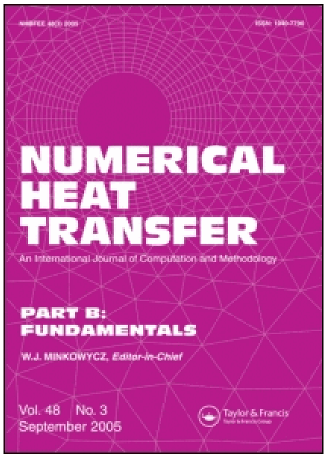

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Design of High-Order Difference Scheme and Analysis of Solution Characteristics—Part I: General Formulation of High-Order Difference Schemes and Analysis of Convective Stability

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DESIGN OF HIGH-ORDER DIFFERENCE SCHEME AND ANALYSIS OF SOLUTION CHARACTERISTICS—PART I: GENERAL FORMULATION OF HIGH-ORDER DIFFERENCE SCHEMES AND ANALYSIS OF CONVECTIVE STABILITY

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In this article, a general design method of second-order difference schemes is presented. By this method, we can easily design any second-order or higher-order difference scheme instead of using complex Lagrange interpolation methods or spline functions. Moreover, it is proved that all existing second-order difference schemes in numerical heat transfer fit this general design style. In addition, based on this general style of second-order scheme, the general style of a second-order absolutely stable scheme is deduced, and the stability definitions are shown in a normalized variable diagram. Finally, through studying the solution characteristics of 14 second-order difference schemes, it is found that, to second order precise, absolutely stable schemes obtained from the general method can achieve good convergence even when the grid Pelect number reaches 100,000. However, at the same time, the false diffusion of the scheme tends to increase along with the increasing value of a_i (coefficient in the interface variable definition).

1. INTRODUCTION

In the numerical solution of convective-diffusive equations, the discretization of the convective terms is one of the most challenging and interesting tasks, since the discretization schemes for the convective terms in the Navier-Stokes equations and scalar transport equations are connected directly to the solution accuracy, efficiency, and convergency. A large number of studies have been conducted in connection with the discretization schemes for the convection terms. Here we focus our attention mainly on the numerical simulation of conventional fluid flow and heat transfer problems. By "convectional" we mean those problems in which no sharp gradient (such as sharp gradients of density in a shock) exists in the computational domain. Most incompressible fluid flow and heat transfer problems in engineering are of this category. Previous studies on the convective term discretization focused

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NOMENCLATURE

a_{i0}	critical value of "narrow sense" absolutely stable scheme
$a_p,\ a_{N,E,W,S}$	coefficient in discretization equation
D	diffusive conductance
E	time-step multiple
Er	channel expansion ratio
F	flow rate at the interface
$P_{\Delta x},\ P_{\Delta y}$	grid Peclet number in the x and y directions
R	residual of discretization equation
Re	Reynolds number
S	nominal source term
$u,\ v$	velocity components in the x and y directions
$U,\ V$	nondimensional Velocity components in the x and y directions
U_0	sliding velocity of the lid
x	coordinate
α	underrelaxation factor
β	parameter indicating the percentage of central difference
Γ	nominal diffusion coefficient
$\delta x,\ \delta y$	distance between two neighboring grid points in the x and y directions
$\Delta x,\ \Delta y$	distance between two neighboring interface in the x and y directions
ν	fluid kinetic viscosity
ϕ	general dependent variable
Subscripts	
e, n, s, w	interfaces
E, N, S, W	near neighboring grid points
EE, NN	far neighboring grid points
$SS,\ WW$	far neighboring grid points

mainly on two respects. One is the comparison analysis of different schemes with respect to false diffusion, stability, and accuracy. The comparison studies [1–6] indicated that the requirements for numerical stability and computational accuracy are often contradictory for most existing schemes used in computational fluid dynamics/numerical heat transfer (CFD/NHT). Lower-order schemes such as first-order upwind are stable but often lead to severe false diffusion. Second-order schemes, such as central difference (CD) and QUICK, eliminate false diffusion but may produce wiggles and often fail to converge [7]. With respect to the central difference schemes, it is well known that the CD is prone to oscillation when a grid Peclet (or Reynolds) number is beyond a certain value. Even though a recent study [9] has shown that the one-dimensional stability analysis methods (such as in [8]) give only the most severe critical Pelect number, for practical multidimensional cases CD may work well even if the local grid Pelect number is as large as 180 [9]; the presence of the Pelect number limit beyond which oscillation will occur is still an undesirable feature of the scheme. Therefore, a significant amount of research effort has been directed toward the convective discretization schemes, and many remedies have been proposed [5, 10–12]. For example, in [5] it is shown that, compared with CD, the second-order upwind difference scheme (SUD) performs better, and its implementation is recommended. From stability considerations, the SUD is perfect since it is absolutely stable; however, numerical studies have shown that it is somewhat diffusive [13].

The other aspect is the design of a new scheme in order to overcome the shortcomings of existing schemes. For example, some researchers have discussed the essence of discretization of the convective term. Inspection of the discretization process of the convective term, written in a nonconservative form $(u\partial\phi/\partial x)_i$, shows that we actually encounter two kinds of quantities, i.e., convective flow (represented by velocity u) and first-order derivative. The convective flow is totally directional in that only the information upstream can be transferred downstream, and not vice versa. In this regard it is better to adopt directional discretization schemes such as first-order upwind or second-order upstream schemes. However, from the first-order derivative itself, approximations with equal points at the two sides of position i are better than bias approximations in that the information at the two sides may be equally taken into account, especially when diffusion is the dominant mechanism. Thus it is expected that an approximation combining symmetry and nonsymmetry structures of difference formulations with their percentage being adjusted automatically will give better performance. This is the basic consideration in constructing a new discretization scheme for the convection term. Starting from this point, some schemes, which are combinations of several existing schemes, have been designed. These include second-order hybrid (SHYBRID) [12] and stability-controllable second-order difference (SCSD) schemes [14]. In [12], CD, QUICK, and SUD are organized in a general form from SHYBRID. In [14], CD and SUD are combined to generate the SCSD scheme. Based on the SCSD scheme, a new scheme whose stability is guaranteed with at least second-order accuracy is formulated. It is named stability-guaranteed second-order difference (SGSD) [15]. These previous research results have greatly advanced the development of difference scheme construction, but, as concluded in [7], the matter is far from being solved, and the need for further study of the formulation of the discretized convection and diffusion scheme still remains. Moreover, it can be easily found that the previous researches improved the schemes from different points of view and lack some general fundamental considerations. It is the present authors' consideration that all kinds of schemes designed through different ways should be subjected to the same general formulation in mathematics, and possess some general characteristics for the resulting algebraic equations during the solution process. In this article, we propose a general design formulation for different discretized schemes and try to analyze the solution characteristics of all kinds of schemes in order to find some general features. In addition, based on this general formulation of discretized schemes, we analyze the stability character of the scheme in detail.

In the following, the details of construction of the general formulation for different schemes, and a performance comparison for the simulation of lid-driven cavity flow and flow over a back-facing step will be presented. In the numerical simulation presented here, the diffusion term is always discretized by the second-order CD scheme, hence the difference behavior in numerical solutions arises mainly from the different discretizations of the convection term.

It may be interesting to note that even though Godnov et al. had shown that for the pure convection equation only the first-order upwind scheme is bounded and absolutely stable with constant coefficient [16], the major concern shown in this article is on convective-diffusive problems. And the stability character of the discretized schemes is actually the combined results from pure convection and diffusion, which differs greatly from that of the pure convection case. Since all the trouble in the discretization of the convection-diffusion equation is caused by the convection term [6, 7], and the central difference will always be used for the diffusion term, for simplicity of presentation, terms such as convective stability will be used instead of convection-diffusion stability.

2. GENERAL DESIGN FORMULATION OF HIGH-ORDER DIFFERENCE SCHEME FOR CONVECTIVE TERM

The general differential equations for a steady-state convection-diffusion problem of a general variable ϕ can be written in a conservative form as

$$\frac{\partial(\rho u_j \phi)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\Gamma_\phi \frac{\partial \phi}{\partial x_j}\right) + S_\phi \tag{1}$$

where ρ is the density of the fluid, u_j is the jth component of the velocity, Γ_ϕ is the diffusion coefficient, and S_ϕ is the source term for the variable ϕ.

For discretization of the above equation, the finite-volume method is adopted. For simplicity of presentation, the multidimensional problem is represented by a 2-D case. A typical 2-D control volume is shown in Figure 1. For two-dimensional problems, after integration over the control volume, the discretized equation can be obtained as follows:

$$\begin{aligned} F_e\phi_e - F_w\phi_w + F_n\phi_n - F_s\phi_s = S_\phi\,\Delta x\,\Delta y + D_e(\phi_E - \phi_P) - D_w(\phi_W - \phi_P) \\ + D_n(\phi_N - \phi_P) - D_s(\phi_S - \phi_P) \end{aligned} \tag{2}$$

The variables $\phi_{n,\,e,\,w,\,s}$ at control-volume interfaces can be determined by means of an interpolation or extrapolation involving the values of the neighboring

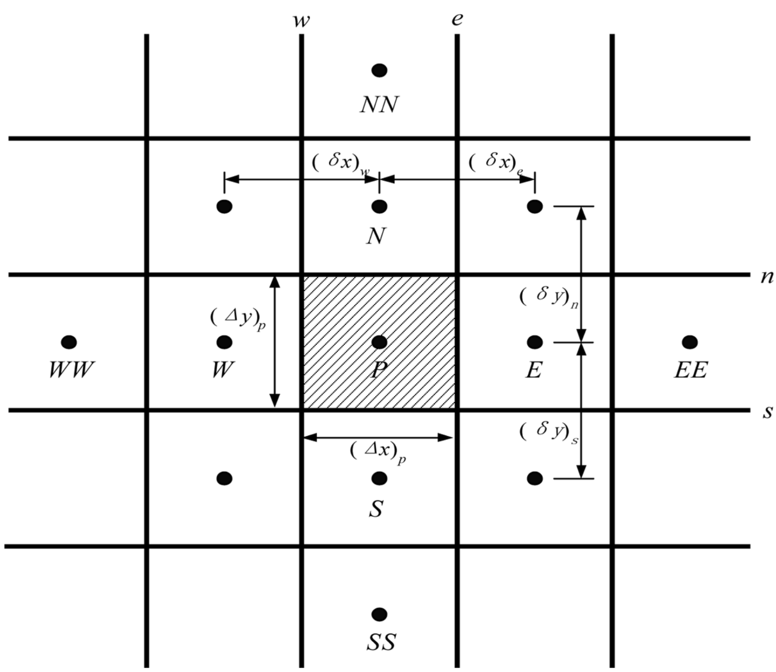


Figure 1. Schematic of the control volume.

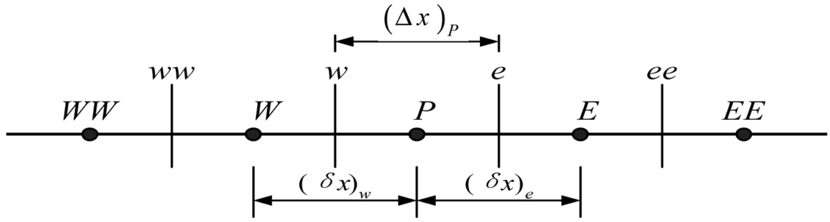


Figure 2. 1-D uniform grid system.

grid points. In the finite-volume method, it is the choice of interpolation or extrapolation method for the interfacial value that determines the scheme.

2.1. Derivation of General Formulation of Second-Order and Higher-Order Difference Schemes

In order to complete the discretization of terms shown in Eq. (2), the interface values of ϕ should be interpolated by the proposed scheme. Here interpolation is discussed for a uniform grid system. Figure 2 shows a 1-D uniform grid system. For example, when $u > 0$, the values of variable ϕ on the east, e, and west, w, interfaces are interpolated as

$$\begin{cases} \phi_e = a_{i-1}\phi_{i-1} + a_i\phi_i + a_{i+1}\phi_{i+1} \\ \phi_w = a_{i-1}\phi_{i-2} + a_i\phi_{i-1} + a_{i+1}\phi_i \end{cases} \tag{3}$$

Schemes defined by the above formulation can automatically fit for the scheme conservativeness condition [6], that is, at any interface, the expressions for the interface values written from its two side grids can fit the following condition (Figure 3):

$$(\phi_i)_e = (\phi_{i+1})_w \tag{4a}$$

And since we take the central difference scheme for the conduction term, we have

$$\left(\frac{\partial \phi_i}{\partial x}\right)_e = \left(\frac{\partial \phi_{i+1}}{\partial x}\right)_w \tag{4b}$$

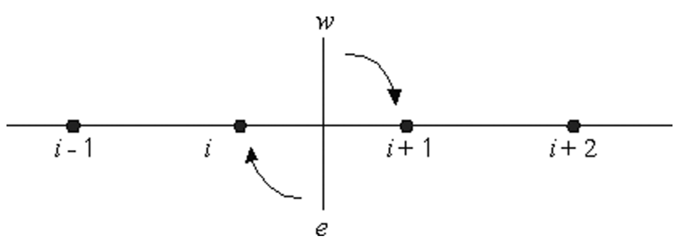


Figure 3. Diagram showing interface continuity of variables.

where the i, $i+1$ grids and e, w interface positions are shown in Figure 3. As demonstrated in [6], once Eqs. (4*a*) and (4*b*) are valid, the proposed scheme possesses the conservation character.

In order to analyze the order of accuracy of the discretized convection term, we adopt the Taylor series expansion method, as this has been done by Leonard in [17]. Then the scheme issue is related to how to discretize the first derivative in the convective term at grid point i, $(\partial\phi/\partial x)_i$. In the finite-volume method, after integration of the convective-diffusive equation on the 1-D grid system shown in Figure 2, the first derivative included in the convective term can be expressed as

$$\left.\frac{\partial\phi}{\partial x}\right|_i = \frac{\phi_e - \phi_w}{\Delta x} \tag{5}$$

By substitution of Eq. (3) into the above expression, we get

$$\left.\frac{\partial\phi}{\partial x}\right|_i = \frac{\phi_e - \phi_w}{\Delta x} = \frac{(a_i - a_{i+1})\phi_i + (a_{i-1} - a_i)\phi_{i-1} + a_{i+1}\phi_{i+1} - a_{i-1}\phi_{i-2}}{\Delta x} \tag{6}$$

The terms in the right-hand side of Eq. (6) are then expressed by the Taylor series expansion at the point i, and the following expression is obtained:

$$\begin{aligned}\left.\frac{\partial\phi}{\partial x}\right|_i &= (a_i + a_{i-1} + a_{i+1})\left.\frac{\partial\phi}{\partial x}\right|_i + (-3a_{i-1} - a_i + a_{i+1})\left.\frac{\partial^2\phi}{\partial x^2}\right|_i \cdot \frac{\Delta x}{2!} \\ &\quad + (7a_{i-1} + a_i + a_{i+1})\left.\frac{\partial^3\phi}{\partial x^3}\right|_i \cdot \frac{\Delta x^2}{3!} \ldots\end{aligned} \tag{7}$$

By neglecting terms with derivatives higher than third order in Eq. (7), we can obtain the following results:

$$\begin{cases} a_{i-1} + a_i + a_{i+1} = 1 \\ -3a_{i-1} - a_i + a_{i+1} = 0 \\ 7a_{i-1} + a_i + a_{i+1} = 0 \end{cases} \longrightarrow \begin{cases} a_i = \frac{5}{6} \\ a_{i-1} = -\frac{1}{6} \\ a_{i+1} = \frac{1}{3} \end{cases} \tag{8}$$

Then, if we substitute the above values of a_{i-1}, a_i, and a_{i+1} into Eq. (3), we can uniquely attain the third-order upwind scheme (TUD) expressed by

$$\begin{cases} \phi_e = -\frac{1}{6}\phi_{i-1} + \frac{5}{6}\phi_i + \frac{1}{3}\phi_{i+1} \\ \phi_w = -\frac{1}{6}\phi_{i-2} + \frac{5}{6}\phi_{i-1} + \frac{1}{3}\phi_i \end{cases}$$

If we just retain the term with the second-order derivative on the right-hand side of Eq. (7), we can obtain the a_{i-1}, a_i, and a_{i+1} values as follows:

$$\begin{cases} a_{i-1} + a_i + a_{i+1} = 1 \\ -3a_{i-1} - a_i + a_{i+1} = 0 \\ 7a_{i-1} + a_i + a_{i+1} \neq 0 \end{cases} \longrightarrow \begin{cases} a_i \neq \frac{5}{6} \\ a_{i-1} = \frac{1}{4} - \frac{a_i}{2} \\ a_{i+1} = \frac{3}{4} - \frac{a_i}{2} \end{cases}$$

Then we can easily find the general formulation of second-order difference schemes as follows:

$$\begin{cases} \phi_e = a_i\phi_i + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-1} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_{i+1} \\ \phi_w = a_i\phi_{i-1} + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-2} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_i \\ a_i \neq \frac{5}{6} \end{cases} \tag{9}$$

where a_i can be any value except $5/6$.

In order to further prove the second-order accuracy of the schemes defined by Eq. (9), we adopt the normalized variable method [18, 19].

Defining the normalized variable,

$$\tilde{\phi} = \frac{\phi - \phi_{i-1}}{\phi_{i+1} - \phi_{i-1}} \tag{10}$$

we can rewrite Eq. (9) via the normalized interface variable:

$$\widetilde{\phi_f} = \left(\frac{3}{4} - \frac{a_i}{2}\right) + a_i\widetilde{\phi_c} = a_i\left(\widetilde{\phi_c} - \frac{1}{2}\right) + \frac{3}{4} \tag{11}$$

Equation (11) shows that the general formulation of Eq. (9) proposed in this article goes through the grid point (0.5, 0.75) in the normalized variable diagram under any a_i value (except $5/6$), which has been proved to be both the sufficient and necessary conditions for a scheme to possesses second-order accuracy [19].

In fact, this derivation method can be extended to the design of any higher-order difference scheme by means of different grid points. For example, when $u > 0$, the values of variable ϕ on the east, e, and west, w, interfaces may be interpolated as

$$\begin{cases} \phi_e = \ldots a_{i-2}\phi_{i-2} + a_{i-1}\phi_{i-1} + a_i\phi_i + a_{i+1}\phi_{i+1} + a_{i+2}\phi_{i+2}\ldots \\ \phi_w = \ldots a_{i-2}\phi_{i-3} + a_{i-1}\phi_{i-2} + a_i\phi_{i-1} + a_{i+1}\phi_i + a_{i+2}\phi_{i+1}\ldots \end{cases} \tag{12}$$

The coefficients in Eq. (12) can also be deduced through the same process as presented above. A companion article will analyze higher-order schemes [20]. In this article our focus is concentrated to the second-order schemes only.

2.2. Derivation of Absolutely Stable Second-Order Difference Scheme

In order to analyze the stability of the above schemes, we analyze them in the explicit scheme of the one-dimensional unsteady convection-diffusion equation in the way similar to that presented in [8]:

$$\rho\frac{\partial\phi}{\partial t} + \rho u\frac{\partial\phi}{\partial x} = \Gamma\frac{\partial^2\phi}{\partial x^2} \tag{13}$$

By substitution of Eq. (9) for the convective term and the central expression for the diffusion term, the discretized form of Eq. (13) can be expressed as follows:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + u\frac{(6a_i - 3)\phi_i^n + (1 - 6a_i)\phi_{i-1}^n + (3 - 2a_i)\phi_{i+1}^n - (1 - 2a_i)\phi_{i-2}^n}{4\,\Delta x} = \Gamma\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\rho\,\Delta x^2} \tag{14}$$

To analyze the scheme stability, the discrete disturbance method is adopted [8]. It is assumed that the original field is uniform and, for simplicity of analysis and without losing generality, the field values are assumed to be everywhere zero. At the point i and time instant n there is a disturbance, designated by ε_i^n, and at any other points and at any subsequent time no disturbances act on the field. Then Eq. (14) is used to predict the effects of this disturbance on the values at point $i \pm 1$ and at instant $n + 1$. Since the diffusion term is discretized by the central difference, and its behavior for transport disturbance is well documented in the literature [6, 8], only the effect of the convective term is analyzed. By writing the above equation without the diffusion term for the points $(i + 1)$ and $(i - 1)$, we can obtain

$$\begin{aligned} \frac{\phi_{i+1}^{n+1} - \phi_{i+1}^n}{\Delta t} &= -u\frac{(6a_i - 3)\phi_{i+1}^n + (1 - 6a_i)\phi_i^n + (3 - 2a_i)\phi_{i+2}^n - (1 - 2a_i)\phi_{i-1}^n}{4\,\Delta x} \\ \frac{\phi_{i-1}^{n+1} - \phi_{i-1}^n}{\Delta t} &= -u\frac{(6a_i - 3)\phi_{i-1}^n + (1 - 6a_i)\phi_{i-2}^n + (3 - 2a_i)\phi_i^n - (1 - 2a_i)\phi_{i-3}^n}{4\,\Delta x} \end{aligned} \tag{14a}$$

Then the effects of the disturbance ε_i^n transported by the convective terms are

$$\begin{cases} \phi_{i+1}^{n+1} = \frac{(6a_i-1)}{4}\left(\frac{u\,\Delta t}{\Delta x}\right)\varepsilon_i^n \\ \phi_{i-1}^{n+1} = \frac{(2a_i-3)}{4}\left(\frac{u\,\Delta t}{\Delta x}\right)\varepsilon_i^n \end{cases} \tag{14b}$$

According to the sign preservation rule [8], it is required that for the scheme to be stable the following condition must be satisfied:

$$\begin{cases} \dfrac{\frac{(6a_i-1)}{4}\left(\frac{u\,\Delta t}{\Delta x}\right)\varepsilon_i^n + \left(\frac{\Gamma\,\Delta t}{\rho\,\Delta x^2}\right)\varepsilon_i^n}{\varepsilon_i^n} \geq 0 \\ \dfrac{\frac{(2a_i-3)}{4}\left(\frac{u\,\Delta t}{\Delta x}\right)\varepsilon_i^n + \left(\frac{\Gamma\,\Delta t}{\rho\,\Delta x^2}\right)\varepsilon_i^n}{\varepsilon_i^n} \geq 0 \end{cases} \tag{15}$$

where the second term in the numerator is the effect transported by the diffusion term. Taking the case for $u > 0$ into consideration, in order to simultaneously satisfy the above two equalities, the following condition must be met:

$$\begin{cases} 6a_i - 1 \geq 0 \\ 2a_i - 3 \geq 0 \end{cases} \rightarrow a_i \geq \frac{3}{2} \tag{16}$$

Thus, for the case of $u > 0$, we can obtain an absolutely stable scheme of second-order accuracy by the following interfacial interpolation formulation consisting of three grid points:

$$\begin{cases} \phi_e = a_i\phi_i + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-1} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_{i+1} \\ a_i \geq \frac{3}{2} \end{cases} \tag{17}$$

The same derivation can be conducted for the case of $u < 0$, and the corresponding expressions can be obtained. Once again, it is noted here that the condition of Eq. (17) for the absolutely stable scheme is the combined actions of convection and diffusion, rather than convection only.

The results of Eq. (17) can be expressed by the normalized variable, and the correspondent expression is

$$\begin{cases} \tilde{\phi}_e = \left(\frac{3}{4} - \frac{a_i}{2}\right) + a_i\tilde{\phi}_i \\ a_i \geq \frac{3}{2} \end{cases} \tag{18}$$

Equation (18) is presented in Figure 4, where the shadow region is its representation. According to the above analysis, it can be stated that if the defining line of a second-order scheme in the normalized variable diagram goes through the grid point Q (0.5, 0.75) and the line is located within the shadow region shown in Figure 4, the scheme must be absolutely stable with a second-order accuracy. In addition, when $a_i = 3/2$, the corresponding scheme is the second-order upwind scheme (SUD) which

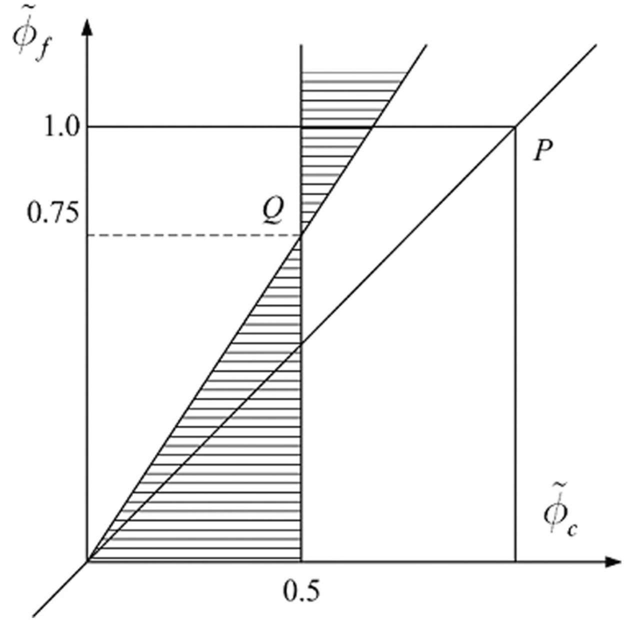


Figure 4. Region of absolutely stable schemes in NVD.

lies at the left boundary of this shadow region and goes through the coordinate origin.

It is well known that the defining line (or characteristic line) of the first-order upwind scheme (FUD) is the diagonal of the normalized variable diagram (NVD), which is absolutely stable, too. This implies that the condition for a scheme to be absolutely stable shown by the shadow region is only sufficient, but not necessary. However, the FUD is only of first-order accuracy. Thus, we can state that any characteristic line going through the grid point Q and located within the shadow region of Figure 4 represents a kind of scheme with absolute stability and second-order accuracy. The critical value of a_i for this absolute stability is designated by a_{i0}.

In addition, we can also give a general discussion of the scheme stability as follows. From Eq. (15), we can deduce the following results:

$$\begin{cases} P_\Delta \geq \frac{4}{1-6a_i}, \ P_{\Delta c} = \frac{4}{1-6a_i} \\ P_\Delta \geq \frac{4}{3-2a_i}, P_{\Delta c} = \frac{4}{3-2a_i} \end{cases} \tag{19}$$

where $P_{\Delta c}$ is the critical Peclet number beyond which the numerical solution of the 1-D convective-diffusive problem will have an oscillating solution.

The variation of the above critical $P_{\Delta c}$ number with the value of a_i is analyzed as follows.

1. When $a_i < -1/2$, Eq. (19) can be further transformed into

$$0 \leq P_\Delta \leq \frac{4}{1-6a_i}, \ P_{\Delta C} = \frac{4}{1-6a_i} > 0 \tag{19a}$$

   When the value of a_i tends to equal $-1/2$, the critical $P_{\Delta c}$ of the above expression can approach its maximal value of 1.
2. When $-1/2 \leq a_i < 3/2$, Eq. (19) can be further reduced into

$$0 \leq P_\Delta \leq \frac{4}{3-2a_i}, \ P_{\Delta C} = \frac{4}{3-2a_i} > 0 \tag{19b}$$

   With increase in a_i, the value of $P_{\Delta c}$ in Eq. (19b) increases from 1 to ∞.
3. When $a_i \geq 3/2$, Eq. (19) leads to the following result:

$$P_{\Delta c} \geq 0 \tag{19c}$$

   In this region, whatever the value of a_i, the corresponding scheme of second-order accuracy is absolutely stable, as indicated before.

From the above analysis we can conclude that, with increase in the coefficient a_i, the corresponding critical $P_{\Delta c}$ also increases up to infinity. Therefore, when designing a second-order scheme, in order to extend the stability range, a larger value of a_i is preferred.

2.3. Relationship of the General Formulation of Second-Order Difference Scheme to Existing Schemes

In the following, the relationship between the general formulation of the second-order scheme with the existing schemes is analyzed.

When $a_i = 1/2$, the general formulation scheme, Eq. (9), leads to the central difference scheme (CD):

$$\phi_e = \frac{1}{2}\phi_i + \frac{1}{2}\phi_{i+1} \tag{19d}$$

When $a_i = 3/4$, the QUICK scheme is obtained:

$$\phi_e = \frac{3}{4}\phi_i - \frac{1}{8}\phi_{i-1} + \frac{3}{8}\phi_{i+1} \tag{19e}$$

When $a_i = 5/6$, the TUD scheme is reached:

$$\phi_e = \frac{3}{4}\phi_i - \frac{1}{8}\phi_{i-1} + \frac{3}{8}\phi_{i+1} \tag{19f}$$

When $a_i = 1$, the Fromm scheme is obtained [21]:

$$\phi_e = \phi_i - \frac{1}{4}\phi_{i-1} + \frac{1}{4}\phi_{i+1} \tag{19d}$$

When $a_i = 3/2$, the absolutely stable SUD scheme results:

$$\phi_e = \frac{3}{2}\phi_i - \frac{1}{2}\phi_{i-1} \tag{19g}$$

It can be seen that all the existing schemes with second-order accuracy can be deduced from the general formulations with some specific value of a_i.

As the matter of fact, some combined schemes designed by the flux blending method can also be deduced from the present general formulation. Consider the SCSD scheme [14] as an example, which is defined by

$$\phi_e = \beta\phi_e^{\mathrm{CD}} + (1-\beta)\phi_e^{\mathrm{SUD}} \qquad (0 \le \beta \le 1) \tag{20a}$$

where the superscripts CD and SUD designate the central difference and the second-order upwind difference, and the weighting coefficient β is a prespecified parameter. By substituting Eqs. (19*d*) and (19*g*) into Eq. (20*a*), we get

$$\phi_e = \beta\phi_e^{\mathrm{CD}} + (1-\beta)\phi_e^{\mathrm{SUD}} = \left(\frac{3}{2} - \beta\right)\phi_i - \frac{(1-\beta)}{2}\phi_{i-1} + \frac{\beta}{2}\phi_{i+1} \tag{20b}$$

In the above expression, we let

$$\left(\frac{3}{2} - \beta\right) = a_i \tag{20c}$$

and we get

$$\beta = \left(\frac{3}{2} - a_i\right) \tag{20d}$$

By back substitution into Eq. (20*b*), we have

$$\phi_e = a_i\phi_i + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-1} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_{i+1} \tag{20e}$$

It can be easily found that Eq. (20*e*) is identical to Eq. (9). In the SCSD the range of β varies from 0 to 1; the corresponding variation range of a_i is from 0.5 to 1.5. Then the SCSD scheme can be stated as follows:

$$\begin{cases} \phi_e = a_i\phi_i + \left(\frac{1}{4} - \frac{a_i}{2}\right)\phi_{i-1} + \left(\frac{3}{4} - \frac{a_i}{2}\right)\phi_{i+1} \\ \frac{1}{2} \le a_i \le \frac{3}{2} \end{cases} \tag{20f}$$

Based on this scheme, a further development was made in [15], in which the so-called stability-guaranteed second-order difference scheme (SGSD) was proposed. In the SGSD an expression for β was proposed:

$$\beta = \frac{2}{(2 + |P_\Delta|)} \qquad 0 \le \beta \le 1 \tag{20g}$$

According to the relationship of Eq. (20*d*), the corresponding variation range of the coefficient a_i is

$$\frac{1}{2} \le a_i \le \frac{3}{2} \qquad \text{and} \qquad a_i = \frac{3}{2} - \frac{2}{(2 + |P_\Delta|)} \tag{20h}$$

It is worth noting that, superficially, Eq. (20*h*) is in contradiction to the absolute stability of the SGSD. Actually, the infinitely large critical Peclet number of the SGSD is obtained when it takes the form of the second-order upwind scheme corresponding to $a_i = 3/2$. Thus it is the value of $a_i = 3/2$ at which the conditional stability and the absolute stability merge.

It is worth noting here that the stability issue studied in this article is the convective stability, i.e., it is related to the discretization of the convective term and has nothing to do with the transient term. Thus, in our above analysis, only the steady governing equation, Eq. (1), is used. As analyzed in [6], in the discretization of the transient governing equation, another type of stability may be involved if the explicit scheme is used, which is related to the time step. Whether the explicit or the implicit scheme is used for the discretization transient governing equation, the convective stability issue is the same. And the results obtained in this article can be used for both the explicit and implicit schemes.

3. ANALYSIS OF SOLUTION CHARACTERISTICS OF SECOND-ORDER DIFFERENCE SCHEMES

3.1. NVD Divisions of Second-Order Difference Schemes with Variation of a_i

In is found that with different sign and values of a_i, the signs of a_{i-1} and a_{i+1} of Eq. (3) are different, leading to different solution characteristics of the coefficient matrix with different convective schemes. Accordingly, 14 schemes with different sign and value of a_i are designed and divided into four regions as shown in Table 1 and illustrated in Figure 5, with the normalized variable defined in Eq. (10). Moreover, their solution characteristics are analyzed through computation examples as follows.

3.2. Test Calculations for Two Benchmark Problems

In this section, the numerical solutions of two benchmark problems using the 14 convective schemes are presented and compared. First, numerical calculations are performed for lid-driven cavity flow, for which Ghia et al. [22] provided detail solution results using a multigrid method. The Reynolds number is defined as $\mathrm{Re} = U_0 L/\nu$, where L is the length of the square-enclosure side wall, U_0 is the speed of the sliding lid, and ν is the fluid kinetic viscosity. Second, numerical computations are conducted for the flow over a backward-facing step for Reynolds numbers 50 and 150, and its reattachment length is calculated by two kinds of grids under Reynolds number 100.

Table 1. Fourteen second-order difference schemes ($u_e > 0$)

Region and scheme characteristics	No.	Origin definition	Normalized variable definition
	0	$\phi_e = \phi_i$ (FUD)	$\tilde{\phi}_e = \tilde{\phi}_i$
I	1	$\phi_e = -2\phi_i + \frac{5}{4}\phi_{i-1} + \frac{7}{4}\phi_{i+1}$ $(a_i = -2)$	$\tilde{\phi}_e = -2\tilde{\phi}_i + \frac{7}{4}$ $(a_i = -2)$
$a_i < 0$	2	$\phi_e = -\phi_i + \frac{3}{4}\phi_{i-1} + \frac{5}{4}\phi_{i+1}$ $(a_i = -1)$	$\tilde{\phi}_e = -\tilde{\phi}_i + \frac{5}{4}$ $(a_i = -1)$
$a_{i-1}, a_{i+1} > 0$	3	$\phi_e = -\frac{1}{4}\phi_i + \frac{3}{8}\phi_{i-1} + \frac{7}{8}\phi_{i+1}$ $(a_i = -\frac{1}{4})$	$\tilde{\phi}_e = -\frac{1}{4}\tilde{\phi}_i + \frac{7}{8}$ $(a_i = -\frac{1}{4})$
II	4	$\phi_e = \frac{1}{4}\phi_{i-1} + \frac{3}{4}\phi_{i+1}$ $(a_i = 0)$	$\tilde{\phi}_e = \frac{3}{4}$ $(a_i = 0)$
$0 \le a_i < 0.5$	5	$\phi_e = 0.2\phi_i + 0.15\phi_{i-1} + 0.65\phi_{i+1}$ $(a_i = 0.2)$	$\tilde{\phi}_e = 0.2\tilde{\phi}_i + 0.65$ $(a_i = 0.2)$
$a_{i-1}, a_{i+1} > 0$	6	$\phi_e = 0.4\phi_i + 0.05\phi_{i-1} + 0.55\phi_{i+1}$ $(a_i = 0.4)$	$\tilde{\phi}_e = 0.4\tilde{\phi}_i + 0.55$ $(a_i = 0.4)$
III	7	$\phi_e = 0.5\phi_i + 0.5\phi_{i+1}$ $(a_i = 0.5$ CD$)$	$\tilde{\phi}_e = 0.5\tilde{\phi}_i + 0.5$ $(a_i = 0.5)$
$0.5 \le a_i < 1.5$	8	$\phi_e = \frac{3}{4}\phi_i - \frac{1}{8}\phi_{i-1} + \frac{3}{8}\phi_{i+1}$ $(a_i = \frac{3}{4}$ QUICK$)$	$\tilde{\phi}_e = \frac{3}{4}\tilde{\phi}_i + \frac{3}{8}$ $(a_i = \frac{3}{4})$
$a_{i-1} \le 0, a_{i+1} > 0$	9	$\phi_e = \frac{5}{6}\phi_i - \frac{1}{6}\phi_{i-1} + \frac{1}{3}\phi_{i+1}$ $(a_i = \frac{5}{6}$ TUD$)$	$\tilde{\phi}_e = \frac{5}{6}\tilde{\phi}_i + \frac{1}{3}$ $(a_i = \frac{5}{6})$
	10	$\phi_e = \phi_i - \frac{1}{4}\phi_{i-1} + \frac{1}{4}\phi_{i+1}$ $(a_i = 1$ Fromm$)$	$\tilde{\phi}_e = \tilde{\phi}_i + \frac{1}{4}$ $(a_i = 1)$
IV	11	$\phi_e = \frac{3}{2}\phi_i - \frac{1}{2}\phi_{i-1}$ $(a_i = \frac{3}{2}$ SUD$)$	$\tilde{\phi}_e = \frac{3}{2}\tilde{\phi}_i$ $(a_i = \frac{3}{2})$
$a_i \ge 1.5$	12	$\phi_e = 2\phi_i - \frac{3}{4}\phi_{i-1} - \frac{1}{4}\phi_{i+1}$ $(a_i = 2)$	$\tilde{\phi}_e = 2\tilde{\phi}_i - \frac{1}{4}$ $(a_i = 2)$
$a_{i-1} < 0, a_{i+1} \le 0$	13	$\phi_e = 4\phi_i - 1.75\phi_{i-1} - 1.25\phi_{i+1}$ $(a_i = 4)$	$\tilde{\phi}_e = 4\tilde{\phi}_i - \frac{5}{4}$ $(a_i = 4)$
	14	$\phi_e = 10\phi_i - 4.75\phi_{i-1} - 4.25\phi_{i+1}$ $(a_i = 10)$	$\tilde{\phi}_e = 10\tilde{\phi}_i - 4.25$ $(a_i = 10)$

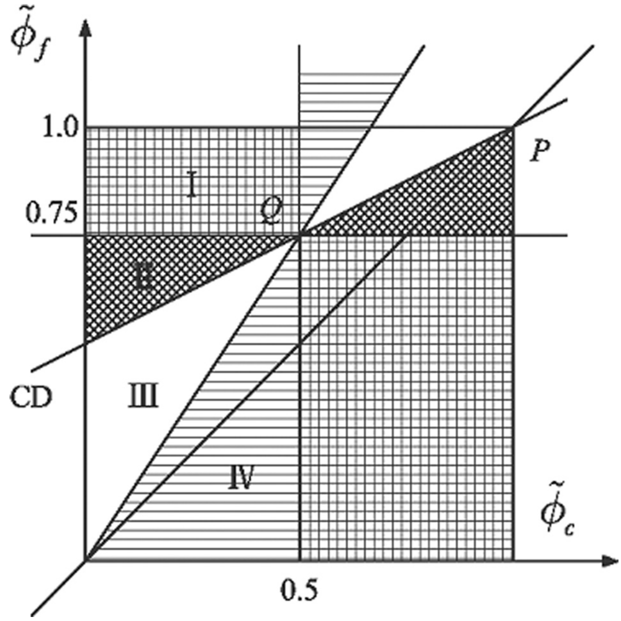


Figure 5. Region division of different schemes in NVD.

The resulting discretization equations are solved by the SIMPLER algorithm, in which the internal iterative method is the alternative direction implicit (ADI) method without the block-correction technique [23]. In order to thoroughly analyze the solution characteristics of the 14 schemes, we calculate the CPU time under different underrelaxation factors α. For convenience of presentation, the time-step multiple E is used in the following presentation, which is related to the underrelaxation factor α by Eq. (21) [24]:

$$E = \frac{\alpha}{1-\alpha} \qquad (0 < \alpha < 1) \tag{21}$$

In calculating the two benchmark problems, we mainly examine three aspects of the solution characteristics for the above 14 schemes, namely, convergence, false diffusion, and stability.

The introduction of the time-step multiple E (ETIME) is just for presentation purpose. It can be easily found from its definition in Eq. (21) that its variation range is much wider than that of the underrelaxation factor, 0 to ∞ versus 0 to 1. When graphic presentation is used, the adoption of E as the abscissa is much more convenient than the adoption of α as the abscissa.

3.2.1. Lid-driven cavity flow. Three uniform grids consisting of 42×42, 72×72, and 102×102 nodes and two Re numbers of 50 and 1,000 are adopted. Each calculation is terminated when the relative control-volume maximum residual of the discretized continuity equations becomes less than 3×10^{-8}.

The comparison of the convergence characteristics and CPU time (second) for the 14 schemes is shown in Table 2. From the table, the following two features may

Table 2. Comparisons of convergence characteristics and CPU time for 14 schemes adopted in lid-driven cavity flow

α	0.1	0.3	0.5	0.7	0.9
ETIME	0.1111	0.4286	1	2.3333	9
Re = 50					
Grid 42×42					
0	12	6.3	3.34	1.8	1.1
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	12.6	6.5	3.52	1.86	1.14
II					
4	12.6	6.5	3.5	1.8	1.1
5	12.7	6.6	3.7	1.9	1.1
6	12.8	6.5	3.6	1.9	1.1
III					
7	12.8	6.7	3.6	1.8	1.1
8	12.9	6.5	3.6	1.9	1.1
9	12.8	6.5	3.6	1.9	1.2
10	12.9	6.6	3.6	1.9	1.2
IV					
11	13	6.7	3.6	1.9	1.1
12	13	6.6	3.7	1.9	1.2
13	13.4	6.8	3.7	2	—
14	14.3	—	—	—	—
Grid 72×72					
0	73.1	50	30.5	16.3	10
I					
1	—	—	—	—	—
2	74.3	51.7	31.2	16.7	10.3
3	74.1	52	31.3	16.8	10.3
II					
4	73	50.9	30.8	16.8	10
5	74.4	51.9	31.2	17	10.1
6	74.4	51.9	31.5	17.1	10.3
III					
7	74.8	52	31.4	17.1	10.2
8	74.3	52.2	31.4	17.1	10.3
9	75.3	52.2	31.6	17.2	10.2
10	75	52	31.6	17	10.2
IV					
11	75	52.1	31.7	17.1	10.2
12	75.1	52.4	31.7	17.2	10.2
13	75.4	52.7	31.7	17	10.3
14	79.1	54	—	—	—
Grid 102×102					
0	211.2	153.7	108.5	62.7	36.8
I					
1	218.8	164	113.3	63.5	37
2	222.3	163	114	65	37.8
3	224.2	165.8	114.2	65.4	38.2

(*Continued*)

Table 2. Continued

α	0.1	0.3	0.5	0.7	0.9
ETIME	0.1111	0.4286	1	2.3333	9
II					
4	221	163.6	112.7	64.6	37.2
5	224	166.2	114.6	65.2	37.9
6	224.5	165.8	114.8	65.4	37.9
III					
7	224	166.2	114.4	65.2	37.7
8	224.5	166.1	115.5	65.3	37.9
9	224.1	166	115.2	65.6	37.9
10	224.8	166.2	115.6	65.5	38
IV					
11	224.7	165.7	115.5	65.2	37.8
12	225	166.4	115.9	65.7	37.9
13	226	168	116.1	67.8	37.9
14	228.6	171.5	120	—	—
Re = 1,000					
Grid 42 × 42					
0	21	7.8	3.9	2	0.95
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	39.9	16.6	8.8	4.9	2.5
III					
7	41.1	17.4	9	5	2.7
8	39	16.8	9	5	2.8
9	37.8	16.6	8.8	5	2.8
10	34.1	15.6	8.6	4.9	3
IV					
11	35	13.7	7.6	4.3	3.2
12	41.5	18.6	9.7	—	—
13	—	—	—	—	—
14	—	—	—	—	—
Grid 72 × 72					
0	173.3	70.3	37	19.2	10.3
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	243.8	125	68.8	37.1	17
III					
7	238.8	124.3	68.5	37	17.1
8	234	121.9	67.4	36.4	17.3

(*Continued*)

Table 2. Continued

α ETIME	0.1 0.1111	0.3 0.4286	0.5 1	0.7 2.3333	0.9 9
9	232.7	121.2	67	36.3	17.2
10	230.5	119.3	66.2	35.9	17.3
IV					
11	225.9	110.3	62.3	34.2	17.3
12	228.6	90.2	53.8	31	—
13	292	—	—	—	—
14	—	—	—	—	—
Grid 102×102					
0	620.3	272.9	149.2	78	45.4
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	751.5	410.3	234.2	127.3	55.4
III					
7	752.4	407.7	233.8	127.2	55.6
8	756.4	405.3	232.4	126.3	55.3
9	756.3	404.4	231.5	126	55.3
10	758.4	400.3	230.9	125.7	55.3
IV					
11	762	386.5	225.5	122.9	54.9
12	771.6	367.1	218.1	119.6	—
13	816.8	—	—	—	—
14	—	—	—	—	—

Missing numbers correspond to cases that did not converge.

be noted. First, the CPU time for different schemes under the same grids and Re number is almost the same, which is especially evident at lower Re number. Second, the schemes with smaller value of a_i can well achieve convergence under lower Re number and fine grids. However, when the Re number is increased and the grid number is decreased, the convergence of these schemes gradually becomes worse. This is consistent with the above analysis. As indicated above, the schemes with smaller values of a_i have smaller critical grid Pelect number, so they easily become divergent with increase of Re number and the distance between neighboring grids.

Tables 3 and 4 list the relative errors of the present numerical results for seven schemes (schemes 6–12; results for the rest of the schemes cannot be obtained for this problem) compared with the results of Ghia et al. for the centerline velocities of u (x direction) and v (y direction) by 42×42 grids under $\mathrm{Re} = 1{,}000$. Generally speaking, the numerical accuracy of the second-order scheme becomes worse with the increase of a_i of the scheme; in another words, false diffusion will become more serious with increasing a_i. As indicated in Section 2.2, scheme stability is enhanced with an increase of a_i. It may be that for the nominal second-order schemes defined by Eq. (9), there is a contradiction between the scheme numerical accuracy and the scheme stability.

Table 3. Relative error of centerline u velocity obtained using uniform grid (42×42), %

x	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0
0.0625	4.0512	3.3908	2.3987	2.1275	1.6828	1.1943	2.2771
0.0703	4.4368	3.7459	2.7056	2.4176	1.9382	1.3363	2.2946
0.0781	4.6365	3.915	2.8266	2.5217	2.0076	1.2923	2.1262
0.0938	4.8926	4.1133	2.9282	2.5876	1.9974	1.0191	1.5488
0.1563	5.2201	4.3385	2.9306	2.4804	1.6228	−0.4937	−1.5729
0.2266	3.5358	2.8623	1.7765	1.3909	0.5978	−1.8473	−4.1654
0.2344	3.2853	2.6462	1.6201	1.251	0.4849	−1.9298	−4.3138
0.5	−0.20459	−0.09201	0.23241	0.32164	0.4732	0.732	0.68641
0.8047	−3.2018	−2.4109	−1.0137	−0.5221	0.5079	4.0627	8.1091
0.8594	−4.3771	−3.3033	−1.1739	−0.4037	1.1683	5.3967	7.899
0.9063	−6.359	−5.4338	−4.5361	−4.3884	−4.3201	−5.4522	−7.8811
0.9453	−2.8008	−3.2769	−5.6231	−6.3696	−7.7609	−11.1828	−14.1097
0.9531	−2.1201	−2.8888	−5.4695	−6.2078	−7.5177	−10.5284	−13.0213
0.9609	−0.8685	−1.9298	−4.745	−5.475	−6.7035	−9.3029	−11.362
0.9688	−0.8394	−1.7188	−3.9454	−4.5105	−5.4492	−7.3972	−8.9501
1	0	0	0	0	0	0	0
Mean error	9.203955	8.277747	8.5134441	8.6149595	8.6953069	8.564648	9.0498876

Table 4. Relative error of centerline v velocity obtained using uniform grid (42×42), %

y	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0
0.0547	−2.0614	−2.4511	−4.2767	−4.9678	−6.4337	−11.22667	−16.03585
0.0625	−2.3395	−2.7227	−4.6029	−5.3193	−6.8489	−11.95314	−17.26247
0.0703	−2.8157	−3.1274	−4.8872	−5.5715	−7.0543	−12.1868	−17.8325
0.1016	−4.4745	−4.4399	−5.5296	−6.0175	−7.1705	−11.9729	−18.5872
0.1719	−5.6533	−4.7869	−3.5758	−3.2382	−2.7156	−2.8394	−6.6217
0.2813	−2.4604	−1.6531	0.199	0.9178	2.4945	8.164	14.6027
0.4531	−0.85912	−0.5017	0.3801	0.7152	1.4439	4.144	7.8353
0.5	−0.33959	−0.1097	0.53376	0.78105	1.32425	3.39763	6.3646
0.6172	0.92921	0.83032	0.85627	0.87715	0.94177	1.38994	2.45543
0.7344	2.2419	1.7969	1.1615	0.9423	0.502	−0.7467	−1.6506
0.8516	4.203	3.3243	1.8382	1.3213	0.2707	−2.9188	−5.994
0.9531	5.2228	2.4054	0.9021	0.3819	−0.6762	−3.9633	−7.4293
0.9609	6.5751	3.5632	2.2715	1.8173	0.8849	−2.0727	−5.2756
0.9688	3.1419	1.8768	0.905	0.5591	−0.1558	−2.4626	−5.0295
0.9766	0.237	1.1843	0.5547	0.3261	−0.1514	−1.7362	−3.5825
1	0	0	0	0	0	0	0
Mean error	9.3378763	7.7837948	8.0217888	8.2704733	8.9081041	11.9684868	16.9691564

Table 5. Predicted reattachment lengths

Scheme	8	9	10	11	12
Grid 62×32	6.197	6.213	6.244	6.342	6.438
Grid 92×47	6.205	6.212	6.224	6.265	6.307

Table 6. Comparisons of convergence characteristics and CPU time for flow over a backward-facing step

α	0.1	0.3	0.5	0.7	0.9
ETIME	0.1111	0.4286	1	2.3333	9
Re = 10					
Grid 62 × 32					
0	6.3	4.8	4.1	3.1	—
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	6.6	4.7	4	3.2	—
III					
7	6.6	4.8	4.1	3.1	—
8	6.7	4.8	4.1	3.2	—
9	6.7	4.8	4	3.1	—
10	6.6	4.7	4.2	3.2	—
IV					
11	6.7	4.8	4	3.2	—
12	6.7	4.8	4.1	3.1	—
13	6.6	4.8	4	3.1	—
14	6.5	4.6	4.1	—	—
Grid 102 × 52					
0	46.9	36.6	30.4	22.3	—
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	47.2	36.8	29.8	22.4	—
III					
7	47.3	36.7	29.8	22.5	—
8	47.3	36.8	29.8	22.3	—
9	47.3	36.8	30	22.5	—
10	47.2	36.7	29.8	22.3	—
IV					
11	47.3	36.8	29.7	22.4	—
12	48.3	36.8	29.8	22.4	—
13	48.2	36.9	29.8	22.4	—
14	49.3	36.9	30.7	—	—
Re = 150					
Grid 62 × 32					
0	8.3	4.4	3.9	3	1.7
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—

(*Continued*)

Table 6. Continued

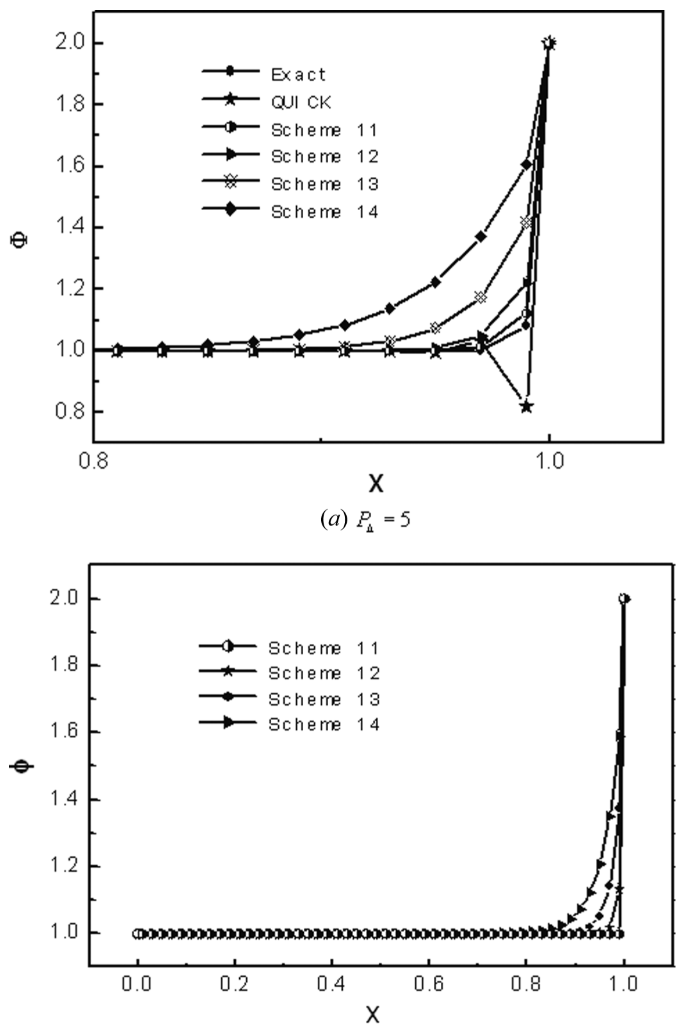
α ETIME	0.1 0.1111	0.3 0.4286	0.5 1	0.7 2.3333	0.9 9
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	33.8	15.6	13.4	30.7	—
9	12.3	6	4.2	3	2.3
10	12.2	5.9	4	3.1	1.8
IV					
11	11.6	5.7	3.8	3	1.7
12	11.3	5.5	4	3	—
13	11.3	5.2	—	—	—
14	13.7	—	—	—	—
Grid 102×52					
0	62.1	36.3	28.9	21.8	—
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	99.1	46.6	35.3	22	—
9	76.3	45	34.6	21.9	—
10	77	40.3	33.8	22	—
IV					
11	64.5	41.2	29.1	22	—
12	62.7	40.2	30.7	21.9	—
13	64.9	39.4	—	—	—
14	61	—	—	—	—

Missing numbers correspond to cases that did not converge.

3.2.2. Flow over a backward-facing step. There are two important parameters that exert great influence on the fluid mechanics in the two-dimensional backward-facing step geometry, i.e., the Reynolds number Re and the channel expansion ratio ER (the ratio of the channel widths downstream and upstream) [25]. Here the case of Re = 100, ER = 1.5 is taken as the test example. Under these conditions, the predicted reattachment length calculated by Kondoh et al. is 6.3 [25], which is slightly longer than the experimental value of 6.0. In the present study, the grids used are uniform and consist of 62×32 and 92×47 nodes. The predicted reattachment lengths obtained by various schemes and grids are listed in Table 5.

In addition, in order to analyze the solution characteristics of the 14 schemes, in Table 6 we present the CPU time for different values of α for two uniform grids consisting of 62×32 and 102×52 nodes and two Re numbers, 10 and 100. From this table we obtain similar conclusions as those from the lid-driven cavity flow problem. For simplicity of presentation, these features will not be restated here.

From the above analysis, it can be clearly observed that for the 14 nominal second-order schemes, the false diffusion of the scheme will become serious with increasing a_i. In order to enhance the numerical accuracy, we need to design a scheme using a smaller a_i; in contrast, in order to increase the critical grid Pelect number, i.e., to enhance the scheme stability, we have to design a



(*a*) $P_\Delta = 5$

(*b*) $P_\Delta = 100{,}000$

Figure 6. Stability examination of difference schemes.

scheme using a bigger a_i. Thus, as a compromise between accuracy and stability of the second-order difference schemes, we propose that the design range of a_i should be within 0.5 to 2.

3.3. 1-D Steady Convective-Diffusion Problem

In order to demonstrate numerically the scheme absolute stability when $a_i \geq a_{i0}$, we calculate the 1-D steady-state convection-diffusion problem under the condition of $\phi(0) = 1$ and $\phi(0) = 2$. Figure 6 shows the numerical results for QUICK, SUD, and schemes 12, 13, and 14. Figure 6(*a*) shows that the results of the QUICK scheme are oscillating when the grid Pelect number is 5, but the results of the SUD, and of schemes 12, 13, and 14, are reasonable. Figure 6(*b*) shows that even when the grid Pelect number reaches 100,000, the absolutely stable schemes SUD, 12, 13, and 14 still give physically plausible solutions. However, with increasing a_i, the false diffusion increases.

4. CONCLUSIONS

In this article, a general formulation of the interfacial interpolation for the discretized convective term is presented and a comprehensive analysis of its characteristics is conducted. The major findings can be summarized as follows.

1. A general formulation of the second-order and any higher-order difference schemes for the convective term is proposed. The general formulation of the second-order schemes can unify all existing schemes of the second-order type. It is helpful to thoroughly analyze the solution characteristics of various schemes, that is, the convergence, false diffusion, and stability.
2. Based on the general formulation of the second-order difference schemes, a formulation for an absolutely stable scheme is presented. Moreover, numerical test proves that the absolutely stable schemes can give physically plausible numerical solutions even if the grid Peclet number is increased to 100,000 for 1-D convective diffusive problem.
3. The characteristics of 14 schemes constructed by the general formulation are examined for the aspects of accuracy, economic, and stability. It is found for the two benchmark problems tested that the CPU time for the different schemes is almost the same. However, with increase of the value of a_i under higher Re, the false diffusion of the schemes becomes serious while their stability become better. For the second-order accuracy schemes, there is a serious contradiction between the scheme numerical accuracy and the scheme stability. And it is recommended that the value of a_i be within 0.5 to 2 as a compromise between stability and accuracy.

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