Revisiting two-dimensional turbulence by Lattice Boltzmann Method

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Abstract: In this paper, Lattice Boltzmann Method (LBM) is used to investigate the freely- decaying two-dimensional (2D) turbulence. The main purpose focuses on recent mathematical results about a central assumption in the theory of two-dimensional turbulence proposed by Batchelor. The numerical results found an approximate exponent of energy spectrum $E(k) \sim k^{-3.5}$ at the high mesh resolutions and an approximate decaying exponent of enstrophy $\langle \omega^2 \rangle \sim T^{-2/3}$. In addition, an approximate exponent of integral length scale $l \sim T^{1/2}$ is observed. These results validate the recent new results of 2D turbulence and are different from the results derived by Batchelor's 2D model.

Keywords: two-dimensional turbulence; kinetic energy; enstrophy; integral length scale; LBM; lattice Boltzmann method.

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1 Introduction

Lattice Boltzmann Method (LBM) has attracted much attention since 1990s. It has achieved a great success in simulations of fluid flow problems. In particular, the natural advantage of lattice-based models for performing parallel computation and handling complex geometries problems, with local operations, indicates that the lattice models may provide an alternative efficient numerical scheme for studying many complex phenomena (Qian et al., 1992; Qian and Zhou 1998), including turbulence. In fact, LBM has been widely used to simulate turbulent fluid flows (Succi, 2001; Orszag et al., 2006) and the results offer us some fundamental understanding of turbulence (Chen et al., 2003, 2004). So far, for Direct Numerical Simulation (DNS) of turbulence, most researchers still use Navier-Stokes equations. Boltzmann equation describes the phenomena of turbulence in the statistical physics framework, which possesses some inherent consistency with physical processes. It is in this aspect that Boltzmann equation has its particular advantage in the simulation of turbulence. In this paper, the complete Galilean-invariant D2Q17 LBM model (Qian and Zhou, 1998) will be adopted to simulate 2D free-decaying turbulence.

It is well-known that much progress has been made on the study of 2D turbulence, but still, a number of fundamental questions remain unanswered and dubitable (Tabeling, 2004). Most characteristics of the energy cascade process, seemingly, need to be reconsidered. The inverse cascade is much less understood. Especially, the exponent of kinetic spectrum is not very clear. The theoretical understanding of these phenomena is still in its developing stage. Compared with 3D turbulence, 2D turbulence exhibits a rich variety of phenomena. In three dimensional case, the energy cascade goes uniquely from large eddies to small eddies, while in two dimensional case, there exist several types of cascades, and turbulence may exist without cascade at all (Tabeling, 2004). From this point of view, two-dimensional turbulence is wealthier than the three-dimensional one.

In 1969, Batchelor (1969), proposed a model of 2D turbulence, in which there is a self-similar decay of energy spectrum. In this model it is anticipated that there exists a finite, non-zero value of the enstrophy dissipation and an invariant kinetic energy in the limit of $\mathrm{Re} \to \infty$ (or viscosity parameter $\nu \to 0$). Recent mathematic analysis and numerical simulations showed that the enstrophy dissipation decays with $\mathrm{Re} \to \infty$ as $(\ln\mathrm{Re})^{-1}$ (Choung and David, 2006; Dmitruk and Montgomery, 2005). In addition observations and analysis (Dmitruk and Montgomery, 2005; Bartello and Warn, 2005; McWilliams, 1984) suggest that the two-dimensional turbulence of $\mathrm{Re} \to \infty$ remembers the peak value of vorticity, which is supported by the theoretical analysis of McWilliams (1984), Legras et al. (2001) and Mariotti et al. (2001) but not indicated in Batchelor's 2D model. The above two issues may imply that the central assumption made in the Batchelor's theory of 2D turbulence is questionable and this stimulates us to revisit the behaviours of 2D turbulence by LBM.

In order to investigate 2D freely-decaying turbulence, DNS will be taken in a periodic domain. Grid systems with large resolutions of mesh (1024^2, 2048^2 and 4096^2) are adopted to simulate the turbulence field for very high Re. The initial velocity of 2D turbulence will be initialised by random fields, which are established in Fourier spectrum space. The peak of initial energy spectrum will be set at low wave number area such that no significant inverse energy cascade takes place over short simulation duration.

This paper is organised as follows. In Section 2, the basic theory of LBM is reviewed. In Section 3, the fundamental theory of 2D turbulence is recalled. In Section 4, the plans of the simulation are established and various numerical results are given. Finally, a conclusion of the simulation results is presented.

2 A review of Lattice Boltzmann Method

We now introduce the lattice BGK model as a solver for the incompressible Navier-Stokes equations. The LBM is built up from the lattice gas cellular automata models (Succi, 2001). The numerical scheme of LBM is established based on a finite discrete-velocity model of the Boltzmann equation as follows

$$f_i(\vec{x} + \delta t\vec{c_i}, t + \delta t) - f_i(\vec{x}, t) = \Omega_i \tag{1}$$

where f_i denotes the single-particle distribution function along the direction $\vec{c_i}$, and $\vec{c_i}$ is an element of the discrete velocity set $V = \{\vec{c_0}, \ldots, \vec{c_n}\}$. Ω_i represents the collision operator. The macroscopic variables, the density ρ and the velocity $\vec{u}$, are defined locally by the distribution functions as follows

$$\rho(\vec{x}, t) = \sum_{i=0}^{n} f_i(\vec{x}, t) = \sum_{i=0}^{n} f_i^{(\mathrm{eq})}(\vec{x}, t), \tag{2}$$

$$\vec{u}(\vec{x}, t) = \frac{1}{\rho(\vec{x}, t)} \sum_{\vec{c_i} \in V} \vec{c_i} f_i(\vec{x}, t) = \frac{1}{\rho(\vec{x}, t)} \sum_{\vec{c_i} \in V} \vec{c_i} f_i^{(\mathrm{eq})}(\vec{x}, t). \tag{3}$$

For the standard LBM, the collision operator is defined by the so-called BGK collision

$$\Omega_i^{BGK} = -\frac{1}{\tau}[f_i(\vec{x}, t) - f_i^{(\mathrm{eq})}(\vec{x}, t)]. \tag{4}$$

The local equilibrium distribution function $f_i^{\mathrm{eq}}(\vec{x}, t)$ is defined by

$$f_i^{(\mathrm{eq})}(\vec{x}, t) = f_i^{L(\mathrm{eq})}(\vec{x}, t) + f_i^{Q(\mathrm{eq})}(\vec{x}, t), \tag{5}$$

where $f_i^{L(\mathrm{eq})}(\vec{x}, t)$ and $f_i^{Q(\mathrm{eq})}(\vec{x}, t)$ denote the linear part and the quadratic part of the equilibrium distribution function, respectively. The linear part is given by

$$f_i^{L(\mathrm{eq})}(\vec{x}, t) = f_i^* \rho \left(1 + \frac{1}{C_s^2} \vec{c}_i \cdot \vec{u}(\vec{x}, t) \right), \tag{6}$$

and the quadratic part is defined by

$$f_i^{Q(\mathrm{eq})}(\vec{x}, t) = f_i^* \frac{1}{2C_s^4} \rho(\vec{u}(\vec{x}, t)\vec{u}(\vec{x}, t)) : \Sigma_i, \tag{7}$$

where C_s is the speed of sound of the model, f_i^* denotes the weight and Σ_i is a second order tensor defined by

$$\Sigma_{i\alpha\beta} = c_{i\alpha} c_{i\beta} - C_s^2 \delta_{\alpha\beta}. \tag{8}$$

Here, the related tensor product definition among n first-order tensors $\vec{a}^{\,i}\ (i = 1, \ldots, n)$ is given as follows

$$(\vec{a}^{\,1} \cdots \vec{a}^{\,n})_{\alpha_1, \ldots, \alpha_n} = \vec{a}^{\,1}_{\alpha_1} \cdots \vec{a}^{\,n}_{\alpha_n}, \tag{9}$$

and the corresponding tensor product between m n-order tensors, $\vec{A}^{\,i}_{\alpha_1 \cdots \alpha_n}\ (i = 1, \ldots, m)$, is given by

$$\vec{A}^{\,1} : \vec{A}^{\,2} : \cdots : \vec{A}^{\,m} = \sum_{\alpha_1, \ldots, \alpha_n = 1}^{d} \vec{A}^{\,1}_{\alpha_1 \cdots \alpha_n} \vec{A}^{\,2}_{\alpha_1 \cdots \alpha_n} \cdots \vec{A}^{\,m}_{\alpha_1 \cdots \alpha_n}, \tag{10}$$

where d denotes the spatial dimension.

For 2D incompressible fluid flows, the popular D2Q9 model (Qian et al., 1992) is widely used to simulate various fluid flow problems. Equations (6) and (7) will lead to the standard D2Q9 equilibrium distribution functions. By using the Chapman-Enskog expansion, the Navier-Stokes equations can be derived up to the second order of Knudsen number at long wavelength and long time limits (Qian et al., 1992),

$$\partial_t \rho + \partial_\alpha (\rho u_\alpha) + O(\delta t^3) = 0, \tag{11}$$

$$\begin{aligned}\partial_t (\rho u_\alpha) + \partial_\beta (\rho u_\alpha u_\beta) = &-\partial_\alpha P + \partial_\beta (\rho \nu \partial_\beta u_\alpha + \rho \varsigma \partial_\alpha u_\beta) \\ &- \sigma \partial_\beta \partial_\gamma (\rho u_\alpha u_\beta u_\gamma),\end{aligned} \tag{12}$$

where

$$\nu = \zeta = C_s^2 \delta t (\tau - 0.5). \tag{13}$$

In Equation (12) P is the thermodynamic pressure and can be determined by the equation of state as follows:

$$P = C_s^2 \rho. \tag{14}$$

The parameter σ in Equation (12) is given by

$$\sigma = \delta t (\tau - 0.5). \tag{15}$$

In Equation (12), the parameter σ in the cubic term is a new term, which is often ignored in the most numerical computation. The physical meaning of this cubic term is the nonlinear response due to the quadratic term in the equilibrium function (Qian and Zhou, 1998).

It has been proved that the cubic term in Equation (12) is a source of non-Galilean invariance. The effect of this cubic term will lead to some nonphysical phenomena for solving Navier-Stokes equations (Qian and Zhou, 1998). In order to eliminate this nonphysical phenomenon, Qian and Zhou (1998) proposed a modified equilibrium to eliminate this cubic term. A cubic tensor about macroscopic velocity is introduced to the equilibrium function,

$$f_i^{C(\mathrm{eq})}(\vec{x}, t) = f_i^* \frac{1}{6C_s^6} \rho(\vec{u}(\vec{x}, t)\vec{u}(\vec{x}, t)\vec{u}(\vec{x}, t)) : T_i \tag{16}$$

where the third order tensor T_i is defined by

$$T_{i\alpha\beta\gamma} = c_{i\gamma}(c_{i\alpha} c_{i\beta} - 3C_s^2 \delta_{\alpha\beta}). \tag{17}$$

The modified equilibrium distribution function $f_i^{(\mathrm{eq})}(\vec{x}, t)$ can be rewritten as follows

$$f_i^{(\mathrm{eq})}(\vec{x}, t) = f_i^{L(\mathrm{eq})}(\vec{x}, t) + f_i^{Q(\mathrm{eq})}(\vec{x}, t) + f_i^{C(\mathrm{eq})}(\vec{x}, t). \tag{18}$$

In order to satisfy the basic physical conservative properties, for two-dimensional case the original 9-velocity discrete space (Qian et al., 1992) is enriched as a 17-velocity discrete space (D2Q17) (Qian and Zhou, 1998). By the D2Q17 model, the moment equation can be recovered as follows

$$\begin{aligned}\partial_t (\rho u_\alpha) + \partial_\beta (\rho u_\alpha u_\beta) = &-\partial_\alpha P + \partial_\beta (\rho \nu \partial_\beta u_\alpha + \rho \varsigma \partial_\alpha u_\beta) \\ &+ \xi \partial_\beta \partial_\gamma (\rho u_\alpha u_\beta u_\gamma),\end{aligned} \tag{19}$$

where ξ is defined by

$$\xi = (\theta - 1)\sigma. \tag{20}$$

In order to eliminate the third-order term in Equation (19), ξ must be equal to 0. In such a way, the cubic nonlinear term is *exactly* cancelled out without introducing any other high order nonlinearities. The details about the values of weight and sound speed can be found in the work of Qian and Zhou (1998).

3 Basic theory and results of 2D free-decay turbulence

It is known that the 2D turbulence encompasses a rich variety of phenomena (Tabeling, 2004). Recently, some researchers (Choung and David, 2006; Dmitruk and Montgomery, 2005; Bartello and Warn, 2005;

McWilliams, 1984; Legras et al., 2001; Mariotti et al., 2001) show that following original results about 2D turbulence are questionable:

- the value of energy spectrum exponents
- the centre assumption of Batchelor's 2D
- turbulence theory
- the values of various scaling exponents.

This situation stimulates us to reconsider the 2D turbulence based on numerical evidences. The results of such researches will redirect us to understand the physical properties and nonlinear interactions in 2D turbulence.

3.1 Batchelor's classical theory of 2D turbulence

In 2D freely-decaying turbulence, the k^{-3}-like (k denotes 'wavenumber') energy spectrum is found according to the Kolmogorov ideas. Batchelor (1969) extended the Kolmogorov's approach to the free decay problem and predicted an energy spectrum in the form

$$E(k) \sim t^{-2}k^{-3}. \tag{21}$$

It is well-known that there does not exist vortex stretching in 2D turbulence. So, the enstrophy $\langle \omega^2 \rangle$ decays as

$$\frac{1}{2}\frac{\mathrm{d}\langle \omega^2 \rangle}{\mathrm{d}t} = -\nu \langle |\nabla \omega|^2 \rangle. \tag{22}$$

It is clear that the enstrophy is bounded by its initial values. For freely-decaying 2D turbulence, the kinetic energy is governed by

$$\frac{1}{2}\frac{\mathrm{d}\langle u^2 \rangle}{\mathrm{d}t} = -\nu \langle \omega^2 \rangle. \tag{23}$$

From Equations (22) and (23), it is seen that the kinetic energy is approximately conservative as $\nu \to 0$ (Batchelor, 1969),

$$\frac{1}{2}\langle u^2 \rangle = \varphi = \text{constant}. \tag{24}$$

Batchelor predicted that the energy spectrum should have the following form

$$E(k,t) = \varphi^{3/2} t g(\varphi^{1/2} k t) \tag{25}$$

where g is a function with a universal form. According to the above similarity relation, the energy of the turbulence is moving into eddies of large length scales, with the representative scale increasing in proportion to t (Batchelor, 1969). Meanwhile, the energy transferred to smaller wavenumbers indicates that the $1/2\langle \omega^2 \rangle$ term decreases with time and the time dependent relation is given by

$$\frac{1}{2}\langle \omega^2 \rangle = \int_0^\infty k^2 E(k)\,\mathrm{d}k = At^{-2}, \tag{26}$$

where

$$A = \int_0^\infty z^2 g(z)\,\mathrm{d}z. \tag{27}$$

According to Batchelor's theory, the similarity relation (Eq. 25) could not be extended to the large wave number range where the viscosity effects are significant. But, Equation (25) still holds in the inertial sub-range because there is only one parameter χ which is the non-zero term of enstrophy dissipation and determined by

$$\chi = -\frac{1}{2}\frac{\mathrm{d}\langle \omega^2 \rangle}{\mathrm{d}t} = 2At^{-3}. \tag{28}$$

So, in the inertial sub-range the vorticity spectrum has the following form

$$\Omega(k,t) \sim (2A)^{2/3} t^{-2} k^{-1} \sim \chi^{2/3} k^{-1}. \tag{29}$$

3.2 Issues of 2D turbulence

The theory of Batchelor (1969) has been examined by Davidson (2004) and some high-resolution numerical results supporting the k^{-1} form of the inertial-range enstrophy spectrum (Dmitruk and Montgomery, 2005). The fundamental support of Batchelor's theory is a basic assumption that there exists a finite, non-zero enstrophy dissipation χ in the limit of $\mathrm{Re} \to \infty$. This assumption implies that the interaction of eddies is local. However, recent mathematical theory (Choung and David, 2006; David et al., 2007) has proved that this assumption is questionable. The enstrophy dissipation satisfies the following inequality

$$\chi = \nu \langle |\nabla \omega|^2 \rangle \le \frac{\|\omega\|_\infty \langle \omega^2 \rangle}{(\ln(\mathrm{Re}))^{1/2}}, \tag{30}$$

where $\|\omega\|_\infty$ is the supremum of the enstrophy. From inequality (30), it is clear that the enstrophy dissipation is dependent on Re. This result contradicts with the 2D turbulence theory proposed by Batchelor (1969).

In 2D turbulence, the mean-strain associated with eddies lying the wave number range $k_0 \le k \le k_\nu$ is related to their enstrophy as follows (David et al., 2007)

$$\frac{1}{2}\langle \omega^2 \rangle = \int_{k_0}^{k_\nu} k^2 E(k)\,\mathrm{d}k \sim \chi^{2/3} \ln\left(\frac{k_\nu}{k_0}\right), \tag{31}$$

where k_0 and k_ν are defined by

$$k_0 = \frac{\int kE(k)\,\mathrm{d}k}{\int E(k)\,\mathrm{d}k} = \frac{\langle \omega^2 \rangle^{1/2}}{\langle u^2 \rangle^{1/2}}, \tag{32}$$

and

$$k_\nu = k_0 \,\mathrm{Re}^{1/2}, \quad \mathrm{Re} = \frac{\langle \omega^2 \rangle^{1/2}}{\nu k_0^2}. \tag{33}$$

The terms of $1/2\langle\omega^2\rangle$ in Equation (31) and $\langle u^2\rangle^{1/2}$ in Equation (32) are assumed as the inviscid invariants of enstrophy and energy, respectively.

From Equation (31), it can be seen that eddy interactions are not local (Davidson, 2004). If it is assumed that the interactions are localised in k-space, a cascade of k^{-3} can be obtained by dimensional analysis. However, we then get into a contradiction. The non-local interactions shown in Equation (31) will undermine the dimensional argument of k^{-3} law.

Furthermore, from Equation (31), it is seen that (Choung and David, 2006)

$$\langle\omega^2\rangle \sim \chi^{2/3}\ln(\mathrm{Re}). \quad (34)$$

If the k^{-1} law of the enstrophy implied in the Batchelor's model is admitted,

$$\frac{1}{2}\langle\omega^2\rangle = \int_{k_0}^{k_\nu} Ck^{-1}\mathrm{d}k = C\ln\left(\frac{k_\nu}{k_0}\right). \quad (35)$$

From the work of Choung and David (2006), we can immediately get

$$\Omega(k) = \frac{\langle\omega^2\rangle}{\ln(\mathrm{Re})}k^{-1}. \quad (36)$$

From Equation (36), the spectrum of $\Omega(k)$ is dependent on Re and is scaled by $\ln(\mathrm{Re})$, which is, obviously, in conflict with Equation (29).

Other important issues are about the decay exponent of the enstrophy and the increase exponent of the integral length scale. From Equation (26), it is known that the enstrophy decaying exponent is –2 with respect to t. The recent results of Victor (2004) suggests an enstrophy decaying exponent of –2/3 with respect to t and the numerical investigates of Chasnov (1997), Clercx and Nielsen (2000) and Ossia and Lesieur (2001) show that the exponent of t are –0.8, –1.0 and –1.1. These results for the decay exponent of the enstrophy are all differ from that of Batchelor. In addition, Batchelor (1969) presented a linear growth in the integral length scale $l \sim t$. But the $l \sim t^{1/2}$ scaling is observed in some simulations and dimension analysis (Davidson, 2004; Lowe and Davidson, 2005).

The above two issues will be re-examined in our numerical simulation.

4 Numerical simulations

Our numerical simulations are implemented by the enriched D2Q17 LBM model (Qian and Zhou, 1998). The initial velocity fields are established by random fields in a spectral space. The initial energy spectrum is given by Dmitruk and Montgomery (2005)

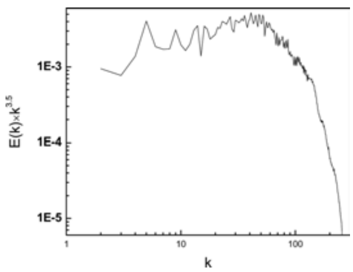
$$E(k) = ak^8\exp(-8\pi(k/8)^2), \quad (37)$$

which peaks at the low wavenumber $k = \sqrt{32/\pi}$. This spectrum will not lead to the significant inverse energy cascade for a short duration simulation. The initial RMS velocity magnitude is 0.005 (lattice unit), which leads a very small Mach number. The viscosity is set so that the viscous wavenumber occurs at 3/4 of the maximum wavenumber $n/2$. This choice could ensure adequate dissipation at high wavenumber to resolve the statistics of $\langle|\Delta\omega|^2\rangle$ and other fine scale quantities (David et al., 2007). The Reynolds numbers for three different mesh systems of 1024^2, 2048^2 and 4096^2 are 1.243×10^3, 6.377×10^3 and 2.304×10^4, respectively.

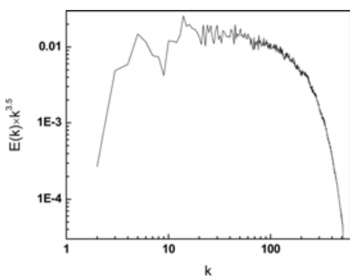
4.1 Energy spectrum

As indicated above the Batchelor's theory (Batchelor, 1969) of 2D turbulence predicted that the kinetic spectrum had a k^{-3} scaling law. The previous investigations of most numerical results supported the scaling law of kinetic energy (Davidson, 2004). However, there exist other controversial results (Davidson, 2004) which show that the energy spectrum scaling laws do not obey k^{-3}, but exhibit some faster decaying exponents compared with k^{-3}. In Figure 1, the energy compensated spectra are presented for three given Reynolds numbers at the fifth eddy turnover time. According to the current computations, it is found that the scaling exponent n of kinetic energy spectra $E(k) \sim k^{-n}$ is about 3.5 at the initial transition period. For the full development turbulent of $\mathrm{Re} = 2.304 \times 10^4$, the scaling exponent n is close to three in Figure 2. Interestingly, it seems that the exponent of $E(k) \sim k^{-n}$ relationship depends on the initial condition. It is implied that the assumption, 2D turbulence remembers only $\langle u^2\rangle$, is questionable as adopted in the Batchelor's model (Batchelor, 1969). This indicates that the 2D turbulence is more complex than understood by the conventional theories.

Figure 1 Energy spectra for three different Re at the fifth eddy turnover time: (a) the energy compensated spectrum for $\mathrm{Re} = 1.243 \times 10^3$; (b) the energy compensated spectrum for $\mathrm{Re} = 6.377 \times 10^3$ and (c) the energy compensated spectrum for $\mathrm{Re} = 2.304 \times 10^4$ (continues on next page)

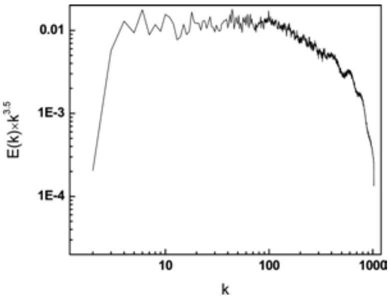


(a)



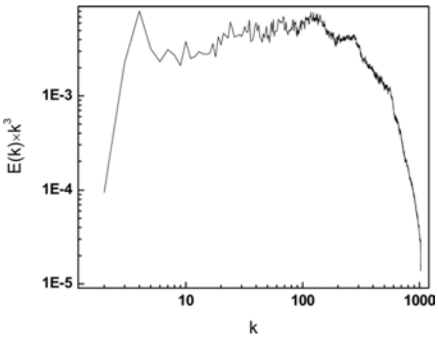
(b)

Figure 1 Energy spectra for three different Re at the fifth eddy turnover time: (a) the energy compensated spectrum for Re = 1.243×10^3; (b) the energy compensated spectrum for Re = 6.377×10^3 and (c) the energy compensated spectrum for Re = 2.304×10^4 (continued)



(c)

Figure 2 Energy spectrum for Re = 2.304×10^4 for full development turbulence



4.2 Integral length scale and vorticity correlations

In Figures 3 and 4, the relations of normalised length scale and eddy turnover time are given. Davidson (2004) and Lowe and Davidson (2005) observed that the growth of the normalised length scale can be well represented by $l \sim T^{1/2}$ (T is normalised by the eddy turn over time). From Figure 3, we note that the normalised length scale is dependent on the initial fields. However, the $l \sim T^{1/2}$ scale is asymptotically approached very well when the turbulence is mature.

Figure 3 Normalised integral scale l/l_0 against eddy turnover time

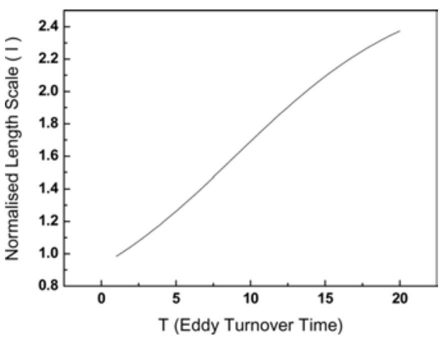
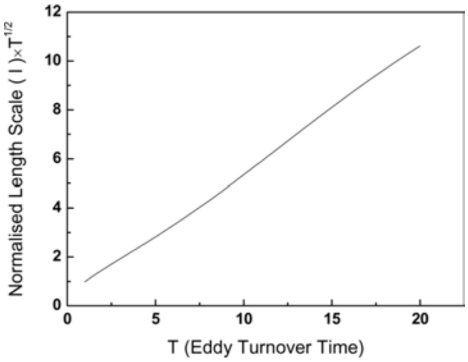


Figure 4 Normalised integral scale l/l_0 multiplied by $T^{1/2}$ against eddy turnover time



Now, we give the ensemble averaged normalised vorticity correlation $\langle \omega \cdot \omega' \rangle$ in Figures 5–7. The arrow in the figures represents the growth in the length scale of the turbulence. The form of the initial conditions determines the nature of the vorticity correlation during the initial evolution of the turbulence (Lowe and Davidson, 2005). The correlation oscillates about *r*-axis when the energy is closely peaked around a narrow range of wave number. From Figures 5–7, the initial correlation values oscillate around 0 over $15 \rightarrow 20$ initial integral length scales. Meanwhile, it is found that the similar form of the correlation is observed after about 20 eddy turnover times.

Figure 5 The ensemble averaged vorticity correlation $\langle \omega \cdot \omega' \rangle / \langle \omega^2 \rangle$ for Re = 1.243×10^3

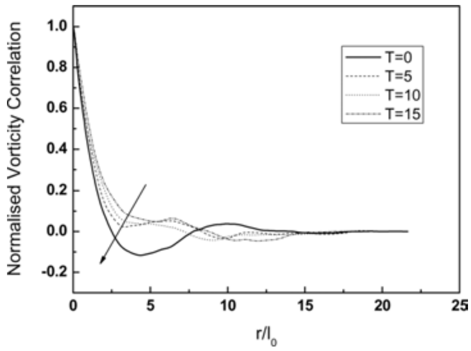


Figure 6 The ensemble averaged vorticity correlation $\langle \omega \cdot \omega' \rangle / \langle \omega^2 \rangle$ for Re = 6.377×10^3

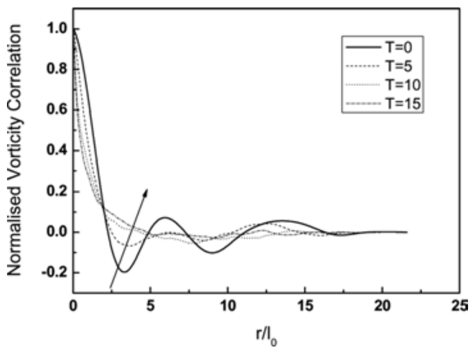
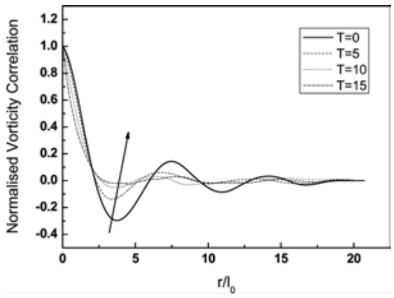


Figure 7 The ensemble averaged vorticity correlation $\langle \omega \cdot \omega' \rangle / \langle \omega^2 \rangle$ for Re = 2.304 × 10^4



4.3 Evolutions of turbulence basic quantities

In Figures 8–10, the variations of normalised vorticity peak, kinetic energy and enstrophy with the eddy turnover time are given. From the figures, the two conserved quantities of kinetic energy and vorticity peak can be observed. It is known that at modest values of Re, the peak vorticity $\overline{\omega}$ is not conserved (Lowe and Davidson, 2005). However, at high Re, $\overline{\omega}$ tends to be located at the centre of coherent vortices, and declines due to diffusion only (McWilliams, 1984). At high Re, the diffusion has a small effect and so $\overline{\omega}$ is expected to be conserved (Legras et al., 2001; Mariotti et al., 2001). Our numerical investigations of the basic quantities for three different high Re endorse these results.

Figure 8 Evolution of normalised basic quantities: vorticity peak $\overline{\omega}$, kinetic energy $\langle u^2 \rangle$ and enstrophy $\langle \omega^2 \rangle$ for Re = 1.243 × 10^3

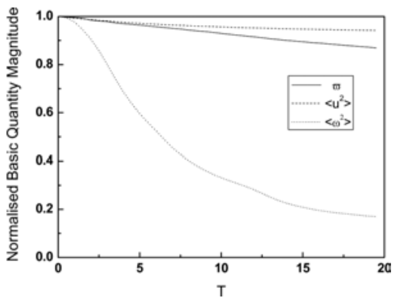


Figure 9 Evolution of normalised basic quantities: vorticity peak $\overline{\omega}$, kinetic energy $\langle u^2 \rangle$ and enstrophy $\langle \omega^2 \rangle$ for Re = 6.377 × 10^3

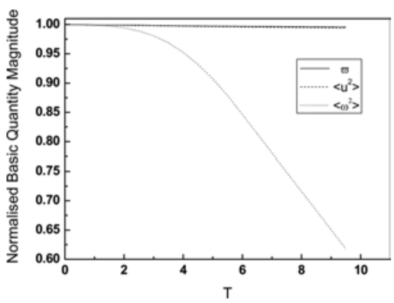
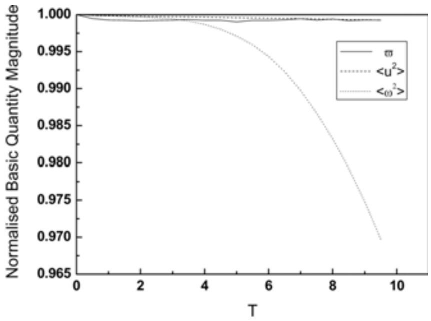


Figure 10 Evolution of normalised basic quantities: vorticity peak $\overline{\omega}$, kinetic energy $\langle u^2 \rangle$ and enstrophy $\langle \omega^2 \rangle$ for Re = 2.304 × 10^4



Another important quantity is the decay exponent of the enstrophy $\langle \omega^2 \rangle$. It is shown by Victor (2004) that in the limit Re → ∞ the total enstrophy in the system obeys a universal asymptotic relation $\langle \omega^2 \rangle \sim T^{-2/3}$. For Re = 1.243 × 10^3, a long time computation was implemented and a decay relation is obtained in Figure 11. It is observed that there exists an approximate relation of $\langle \omega^2 \rangle \sim T^{-2/3}$. In Figures 12–14 the snapshots of coherent vortices are shown. It is clear that the coherent vortices wind up the surrounding vorticity filaments. One possible explanation for $T^{1/2}$ growth of the integral scale in 2D turbulence involves the interaction of the coherent vorticity with the surrounding vorticity filaments (Davidson, 2004; Lowe and Davidson, 2005).

Figure 11 Evolution of the enstrophy for Re = 1.243 × 10^3

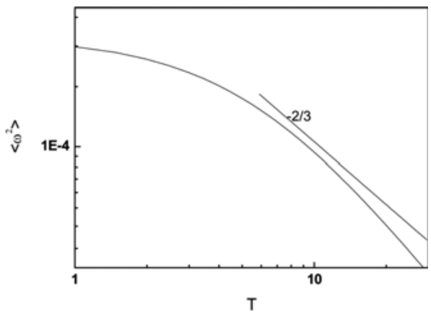


Figure 12 Vorticity snapshot for Re = 1.243 × 10^3 at $T = 5$ (see online version for colours)

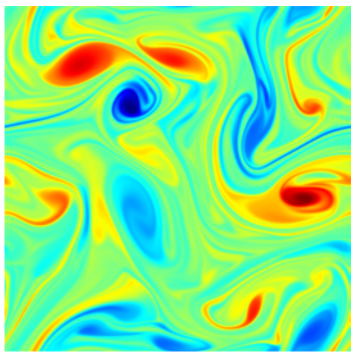


Figure 13 Vorticity snapshot for Re = 6.377×10^3 at $T = 5$ (see online version for colours)

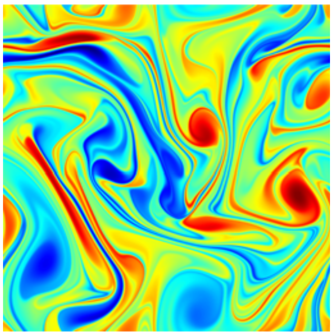
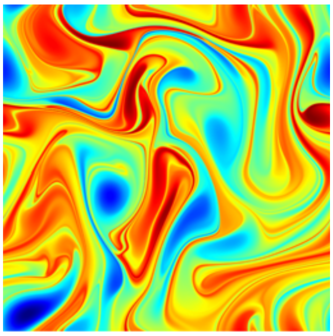


Figure 14 Vorticity snapshot for Re = 2.304×10^4 at $T = 5$ (see online version for colours)



5 Conclusion

In this paper, we revisit the freely-decaying 2D turbulence by LBM. The numerical investigations found an approximate exponent of energy spectrum $E(k) \sim k^{-3.5}$ at the high mesh resolutions and an approximate decaying exponent of enstrophy $\langle \omega^2 \rangle \sim T^{-2/3}$. Also, an approximate exponent of integral length scale $l \sim T^{1/2}$ is observed. These results validate the recent new results of 2D turbulence and are different from the results derived by Batchelor's 2D model. The numerical results demonstrate that LBM can be used to implement fundamental and theoretical researches of turbulence.

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References

Bartello, P. and Warn, T. (2005) 'Self-similarity of decaying two-dimensional turbulence', *J. Fluid Mech.*, Vol. 326, pp.357–372.

Batchelor, G.K. (1969) 'Computational of the energy spectrum in homogeneous two-dimensional turbulence', *Phys. Fluids*, Vol. 12, pp.233–239.

Chasnov, J.R. (1997) 'On the decay of two-dimensional homogeneous turbulence', *Phys. Fluids*, Vol. 9, No. 1, pp.171–180.

Chen, H., Kandasamy, S., Orszag, S.A., Shock, R., Succi, S. and Yakhot, V. (2003) 'Extended Boltzmann kinetic equation for turbulent flows', *Science*, Vol. 301, pp.633–636.

Chen, H., Orszag, S.A., Straroselsky, I. and Succi, S. (2004) 'Expanded analogy between Boltzmann kinetic theory of fluids and turbulence', *J. Fluid Mech.*, Vol. 519, pp.301–314.

Choung, V.T. and David, G.D. (2006) 'Vanishing enstrophy dissipation in two-dimensional Navier-Stokes turbulence in the inviscid limit', *J. Fluid Mech.*, Vol. 559, pp.107–116.

Clercx, H.J.H. and Nielsen, A.N. (2000) 'Vortex statistics for turbulence in a container with rigid boundaries', *Phys. Rev. Lett.*, Vol. 85, No. 4, pp.752–755.

David, G.D., Chuong, V.T. and Richard, K.S. (2007) 'Revisiting Batchelor's theory of two-dimensional turbulence', *J. Fluid Mech.,* Vol. 591, pp.379–391.

Davidson, P.A. (2004) *Turbulence: An Introduction for Scientists and Engineers*, Oxford University Press Inc., New York.

Dmitruk, P. and Montgomery, D.C. (2005) 'Numerical study of the decay of enstrophy in two-dimensional Navier-Stokes fluid in the limit of very small viscosities', *Phys. Fluids*, Vol. 17, p.035114-1-5.

Legras, B., Dritschel, D.G. and Caillol, P. (2001) 'The erosion of a distributed two-dimensional vortex in a background straining flow', *J. Fluid Mech.*, Vol. 441, pp.369–398.

Lowe, A.J. and Davidson, P.A. (2005) 'The evolution of freely-decaying, isotropic, two-dimensional turbulence', *European Journal of Mechanics B/Fluids*, Vol. 24, pp.314–327.

Mariotti, A., Legras, A. and Dritschel, D.G. (2001) 'Vortex stripping and the erosion of coherent structures in two-dimensional McWilliams flows', *Phys. Fluids*, Vol. 6, No. 12, pp.3954–3962.

McWilliams, J.C. (1984) 'The emergence of isolated coherent vortices in turbulent flow', *J. Fluid Mech.*, Vol. 146, pp.21–43.

Orszag, S.A., Chen, H., Succi, S. and Latt, J. (2006) 'Turbulence effects on kinetic equations', *J. Sci. Comput.*, Vol. 28, pp.459–466.

Ossia, S. and Lesieur, M. (2001) 'Large-scale energy and pressure dynamics in decaying two-dimensional incompressible isotropic turbulence', *J. Turbulence*, Vol. 2, No. 013, pp.1–34.

Qian, Y.H. and Zhou, Y. (1998) 'Complete galilean-invariant lattice BGK models for the navier-stokes equation', *Europhys. Lett.*, Vol. 42, pp.359–364.

Qian, Y.H., D'Humieres, D. and Lallemand, P. (1992) 'Lattice BGK models for Navier-stokes equation', *Europhys. Lett.*, Vol. 17, pp.479–484.

Succi, S. (2001) *The Lattice Boltzmann Equation for Fluid Dynamics and Beyond*, Oxford Uniersity Press, Oxford, UK.

Tabeling, P. (2004) 'Two-dimensional turbulence: a physicist approach', *Physics Reports*, Vol. 362, pp.1–62.

Victor, Y. (2004) 'Universal law of enstrophy decay in two-dimensional large-Reynolds-Number turbulence', *Phys. Rev. Lett.*, Vol. 93, No. 1, p.014502-1-4.