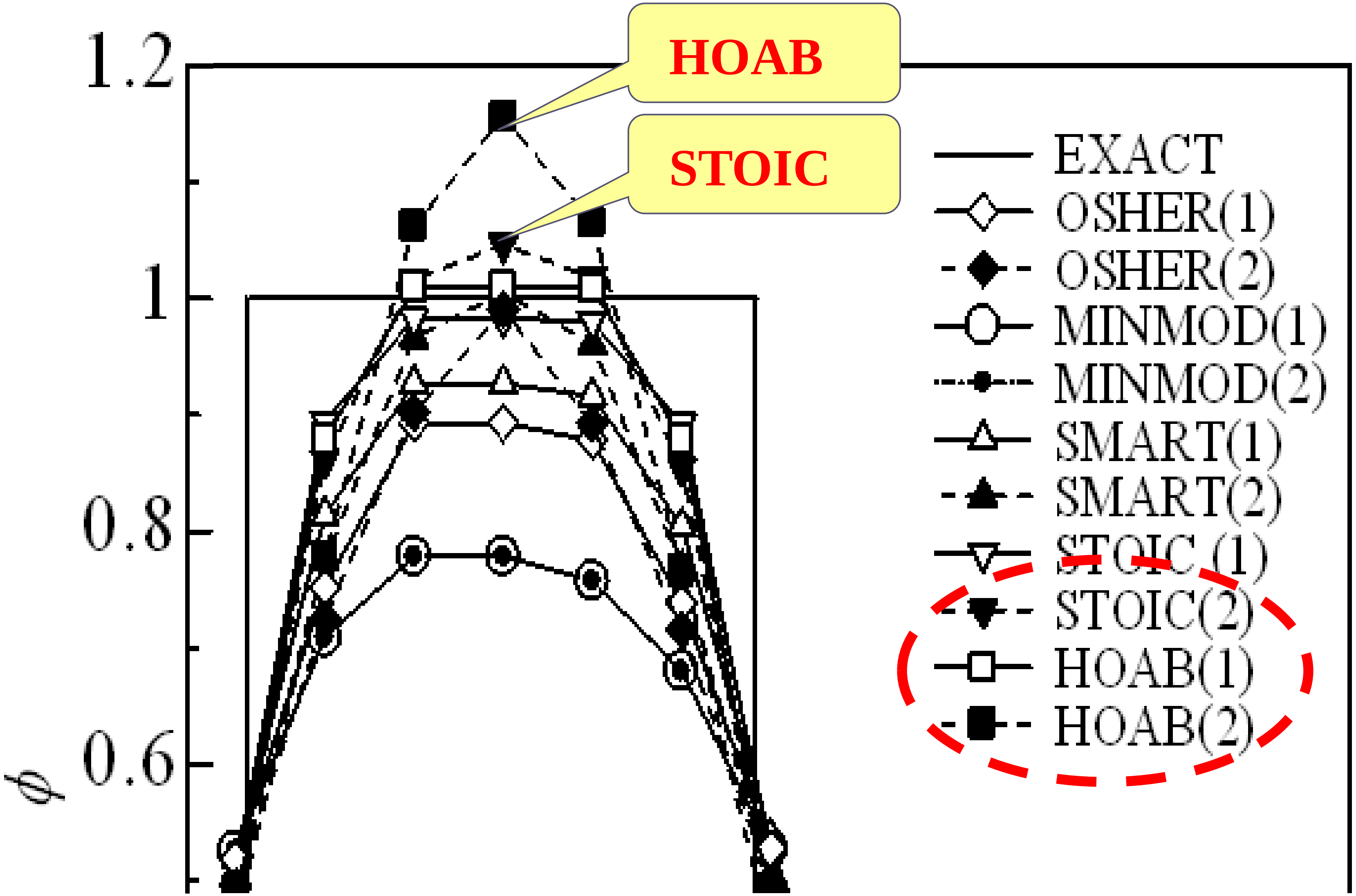
$$\begin{cases} \phi_{i-1/2} = 3.5\phi_i & 0 < \phi_i \le 1/6 \\ \phi_{i-1/2} = 0.5\phi_i + 0.5 & 1/6 < \phi_i \le 0.5 \\ \phi_{i-1/2} = \phi_i + 0.25 & 0.5 < \phi_i \le 0.75 \\ \phi_{i-1/2} = 1 & 0.75 < \phi_i \le 1 \\ \phi_{i-1/2} = \phi_i & \text{elsewhere} \end{cases}$$



还有以下格式定义式中包含 $\tilde{\phi}_f = 1.0$ 的部分：

WACEB STOIC SMART OSHER MUSCL SUPERBEE


1.2
1
0.8
0.6
0.4
0.2
0
ϕ
0
0.2
0.4
0.6
0.8
1
Y
EXACT
OSHER(1)
OSHER(2)
MINMOD(1)
MINMOD(2)
SMART(1)
SMART(2)
STOIC (1)
STOIC(2)
HOAB(1)
HOAB(2)



HOAB STOIC出现越界现象

在$0 \le \tilde{\phi}_C \le 1$

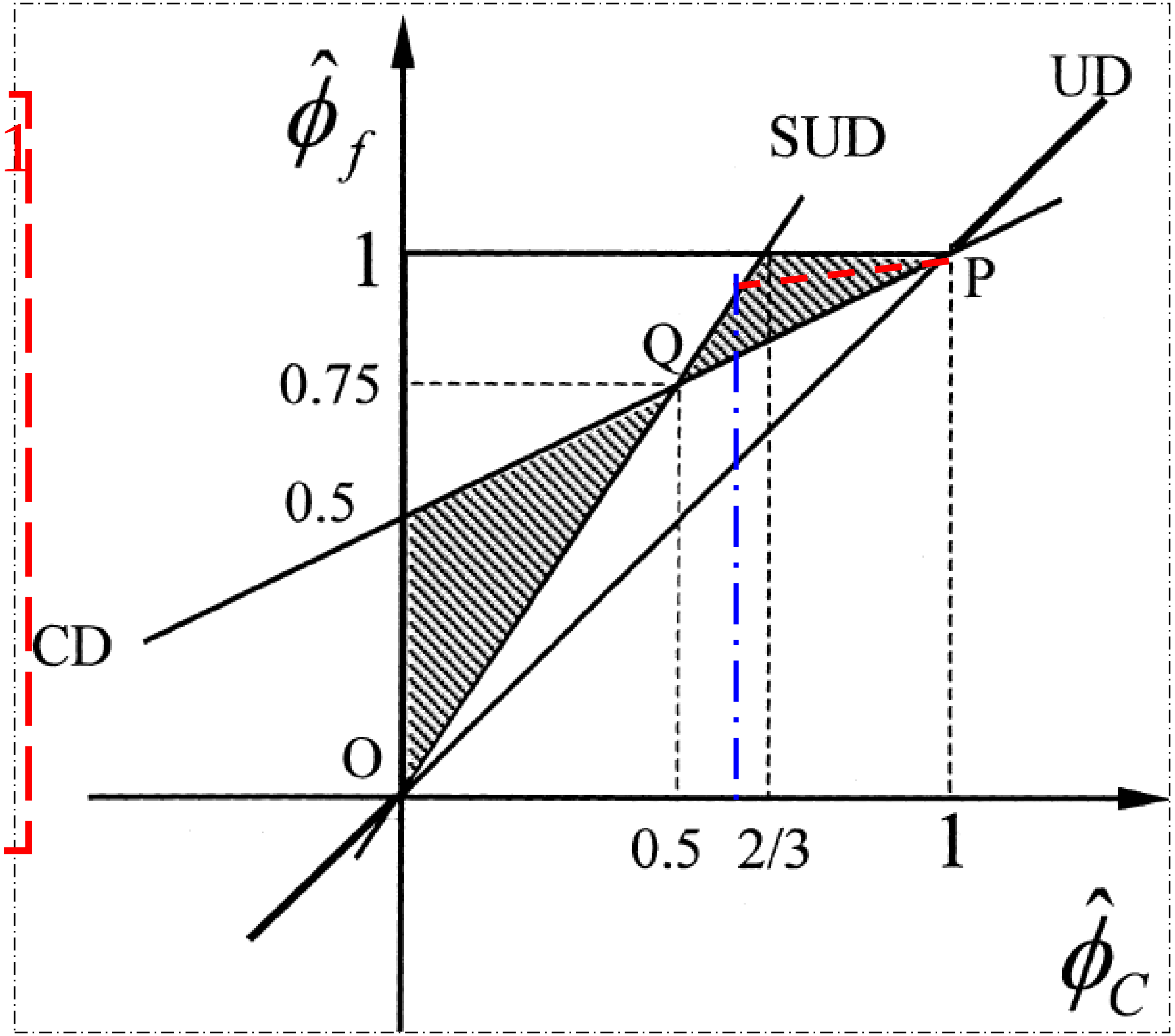
的范围内，

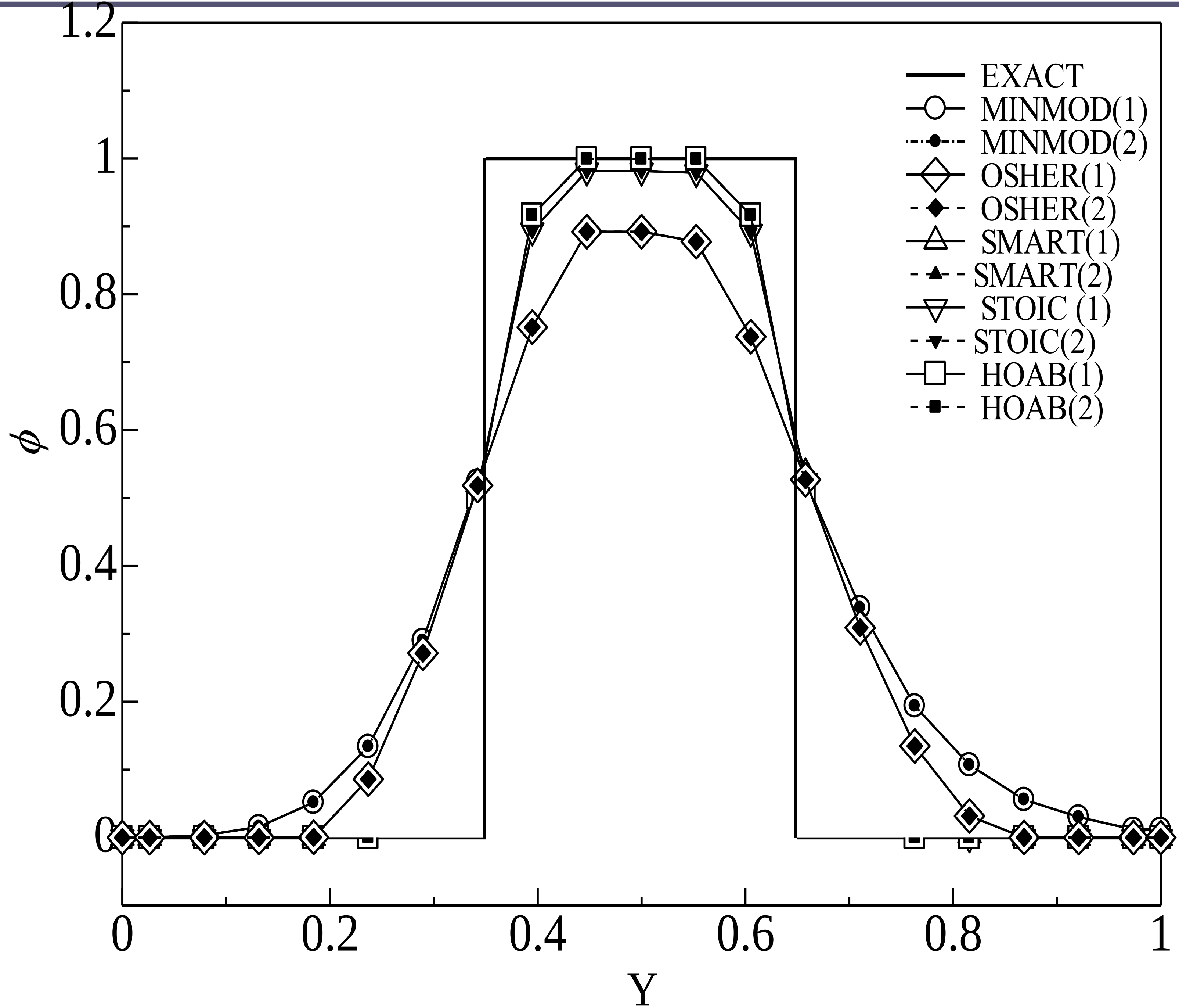
$\tilde{\phi}_f = 1$

只能在

$\tilde{\phi}_c = 1$

取得！




1.2
1
0.8
0.6
0.4
0.2
0
ϕ
0
0.2
0.4
0.6
0.8
1
Y
EXACT
MINMOD(1)
MINMOD(2)
OSHER(1)
OSHER(2)
SMART(1)
SMART(2)
STOIC (1)
STOIC(2)
HOAB(1)
HOAB(2)

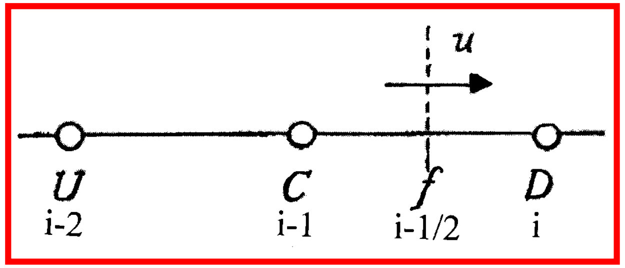
2.3.6 非均匀及非结构化网格上的高阶格式

1. 非均分网格上的高阶格式

引入规整空间坐标，见《计算传热学的近代进展》节3-7

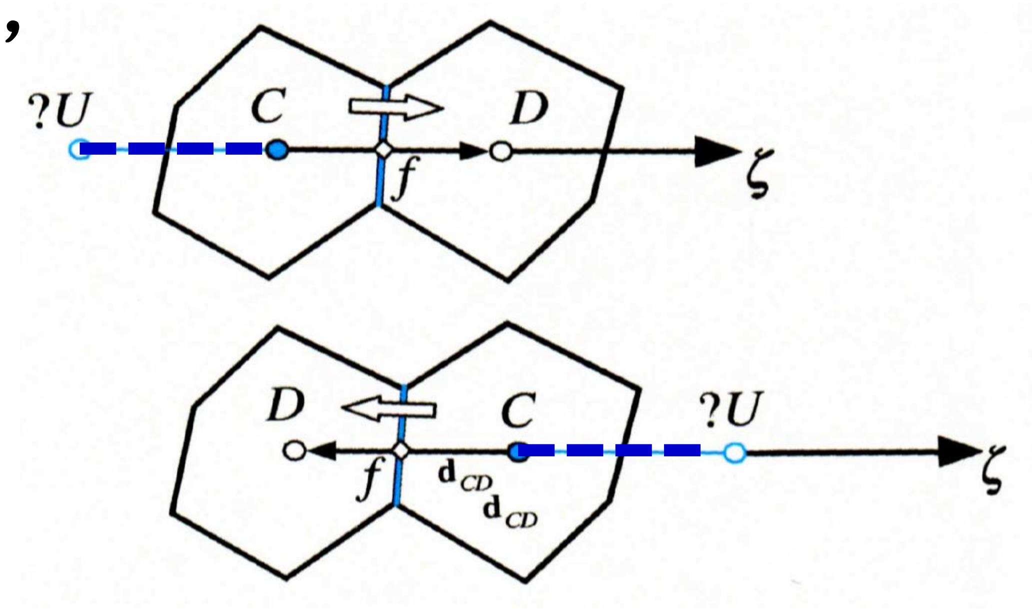
2. 非结构化网格上的高阶格式

在结构化网格中很容易找出上游, 下游及中心点，U,D,C, 从而给出各种格式的定义;



在非结构化网格中通过引入一个虚拟的节点就可采用以前的结果。

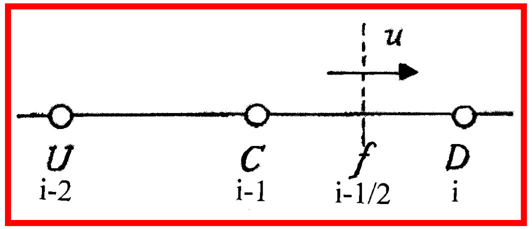
对如下非结构化网格，



在相邻两控制体中点连线的上游取虚拟点U，并使得UC=CD；上游点的被求函数值可插值确定：

$$\phi_D - \phi_U = \nabla\phi_C \bullet \mathbf{d}_{UD} \Rightarrow \phi_U = \phi_D - 2\nabla\phi_C \bullet \mathbf{d}_{CD}$$

就构造出了类似结构化网格中的定义环境：



变量梯度 $\nabla\phi_C$ 的确定方法将在后面介绍。

2.4 构造FVM对流项离散格式的一般方法

2.4.1 Two Basic Questions in the Scheme Design

2.4.2 General Formulation of 2nd-Order Difference Schemes

2.4.3 General Formulation of High-Order Difference Scheme

2.4.4 Derivation of Absolutely Stable Difference Scheme

2.4 构造FVM对流项离散格式的一般方法

2.4.1 Two Basic Questions in the Scheme Design

Many discretization schemes for convective term are proposed in CFD and CHT. However, two problems remain unresolved

1. Is there a general way to construct any-order scheme?

Each scheme is constructed individually, and in some sense, with some personal brainstorm and insight(灵机一动).

2. Can we design an accurate scheme with absolute stability?

Conventionally it is considered that stability and accuracy are a pair of contradictions.

The upwind-based schemes, such as QUICK, are generally accepted as good schemes in the compromise (折衷) between accuracy and stability. Is that true? Is there a better way ?

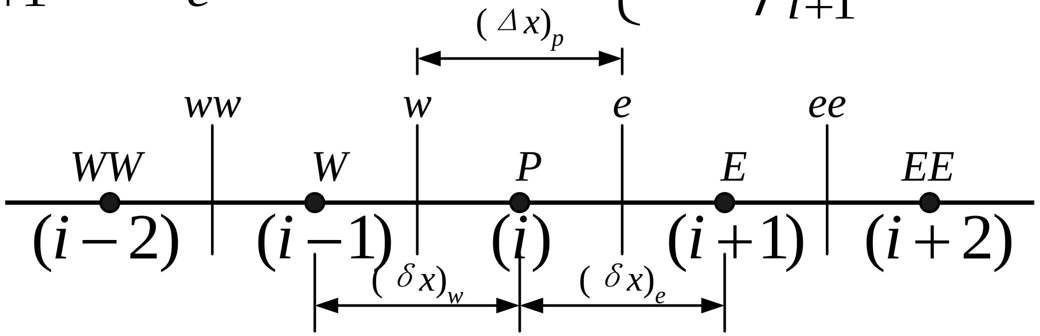
Upwind schemes: More information from “upwind” flow direction should be adopted than the information of downwind.

FUD :

$$\phi_e = \begin{cases} \phi_i & u_e > 0 \\ \phi_{i+1} & u_e < 0 \end{cases}$$

SUD :

$$\phi_e = \begin{cases} 1.5\phi_i - 0.5\phi_{i-1} & u_e > 0 \\ 1.5\phi_{i+1} - 0.5\phi_{i+2} & u_e < 0 \end{cases}$$



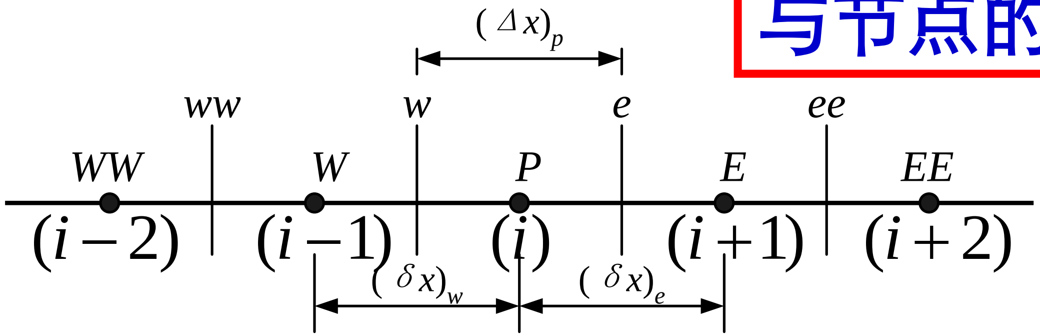
2.4.2 General Formulation of 2nd-Order Difference Schemes

1. General form of interface interpolation

In one-dimensional uniform grid system, when $u>0$ the variable ϕ on the east e and west w interfaces are interpolated:

$$\begin{cases} \phi_e = a_{i-1}\phi_{i-1} + a_i\phi_i + a_{i+1}\phi_{i+1} \\ \phi_w = a_{i-1}\phi_{i-2} + a_i\phi_{i-1} + a_{i+1}\phi_i \end{cases} \quad (1)$$

注意：这里不要求代数方程限于五对角阵，界面插值系数按界面与节点的位置而定。



Uniform grid of 1-D convective–diffusive problem

In FVM, the integration of the convective term leads to following expression :

$$\left.\frac{\partial \phi}{\partial x}\right|_i = \frac{\phi_e - \phi_w}{\Delta x} \qquad (2)$$

精确解

模拟解

Substituting Eq. (1) into the above expression:

$$\left.\frac{\partial \phi}{\partial x}\right|_i = \frac{\phi_e - \phi_w}{\Delta x} = \frac{\left(a_i - a_{i+1}\right)\phi_i + \left(a_{i-1} - a_i\right)\phi_{i-1} + a_{i+1}\phi_{i+1} - a_{i-1}\phi_{i-2}}{\Delta x} \qquad (3)$$

Determining the coefficients, a_{i-1}, a_i, a_{i+1} **by Taylor series expansion for** ϕ_{i-1}, ϕ_{i+1}, ϕ_{i+2}.

2. Taylor series expansion for determining interpolation coefficients

Expanding $\phi_{i-2}, \phi_{i-1}, \phi_{i+1}$ by Taylor series at the point *i* , substituting into Eq.(3) and rearranging:

模拟解

一阶截差

$$\left.\frac{\partial\phi}{\partial x}\right|_i = \overset{=1}{\underline{(a_i + a_{i-1} + a_{i+1})}}\left.\frac{\partial\phi}{\partial x}\right|_i + \overset{=0}{\cancel{\underline{(-3a_{i-1} - a_i + a_{i+1})}}}\left.\frac{\partial^2\phi}{\partial x^2}\right|_i \cdot \frac{\Delta x}{2!}$$

精确解

$$+ \overset{=0}{\cancel{\underline{(7a_{i-1} + a_i + a_{i+1})}}}\left.\frac{\partial^3\phi}{\partial x^3}\right|_i \cdot \frac{\Delta x^2}{3!} + \cdots \qquad \textbf{(4)}$$

保留其余部分，格式具有三阶截差

二阶截差

From three conditions , following equations can be obtained:

$$\begin{cases} a_{i-1}+a_i+a_{i+1}=1 \\ -3a_{i-1}-a_i+a_{i+1}=0 \\ 7a_{i-1}+a_i+a_{i+1}=0 \end{cases} \longrightarrow \boxed{a_i=\frac{5}{6},a_{i-1}=-\frac{1}{6},a_{i+1}=\frac{1}{3}}$$

The third-order upwind scheme (TUD) is obtained:

$$\begin{cases} \phi_e=-\frac{1}{6}\phi_{i-1}+\frac{5}{6}\phi_i+\frac{1}{3}\phi_{i+1} \\ \phi_w=-\frac{1}{6}\phi_{i-2}+\frac{5}{6}\phi_{i-1}+\frac{1}{3}\phi_i \end{cases} \qquad \textbf{(5)}$$

If Δx^2 term(third order derivative) is retained in Eq. (4):

$$\left.\frac{\partial\phi}{\partial x}\right|_i=\underbrace{(a_i+a_{i-1}+a_{i+1})}_{=1}\left.\frac{\partial\phi}{\partial x}\right|_i+\underbrace{(-3a_{i-1}-a_i+a_{i+1})}_{=0}\left.\frac{\partial^2\phi}{\partial x^2}\right|_i\cdot\frac{\Delta x}{2!}+\underbrace{(7a_{i-1}+a_i+a_{i+1})}_{\neq 0}\left.\frac{\partial^3\phi}{\partial x^3}\right|_i\cdot\frac{\Delta x^2}{3!}+\cdots$$

$$\begin{cases} a_{i-1}+a_i+a_{i+1}=1 \\ -3a_{i-1}-a_i+a_{i+1}=0 \\ 7a_{i-1}+a_i+a_{i+1}\neq 0 \end{cases} \longrightarrow \boxed{a_i\neq\frac{5}{6},\ a_{i-1}=\frac{1}{4}-\frac{a_i}{2},\ a_{i+1}=\frac{3}{4}-\frac{a_i}{2}}$$

Then the general formulation of second-order difference schemes is obtained as follows:

$$\begin{cases} \phi_e=a_i\phi_i+\left(\frac{1}{4}-\frac{a_i}{2}\right)\phi_{i-1}+\left(\frac{3}{4}-\frac{a_i}{2}\right)\phi_{i+1} \\ \phi_w=a_i\phi_{i-1}+\left(\frac{1}{4}-\frac{a_i}{2}\right)\phi_{i-2}+\left(\frac{3}{4}-\frac{a_i}{2}\right)\phi_i \\ a_i\neq\frac{5}{6} \end{cases} \qquad \textbf{(6)}$$

where a_i can be any value but is not equal to 5/6.

Taking ϕ_e as an example to show the results

Relationship of the General Formulation of 2nd/3rd-Order Difference Scheme with Existing Schemes

$a_i = 1/2$	$\phi_e = \frac{1}{2}\phi_i + \frac{1}{2}\phi_{i+1}$	**(CD)**
$a_i = 3/4$	$\phi_e = \frac{3}{4}\phi_i - \frac{1}{8}\phi_{i-1} + \frac{3}{8}\phi_{i+1}$	**(QUICK)**
$a_i = 5/6$	$\phi_e = -\frac{1}{6}\phi_{i-1} + \frac{5}{6}\phi_i + \frac{1}{3}\phi_{i+1}$	**(TUD)**
$a_i = 1$	$\phi_e = \phi_i - \frac{1}{4}\phi_{i-1} + \frac{1}{4}\phi_{i+1}$	**(FROMM)**
$a_i = 3/2$	$\phi_e = \frac{3}{2}\phi_i - \frac{1}{2}\phi_{i-1}$	**(SUD)**
$\frac{1}{2} \le a_i \le \frac{3}{2}$	$\phi_e = a_i\phi_i + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-1} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_{i+1}$	**(SCSD)**

$$a_i = \left(\frac{3}{2} - \beta\right)_i \qquad \phi_e = a_i \phi_i + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-1} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_{i+1} \qquad \textbf{(SGSD)}$$

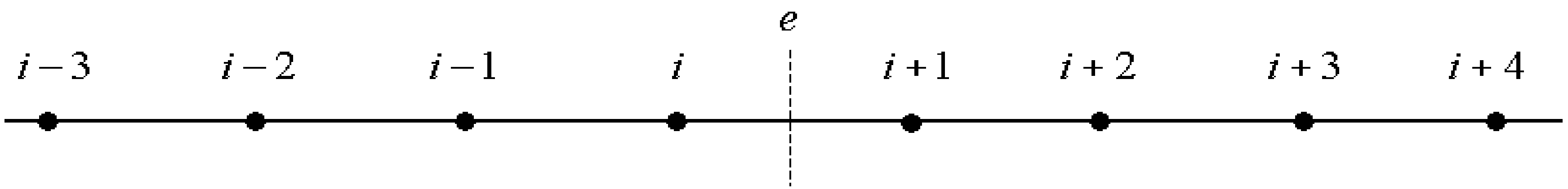
$$\beta = \frac{2}{2 + |P_\Delta|}$$

It is the first time in the literatures that such general discretization schemes for convection term is established For the convection scheme in FVM!

2.4.3 General Formulation of High-Order Difference Scheme

By extending above analysis:

$$u > 0$$



$$\begin{cases}\phi_e = \ldots a_{i-2}\phi_{i-2} + a_{i-1}\phi_{i-1} + a_i\phi_i + a_{i+1}\phi_{i+1} + a_{i+2}\phi_{i+2}\ldots \\ \phi_w = \ldots a_{i-2}\phi_{i-3} + a_{i-1}\phi_{i-2} + a_i\phi_{i-1} + a_{i+1}\phi_i + a_{i+2}\phi_{i+1}\ldots\end{cases} \qquad \mathbf{(7)}$$

$$\left.\frac{\partial\phi}{\partial x}\right|_i = \frac{\phi_e - \phi_w}{\Delta x}$$

$$= \frac{\ldots\ldots + \left(a_{i-n} - a_{i-n-1}\right)\phi_{i-n} + \ldots + \left(a_{i-4} - a_{i-3}\right)\phi_{i-4} + \left(a_{i-3} - a_{i-2}\right)\phi_{i-3} + \left(a_{i-2} - a_{i-1}\right)\phi_{i-2} + \left(a_{i-1} - a_i\right)\phi_{i-1}}{\Delta x}$$

$$+ \frac{\left(a_i - a_{i+1}\right)\phi_i + \left(a_{i+1} - a_{i+2}\right)\phi_{i+1} + \left(a_{i+2} - a_{i+3}\right)\phi_{i+2} + \left(a_{i+3} - a_{i+4}\right)\phi_{i+3} + \ldots + \left(a_{i+n} - a_{i+n+1}\right)\phi_{i+n} + \ldots\ldots}{\Delta x}$$

Expanding $\ldots, \phi_{i-n}, \ldots\phi_{i-2}, \phi_{i-1}, \phi_{i+1}, \ldots\phi_{i+n}, \ldots$ **by Taylor series at the point *i*, and then rewriting, yields:**

精确解

$$\left.\frac{\partial \phi}{\partial x}\right|_i = [(n+1)a_{i-n} - n(a_{i-n} - a_{i-n+1}) - .... - 3(a_{i-3} - a_{i-2}) - 2(a_{i-2} - a_{i-1}) - 1(a_{i-1} - a_i)$$

$= 1$

$$+1(a_{i+1} - a_{i+2}) + 2(a_{i+2} - a_{i+3}) + 3(a_{i+3} - a_{i+4}) + ... + n(a_{i+n} - a_{i+n+1}) + (n+1)a_{i+n+1}]\left.\frac{\partial \phi}{\partial x}\right|_i$$

模拟解

$$+[-(n+1)^2 a_{i-n} + n^2 (a_{i-n} - a_{i-n+1}) + ... + 3^2 (a_{i-3} - a_{i-2}) + 2^2 (a_{i-2} - a_{i-1}) + 1^2 (a_{i-1} - a_i)$$

$= 0$

$$+1^2 (a_{i+1} - a_{i+2}) + 2^2 (a_{i+2} - a_{i+3}) + 3^2 (a_{i+3} - a_{i+4}) + ... + n^2 (a_{i+n} - a_{i+n+1}) + (n+1)^2 a_{i+n+1}]\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i \cdot \frac{\Delta x}{2!}$$

$$+[(n+1)^3 a_{i-n} - n^3 (a_{i-n} - a_{i-n+1}) - ... - 3^3 (a_{i-3} - a_{i-2}) - 2^3 (a_{i-2} - a_{i-1}) - (a_{i-1} - a_i)$$

$= 0$

$$+(a_{i+1} - a_{i+2}) + 2^3 (a_{i+2} - a_{i+3}) + 3^3 (a_{i+3} - a_{i+4}) + ... + n^3 (a_{i+n} - a_{i+n+1}) + (n+1)^3 a_{i+n+1}]\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i \cdot \frac{\Delta x^2}{3!}$$

……

$$+[(n+1)^{(2n+1)} a_{i-n} - n^{(2n+1)} (a_{i-n} - a_{i-n+1}) - ... - 3^{(2n+1)} (a_{i-3} - a_{i-2}) - 2^{(2n+1)} (a_{i-2} - a_{i-1})$$

$= 0$

$$-(a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) + 2^{(2n+1)} (a_{i+2} - a_{i+3}) + 3^{(2n+1)} (a_{i+3} - a_{i+4}) + ... + n^{(2n+1)} (a_{i+n} - a_{i+n+1})$$

$$+(n+1)^{(2n+1)} a_{i+n+1}]\left.\frac{\partial^{(2n+1)} \phi}{\partial x^{(2n+1)}}\right|_i \frac{\Delta x^{2n}}{(2n+1)!}$$

$$+[-(n+1)^{(2n+2)} a_{i-n} + n^{(2n+2)} (a_{i-n} - a_{i-n+1}) + ... + 3^{(2n+2)} (a_{i-3} - a_{i-2}) + 2^{(2n+2)} (a_{i-2} - a_{i-1})$$

$$+(a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) + 2^{(2n+2)} (a_{i+2} - a_{i+3}) + 3^{(2n+2)} (a_{i+3} - a_{i+4}) + ... + n^{(2n+2)} (a_{i+n} - a_{i+n+1})$$

$= 0$

$$+(n+1)^{(2n+2)} a_{i+n+1}]\left.\frac{\partial^{(2n+2)} \phi}{\partial x^{(2n+2)}}\right|_i \frac{\Delta x^{2n+1}}{(2n+2)!} + ...$$

（余项为（2n+2)阶）

The only $2n+2$ order accuracy scheme can be obtained:

$$
\begin{cases}
(n+1)a_{i-n} - n(a_{i-n} - a_{i-n+1}) - \ldots. - 3(a_{i-3} - a_{i-2}) - 2(a_{i-2} - a_{i-1}) - 1(a_{i-1} - a_i) \\
+1(a_{i+1} - a_{i+2}) + 2(a_{i+2} - a_{i+3}) + 3(a_{i+3} - a_{i+4}) + \ldots + n(a_{i+n} - a_{i+n+1}) + (n+1)a_{i+n+1} = 1 \\
-(n+1)^2 a_{i-n} + n^2(a_{i-n} - a_{i-n+1}) + \ldots + 3^2(a_{i-3} - a_{i-2}) + 2^2(a_{i-2} - a_{i-1}) + 1^2(a_{i-1} - a_i) \\
+1^2(a_{i+1} - a_{i+2}) + 2^2(a_{i+2} - a_{i+3}) + 3^2(a_{i+3} - a_{i+4}) + \ldots + n^2(a_{i+n} - a_{i+n+1}) + (n+1)^2 a_{i+n+1} = 0 \\
(n+1)^3 a_{i-n} - n^3(a_{i-n} - a_{i-n+1}) - \ldots - 3^3(a_{i-3} - a_{i-2}) - 2^3(a_{i-2} - a_{i-1}) - (a_{i-1} - a_i) \\
+(a_{i+1} - a_{i+2}) + 2^3(a_{i+2} - a_{i+3}) + 3^3(a_{i+3} - a_{i+4}) + \ldots + n^3(a_{i+n} - a_{i+n+1}) + (n+1)^3 a_{i+n+1} = 0 \\
\cdots\cdots \\
(n+1)^{(2n+1)} a_{i-n} - n^{(2n+1)}(a_{i-n} - a_{i-n+1}) - \ldots - 3^{(2n+1)}(a_{i-3} - a_{i-2}) - 2^{(2n+1)}(a_{i-2} - a_{i-1}) \\
-(a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) + 2^{(2n+1)}(a_{i+2} - a_{i+3}) + 3^{(2n+1)}(a_{i+3} - a_{i+4}) + \ldots + n^{(2n+1)}(a_{i+n} - a_{i+n+1}) \\
+(n+1)^{(2n+1)} a_{i+n+1} = 0 \\
-(n+1)^{(2n+2)} a_{i-n} + n^{(2n+2)}(a_{i-n} - a_{i-n+1}) + \ldots + 3^{(2n+2)}(a_{i-3} - a_{i-2}) + 2^{(2n+2)}(a_{i-2} - a_{i-1}) \\
+(a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) + 2^{(2n+2)}(a_{i+2} - a_{i+3}) + 3^{(2n+2)}(a_{i+3} - a_{i+4}) + \ldots + n^{(2n+2)}(a_{i+n} - a_{i+n+1}) \\
+(n+1)^{(2n+2)} a_{i+n+1} = 0
\end{cases}
$$

If the coefficient of the term $\Delta x^{(2n+1)}\left(\left.\frac{\partial^{(2n+2)}\phi}{\partial x^{(2n+2)}}\right|_i\right)$ is not zero, the general formula for $(2n+1)$-order accuracy schemes can be obtained by solving following equations:

$$\left\{\begin{array}{l}
(n+1)a_{i-n}-n\left(a_{i-n}-a_{i-n+1}\right)-\ldots.-3\left(a_{i-3}-a_{i-2}\right)-2\left(a_{i-2}-a_{i-1}\right)-a_{i-1} \\
+1\left(a_{i+1}-a_{i+2}\right)+2\left(a_{i+2}-a_{i+3}\right)+3\left(a_{i+3}-a_{i+4}\right)+\ldots+n\left(a_{i+n}-a_{i+n+1}\right)+(n+1)a_{i+n+1}=1-a_i \\
-(n+1)^2 a_{i-n}+n^2\left(a_{i-n}-a_{i-n+1}\right)+\ldots+3^2\left(a_{i-3}-a_{i-2}\right)+2^2\left(a_{i-2}-a_{i-1}\right)+a_{i-1} \\
+1^2\left(a_{i+1}-a_{i+2}\right)+2^2\left(a_{i+2}-a_{i+3}\right)+3^2\left(a_{i+3}-a_{i+4}\right)+\ldots+n^2\left(a_{i+n}-a_{i+n+1}\right)+(n+1)^2 a_{i+n+1}=a_i \\
(n+1)^3 a_{i-n}-n^3\left(a_{i-n}-a_{i-n+1}\right)-\ldots-3^3\left(a_{i-3}-a_{i-2}\right)-2^3\left(a_{i-2}-a_{i-1}\right)-a_{i-1} \\
+\left(a_{i+1}-a_{i+2}\right)+2^3\left(a_{i+2}-a_{i+3}\right)+3^3\left(a_{i+3}-a_{i+4}\right)+\ldots+n^3\left(a_{i+n}-a_{i+n+1}\right)+(n+1)^3 a_{i+n+1}=-a_i \\
\cdots\cdots \\
(n+1)^{(2n-1)} a_{i-n}-n^{(2n-1)}\left(a_{i-n}-a_{i-n+1}\right)-\ldots-3^{(2n-1)}\left(a_{i-3}-a_{i-2}\right)-2^{(2n-1)}\left(a_{i-2}-a_{i-1}\right) \\
-a_{i-1}+\left(a_{i+1}-a_{i+2}\right)+2^{(2n-1)}\left(a_{i+2}-a_{i+3}\right)+3^{(2n-1)}\left(a_{i+3}-a_{i+4}\right)+\ldots+n^{(2n-1)}\left(a_{i+n}-a_{i+n+1}\right) \\
+(n+1)^{(2n-1)} a_{i+n+1}=-a_i \\
-(n+1)^{2n} a_{i-n}+n^{2n}\left(a_{i-n}-a_{i-n+1}\right)+\ldots+3^{2n}\left(a_{i-3}-a_{i-2}\right)+2^{2n}\left(a_{i-2}-a_{i-1}\right) \\
+(a_{i-1}-a_i)+\left(a_{i+1}-a_{i+2}\right)+2^{2n}\left(a_{i+2}-a_{i+3}\right)+3^{2n}\left(a_{i+3}-a_{i+4}\right)+\ldots+n^{2n}\left(a_{i+n}-a_{i+n+1}\right) \\
+(n+1)^{2n} a_{i+n+1}\neq 0
\end{array}\right.$$

Thus we have developed a general way for the discretization of the convective term with any order of accuracy for the FVM, obviously the first time in the history of the computational heat transfer. The first problem has been solved.

Then how to guarantee the stability of the discretized form of the convective term?

2.4.4 Derivation of Absolutely Stable Difference Scheme

Taking the general expression for 2nd-order scheme as an example. Eq. (6) is the definition of interpolation for the interface value.

Analyzing the scheme stability via 1-D unsteady convection-diffusion equation:

$$\rho \frac{\partial \phi}{\partial t} + \rho u \frac{\partial \phi}{\partial x} = \Gamma \frac{\partial^2 \phi}{\partial x^2}$$

$$\begin{cases} \phi_e = a_i \phi_i + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-1} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_{i+1} \\ \phi_w = a_i \phi_{i-1} + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-2} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_i \\ a_i \neq \frac{5}{6} \end{cases} \quad (6)$$

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + u \frac{\phi_e^n - \phi_w^n}{\Delta x} = \frac{\Gamma}{\rho}[(\frac{d\phi}{dx})_e^n - (\frac{d\phi}{dx})_w^n] \quad (8)$$

Substituting Eq.(6) into Eq. (8) for the convective term and adopting CD for the diffusion term:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + u \frac{(6a_i - 3)\phi_i^n + (1 - 6a_i)\phi_{i-1}^n + (3 - 2a_i)\phi_{i+1}^n - (1 - 2a_i)\phi_{i-2}^n}{4\Delta x}$$

$$= \Gamma \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\rho \Delta x^2} \quad (9)$$

To analyze the stability, first omitting the diffusion term for $(i+1),(i-1)$ we have following equations:

$$\frac{\phi_{i+1}^{n+1}-\phi_{i+1}^{n}}{\Delta t}=-u\frac{(6a_i-3)\phi_{i+1}^{n}+(1-6a_i)\phi_{i}^{n}+(3-2a_i)\phi_{i+2}^{n}-(1-2a_i)\phi_{i-1}^{n}}{4\Delta x}$$

$$\frac{\phi_{i-1}^{n+1}-\phi_{i-1}^{n}}{\Delta t}=-u\frac{(6a_i-3)\phi_{i-1}^{n}+(1-6a_i)\phi_{i-2}^{n}+(3-2a_i)\phi_{i}^{n}-(1-2a_i)\phi_{i-3}^{n}}{4\Delta x}$$

Assuming that the initial fields are zero and at instant *n* there is only disturbance at point *i*, ε_i^n, and no any disturbance in the subsequent time , then for *i*+1, its effect can be determined:

$$\frac{\phi_{i+1}^{n+1}-\overset{0}{\phi_{i+1}^{n}}}{\Delta t}=-u\frac{(6a_i-3)\overset{0}{\phi_{i+1}^{n}}+(1-6a_i)\overset{\varepsilon_i^n}{\phi_{i}^{n}}+(3-2a_i)\overset{0}{\phi_{i+2}^{n}}-(1-2a_i)\overset{0}{\phi_{i-1}^{n}}}{4\Delta x}$$

$$\phi_{i+1}^{n+1}=(\frac{6a_i-1}{4})(\frac{u\Delta t}{\Delta x})\phi_i^n=(\frac{6a_i-1}{4})(\frac{u\Delta t}{\Delta x})\varepsilon_i^n$$

Similar analysis can be done for (i-1);

The effect of the disturbance ε_i^n **transported by the convective terms can be summarized:**

$$\begin{cases} \phi_{i+1}^{n+1} = \dfrac{(6a_i - 1)}{4}\left(\dfrac{u\Delta t}{\Delta x}\right)\varepsilon_i^n \\ \phi_{i-1}^{n+1} = \dfrac{(2a_i - 3)}{4}\left(\dfrac{u\Delta t}{\Delta x}\right)\varepsilon_i^n \end{cases} \qquad (10)$$

The effect of the diffusion term is $\phi_{i\pm1}^{n+1} = (\dfrac{\Gamma\Delta t}{\Delta x^2})\varepsilon_i^n$

Then the total effects of the convection/diffusion should satisfy the sign preservation principle:

$$\left.\begin{cases} \dfrac{\dfrac{(6a_i-1)}{4}\left(\dfrac{u\Delta t}{\Delta x}\right)\varepsilon_i^n+\left(\dfrac{\Gamma\Delta t}{\rho\Delta x^2}\right)\varepsilon_i^n}{\varepsilon_i^n}\ge 0 \\ \dfrac{\dfrac{(2a_i-3)}{4}\left(\dfrac{u\Delta t}{\Delta x}\right)\varepsilon_i^n+\left(\dfrac{\Gamma\Delta t}{\rho\Delta x^2}\right)\varepsilon_i^n}{\varepsilon_i^n}\ge 0 \end{cases}\right\} \longrightarrow \begin{cases} 6a_i-1\ge 0 \\ 2a_i-3\ge 0 \end{cases} \longrightarrow a_i\ge\frac{3}{2}$$

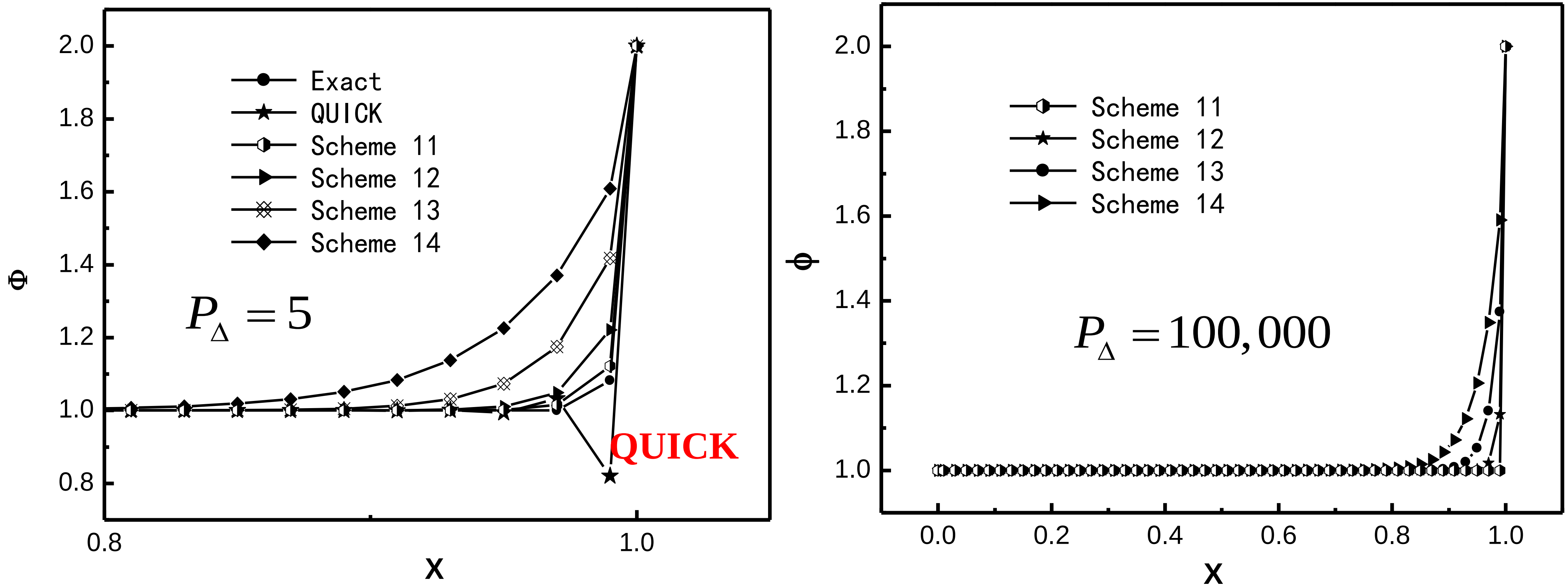
From Eq.(6): $\phi_e=a_i\phi_i+\left(\frac{1}{4}-\frac{a_i}{2}\right)\phi_{i-1}+\left(\frac{3}{4}-\frac{a_i}{2}\right)\phi_{i+1}$

Absolutely stable scheme of second-order accuracy :

$$\boxed{\begin{cases} \phi_e=a_i\phi_i+\left(\dfrac{1}{4}-\dfrac{a_i}{2}\right)\phi_{i-1}+\left(\dfrac{3}{4}-\dfrac{a_i}{2}\right)\phi_{i+1} \\ a_i\ge\dfrac{3}{2} \end{cases}} \qquad \textbf{(11)}$$

Demonstration of Absolutely Stability

1-D diffusion-convection problem with $\phi(0)=1 \quad \phi(1)=2$ were solved with different coefficient $a_i \ge a_{i0}$



Scheme11: $a_i = 1.5$; Scheme12: $a_i = 2.0$; Scheme13: $a_i = 4.0$; Scheme14: $a_i = 10.0$

2.5 对称奇阶格式

2.5.1 “Symmetry & odd-order ” scheme

2.5.2 “Symmetry & 3rd-order ” scheme and demonstrations

2.5.3 Computational time comparisons

2.5.4 Conclusions and summary

Jin W W, Tao W Q. NHT, Part B, 2007, 52(3): 131-254
Jin W W, Tao W Q. NHT, Part B, 2007, 52(3): 255-280

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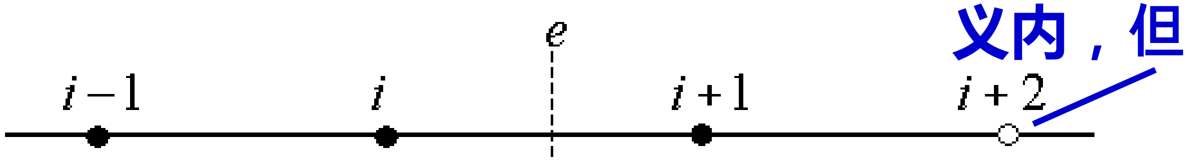
2.5.1 "Symmetry & odd-order " scheme

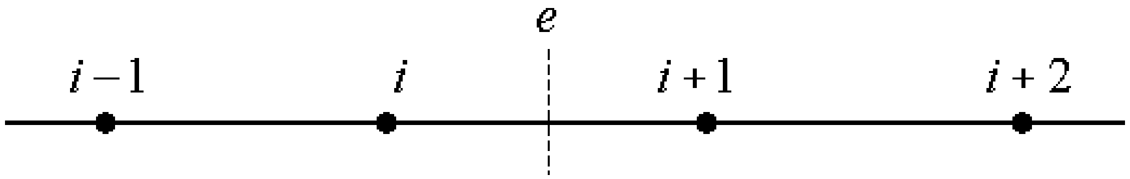
Numerous numerical practices have been conducted for different stencils with the same number of total grids to compare their accuracy, stability and economics.

For simplicity stencils are presented for e-interface at $u_e > 0$:

此点不在格式定义内，但需要内存

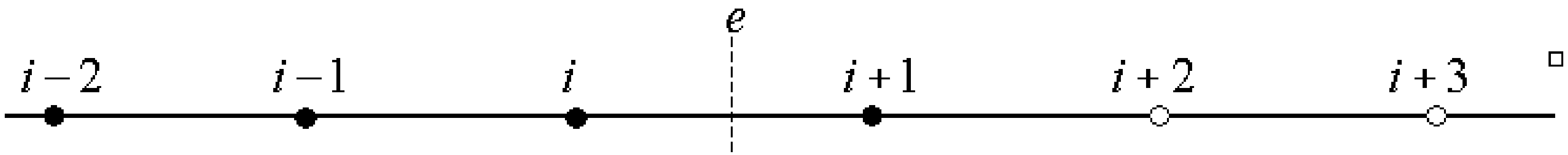


Four-grid stencil of e-interface for Non-Symmetry Second-Order(QUICK)

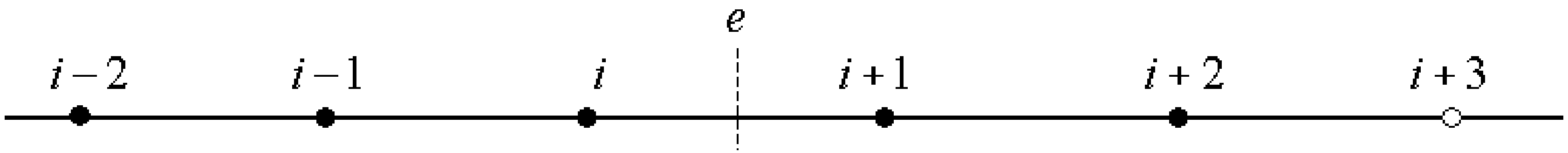


Four-grid stencil of e-interface for Symmetry Third-Order (TS)

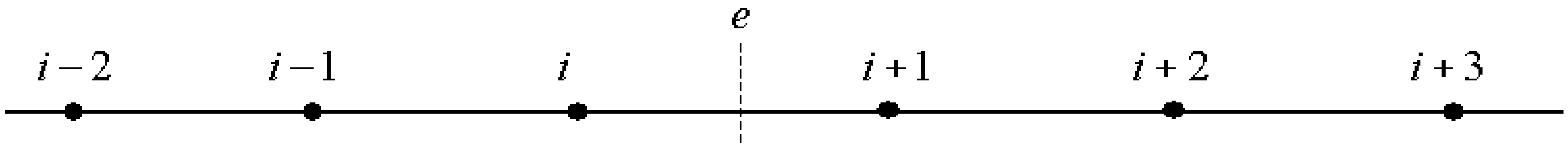
黑点是格式定义中用到的节点。



Six-grid stencil of e-interface for Non-Symmetry Third-Order



Six-grid stencil of e-interface for Non-Symmetry Fourth-Order



Six-grid stencil of e-interface for Symmetry Fifth-Order

For the three schemes of the six-grid stencil of e-interface the required computer spaces are the same, because all the six grids have to be reserved for storing information.

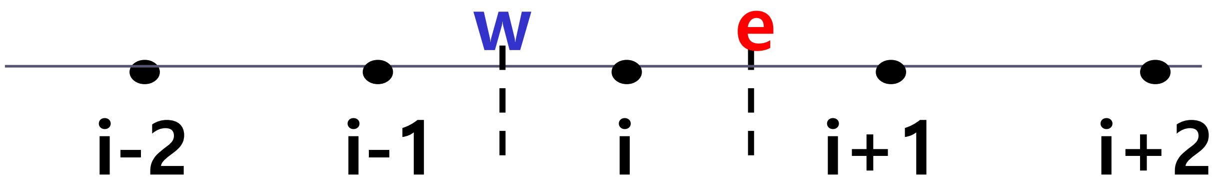
Through numerous numerical simulations it is found:

1. **For any stencil the correspondent scheme can be absolutely stable if the coefficient a_i satisfies the condition obtained by the sign preservation rule; (Answer to 2nd question!)**
2. **With the reduction of the non-symmetry in stencil , the effect of the false diffusion decreases.**
3. **For the same problem simulated, the schemes having the same number of total grids in stencils require almost the same CPU time.**

We thus propose a new idea of constructing " symmetry & odd-order " scheme:

By sign-preservation principle the "Symmetry and Third-order" scheme is absolutely-stable when $a_i \geq 13/12$

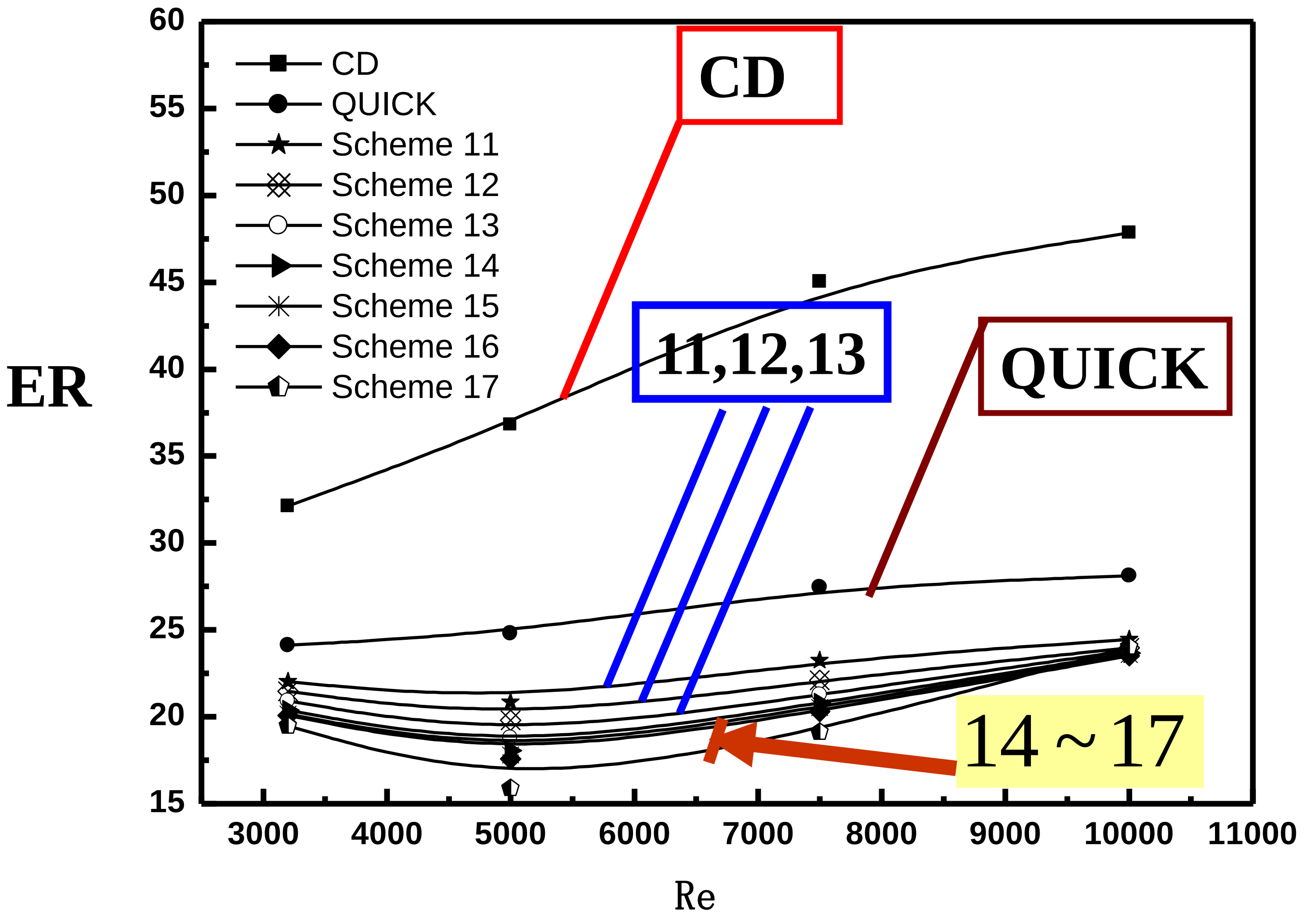
$$\begin{cases} \phi_e = \dfrac{(2-6a_i)}{18}\underline{\phi_{i-1}} + a_i\underline{\phi_i} + \dfrac{(21-18a_i)}{18}\underline{\phi_{i+1}} + \dfrac{(6a_i-5)}{18}\underline{\phi_{i+2}} \\ \phi_w = \dfrac{(2-6a_i)}{18}\underline{\phi_{i-2}} + a_i\underline{\phi_{i-1}} + \dfrac{(21-18a_i)}{18}\underline{\phi_i} + \dfrac{(6a_i-5)}{18}\underline{\phi_{i+1}} \\ a_i \geq \dfrac{13}{12} \qquad \text{Denoting 13/12 as } a_{i0}\,! \end{cases}$$



Full stencil of the absolutely-stable "symmetry & 3rd-order"schemes

Followings are some simple demonstrations:

For lid-driven cavity flow at Re=3200、5000、7500 and 10000 with 24×24 grid system：



$11: a_i = 5/6$

$12: a_i = 11/12$

$13: a_i = 1$

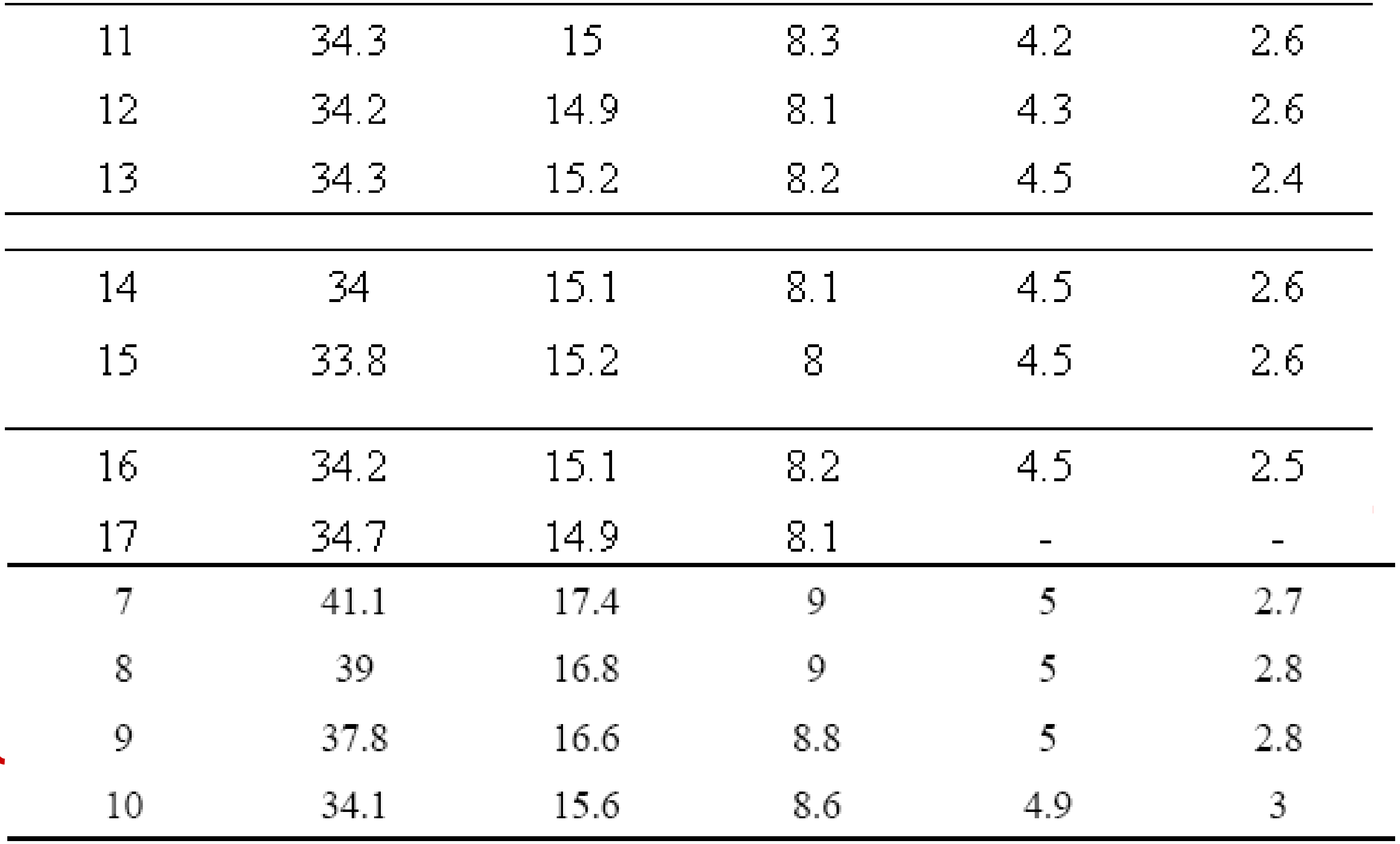
$14: a_i = 13/12$

$15: a_i = 9/8$

$16: a_i = 7/6$

$17: a_i = 5/3$

Computational time at Re=1000 of "symmetry & 3rd-order" and "non-symmetry 2nd-order" schemes (Grid 42×42)



α	0.1	0.3	0.5	0.7	0.9
11	34.3	15	8.3	4.2	2.6
12	34.2	14.9	8.1	4.3	2.6
13	34.3	15.2	8.2	4.5	2.4
14	34	15.1	8.1	4.5	2.6
15	33.8	15.2	8	4.5	2.6
16	34.2	15.1	8.2	4.5	2.5
17	34.7	14.9	8.1	-	-
7	41.1	17.4	9	5	2.7
8	39	16.8	9	5	2.8
9	37.8	16.6	8.8	5	2.8
10	34.1	15.6	8.6	4.9	3

Scheme

Symmetry 3rd-order (schemes 11–17)

non-symmetry 2nd-order (schemes 7–10)

2.5.2 "S-T " Scheme Further Demonstrations

The third order accurate scheme is absolutely stable but use the same stencil grids as that of QUICK scheme. Taking the case of $a_i = 9/8$ (>13/12) as an example, interpolations are as:

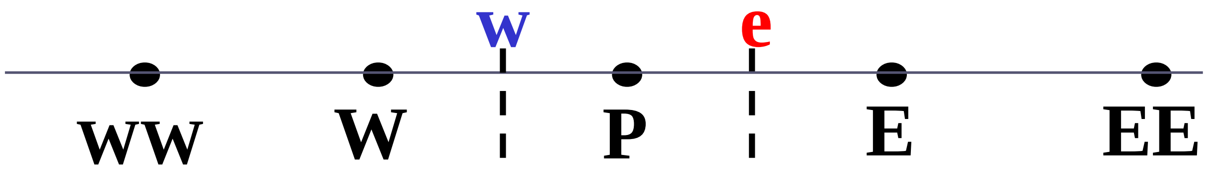
$$u_e > 0 \quad \phi_e = -\frac{19}{72}\underline{\phi_W} + \frac{9}{8}\underline{\phi_P} + \frac{1}{24}\underline{\phi_E} + \frac{7}{72}\underline{\phi_{EE}}$$

$$u_e < 0 \quad \phi_e = -\frac{19}{72}\underline{\phi_{EE}} + \frac{9}{8}\underline{\phi_E} + \frac{1}{24}\underline{\phi_P} + \frac{7}{72}\underline{\phi_W}$$

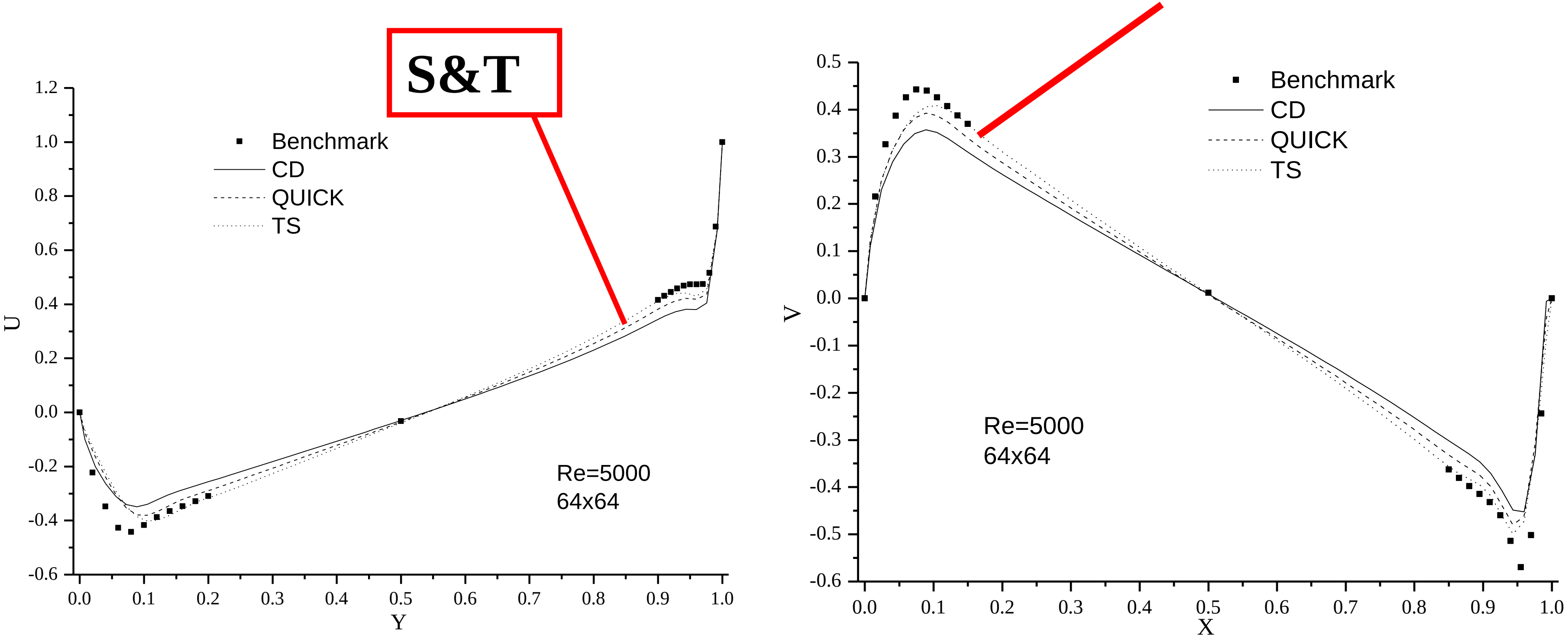
$$u_w > 0 \quad \phi_w = -\frac{19}{72}\phi_{WW} + \frac{9}{8}\phi_W + \frac{1}{24}\phi_P + \frac{7}{72}\phi_E$$

$$u_w < 0 \quad \phi_w = -\frac{19}{72}\phi_E + \frac{9}{8}\phi_P + \frac{1}{24}\phi_W + \frac{7}{72}\phi_{WW}$$

无论流速大于0还是小于0，同一界面插值的节点不变，但系数改变！



1 .Lid-Driven cavity problem



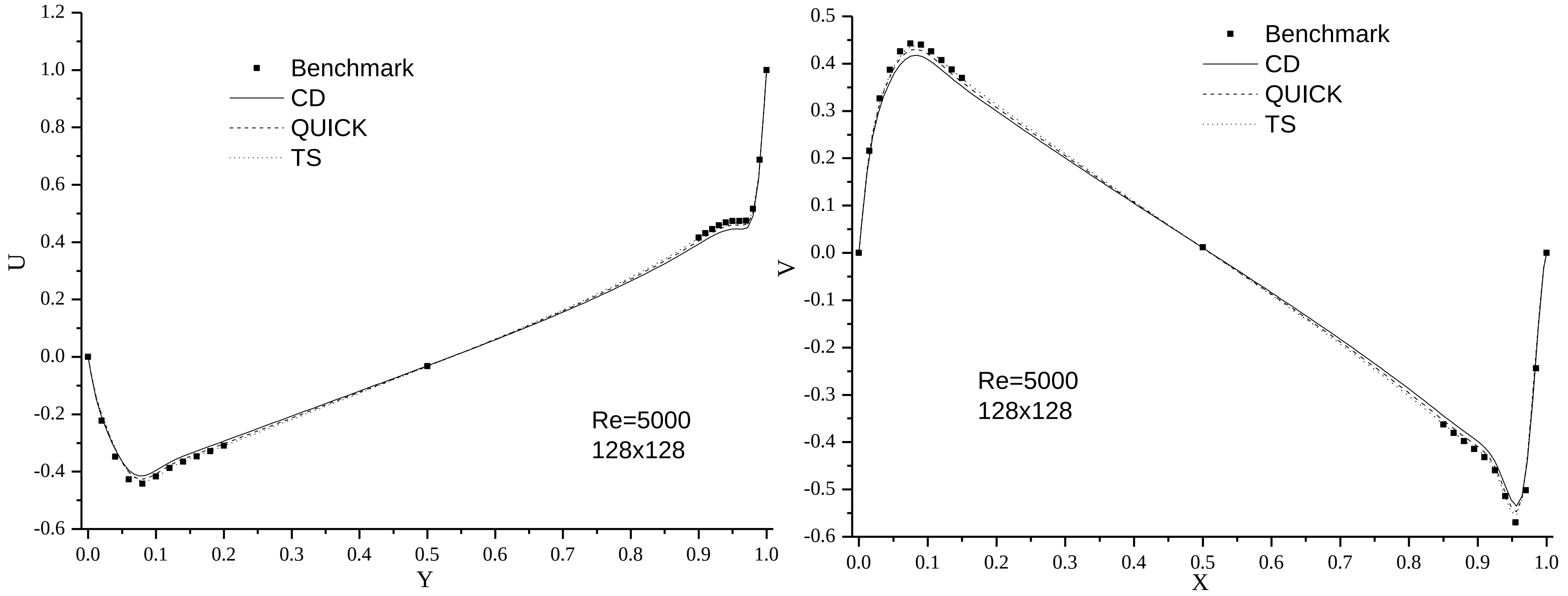
S&T

S&T

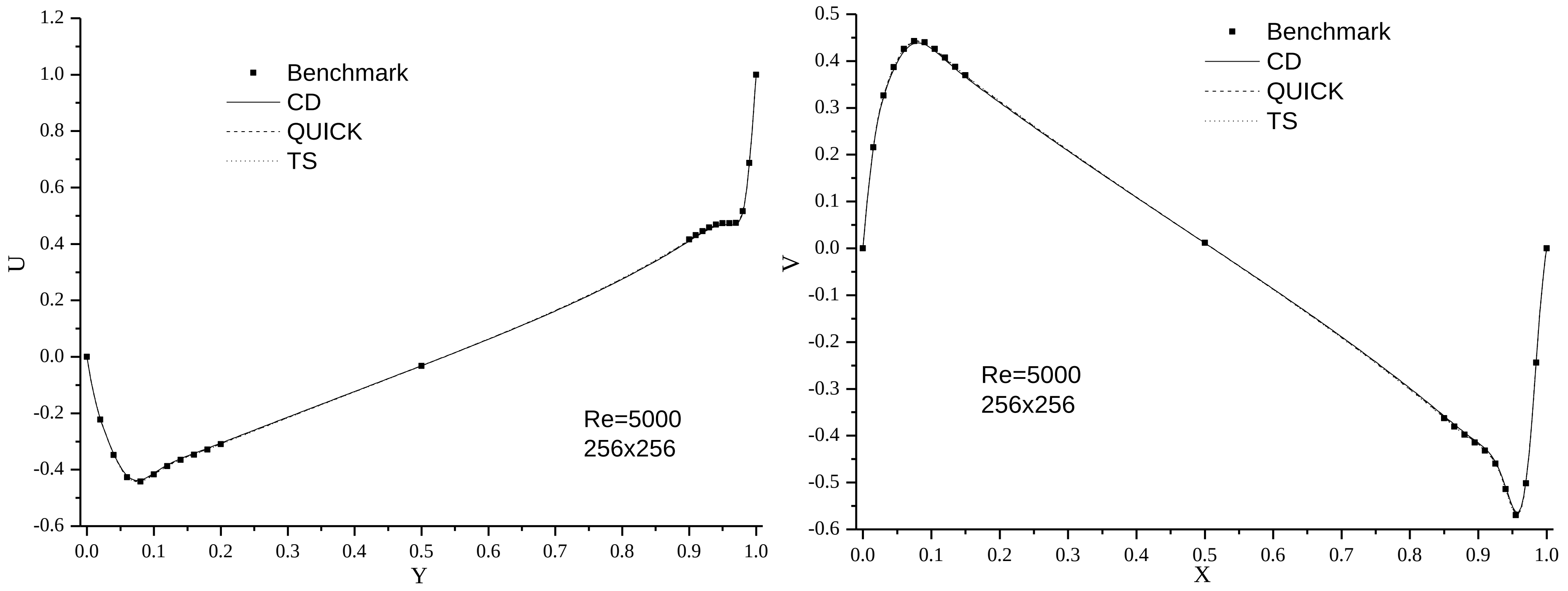
Computed velocity along the line passing through the geometric centre of the cavity at Re=5000, 64x64 grid system.

U. Ghia, K.N. Ghia and C.T. Shin. High-Re Solution for Incompressible Flow Using the Navier-Stokes Equations and a Multigrid Method. *J. Comput. Phys.*, 1982, vol.48, pp.387-411.

E. Ertuk, T.C. Corke and C. Gokcol. Numerical Solutions of 2-D Steady Incompressible Driven Cavity Flow at High Reynolds Numbers, *Int. J. Numer. Meth. Fluids*, 2005, vol.48,pp.747-774



Computed velocity along the line passing through the geometric centre of the cavity at Re=5000, 128x128 grid system.



Computed velocity along the line passing through the geometric centre of the cavity at Re=5000, 256x256 grid system.

Average relative error comparison

Grid system of 64×64:

QUICK －32% more accurate than CD ;

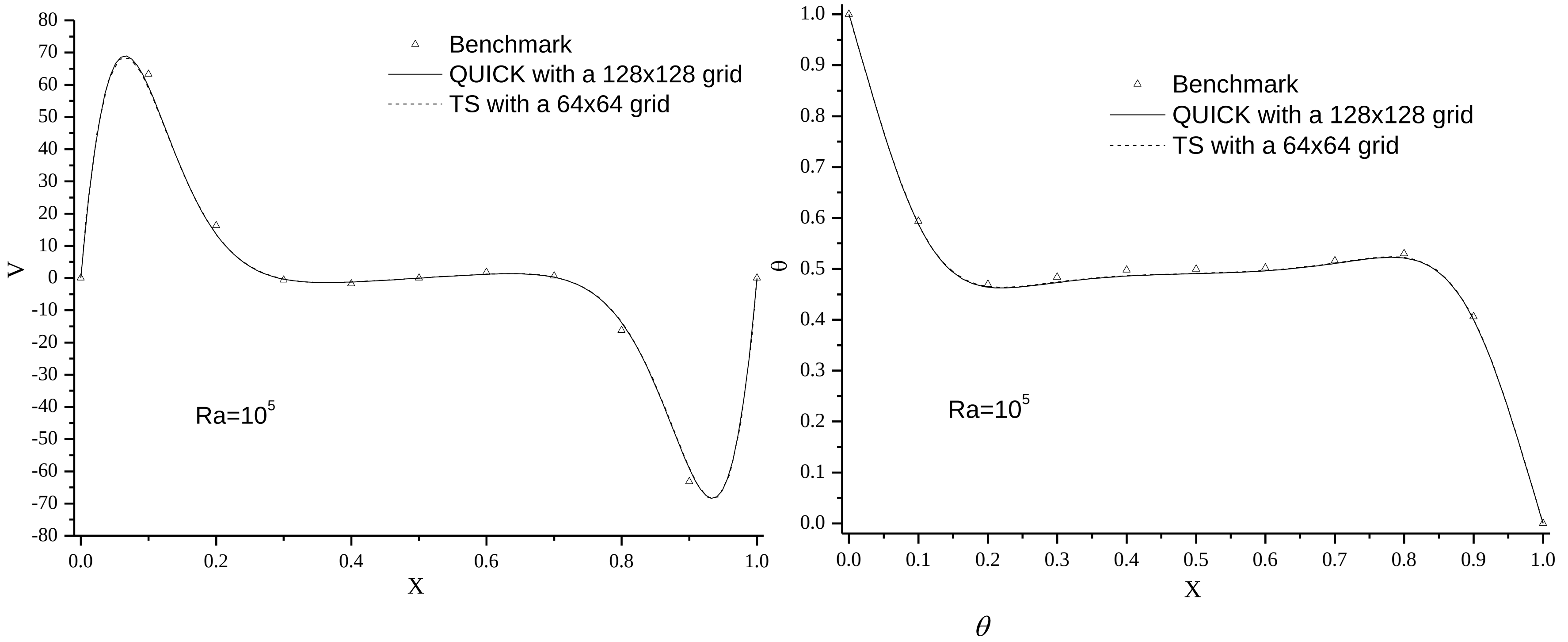
S&T scheme － 22% more accurate than QUICK

Grid system of 128×128:

QUICK－33% more accurate than CD scheme,;

S&T－ 28% more accurate than QUICK scheme.

2. Natural convection in a square cavity



V velocity and dimensionless temp. along the center vertical line of the cavity at Ra=10^5(64X64 for S-T vs. 128X128 for QUICK)

D.C. Wan, B.S.V. Patnaik and G.W. Wei. A New Benchmark Quality Solution for the Buoyancy-Driven Cavity by Discrete Singular Convolution. Numer. Heat Transfer B. 2001,Vol.40, pp.199-228

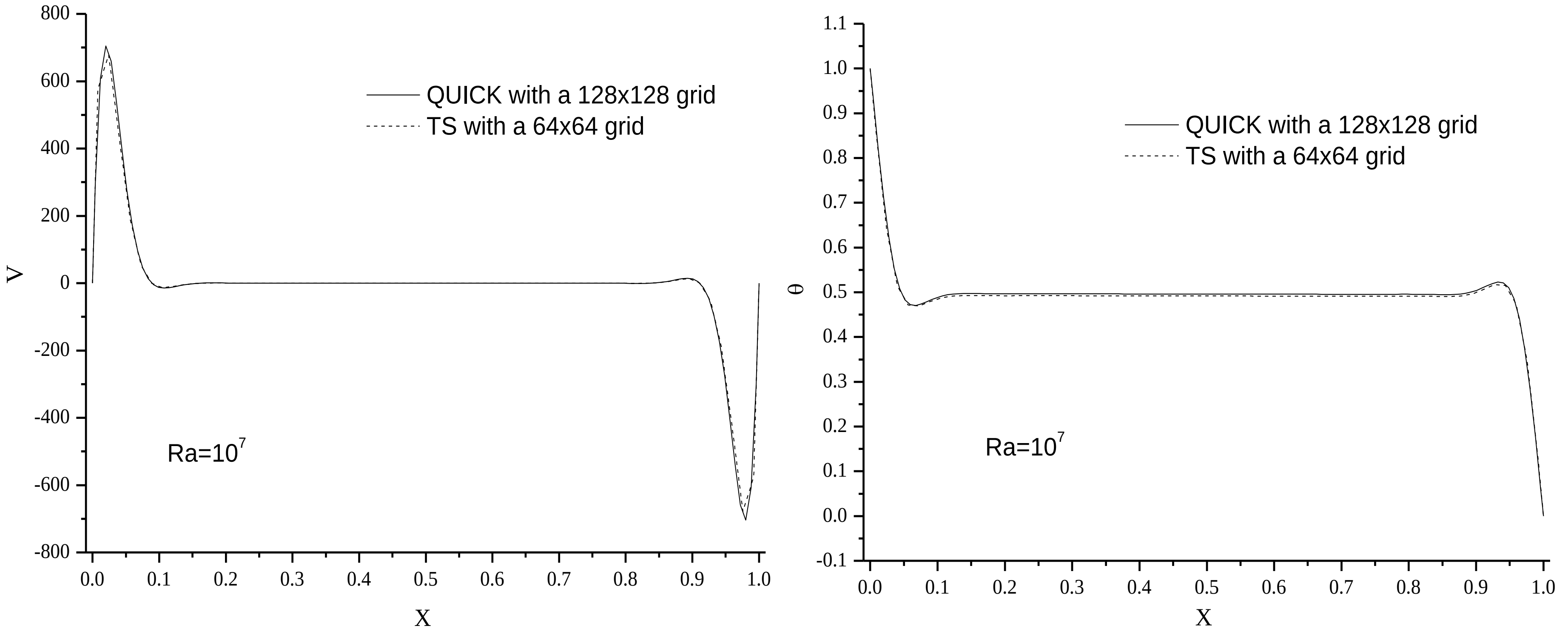
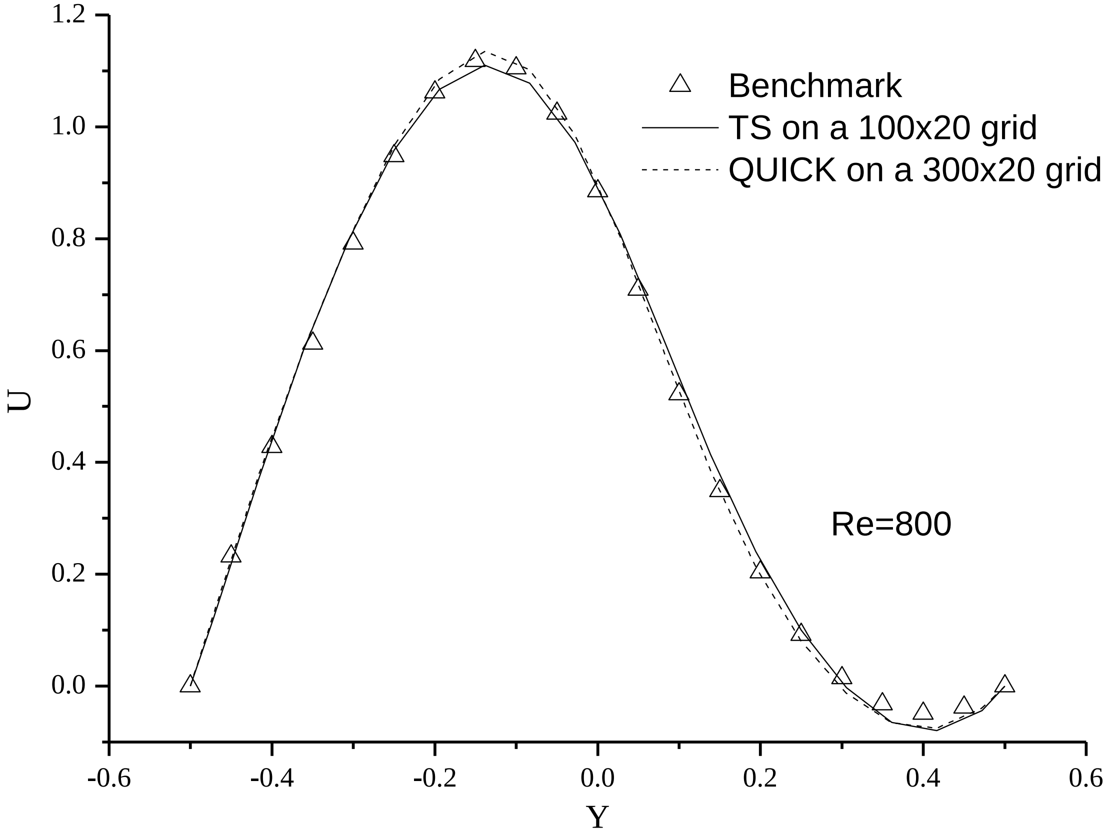


Fig. V velocity and dimensionless temp. along the center vertical line of the cavity at Ra=10^7

Solutions from QUICK with 128x128 grids are almost identical to the solution by ST(TS) with 64x64 grids.

3. Flow over a backward facing step



Computation on a grid of 100X20 by QUICK can not reach converged solutions . A 300 X20 grid must be used for QUICK

D.K.Gartling. A Test Problem for Outflow Boundary Conditions-Flow Over a Backward-Facing Step. Int. J. Numer. Methods Fluids. 1990,11:953-967

2.5.3 Computational time comparisons

Lid-Driven (Re=5000)

128 by 128 grids: QUICK－**102s**. ST－ **108s**.

256 by 256 grid: QUICK－**814s**. ST－ **835s**.

Natural convection ($Ra=10^5$)

64 by 64: QUICK－**15s**. ST－**17s.**

128 by128: QUICK－**152 s**. ST－**220s**.

Flow over backward-step

100 by 20: QUICK －**Divergence!** ST－ **0.7s**.

300 by 20 QUICK －**3.3s**. ST－**2.7s.**

Jin W W, Tao W Q. Numerical Heat Transfer, Part B, 2007, 52(3): 131-254
Jin W W, Tao W Q. Numerical Heat Transfer, Part B, 2007, 52(3): 255-280

推荐阅读(7,8)

2.5.4 Conclusions and summary

1. Following conclusions can be drawn:

1) ST scheme can provide more accurate results than the QUICK scheme with unconditional stability while it consumes almost the same computation cost as the QUICK scheme.

2) “Symmetry and Odd-Order Schemes” possess higher accuracy, absolutely stability and acceptable consumption in computational source .

Then a simple question may be raised: why we should still insist in the adoption of the upwind-based schemes?

2. FVM对流项离散格式特性总结

1) 守恒性（Conservation）

采用控制容积积分法导出的、而且界面插值具有连续性离散方程具有守恒性，可认为是FVM的固有属性；

2) 迁移性（Transportiveness）

只能将扰动向下游传递；

3) 相容性（Consistency）

当步长趋于 0 时 FVM 方程趋近于对应的微分方程；

4) 收敛性（Convergence）

当步长趋于 0 时FVM的解趋近于对应的微分方程的分析解；

5) 稳定性（Stability）

在求解过程中引入的数值误差不会被不断放大以致使数值解变得无界；

6) 精确性（Accuracy）

数值解接近于微分方程分析解的程度；

7) 有界性（Boundness）

数值解的值不会超出物理问题规定的上下限；

8) 经济性（Boundness）

获得数值解所耗费的计算机资源的多少。

推荐阅读

[1] Hayase T, Humphery J A C, Grief A R. A constantly formulated QUICK scheme for fast and stable convergence using finite-volume iterative calculation procedures. J Comput Physics, 1992,93:108-119

[2] Li Z Y, Tao WQ. A new stability-guaranteed second-order difference scheme. Numerical Heat Transfer-Part B, 2002, 42 (4): 349-365

[3] Hou P L, Tao W Q, Yu M Z., Refinement of the convective boundedness criterion of Gaskell and Lau, Engineering Computations, 20(2003) 1023-1043

[4] Jin W W, Tao W Q. Design of high-order difference scheme and analysis of solution characteristics—Part I: General formulation of high-order difference schemes and analysis of convective stability Numerical Heat Transfer, Part B, 2007, 52(3): 255-280

[5] Jin W W, Tao W Q. Design of high-order difference scheme and analysis of solution characteristics—Part II: A kind of third-order difference scheme and new scheme design theory, Numerical Heat Transfer, Part B, 2007, 52(3): 131-254

同舟共济
渡彼岸!

People in the same boat help each other to cross to the other bank, where….