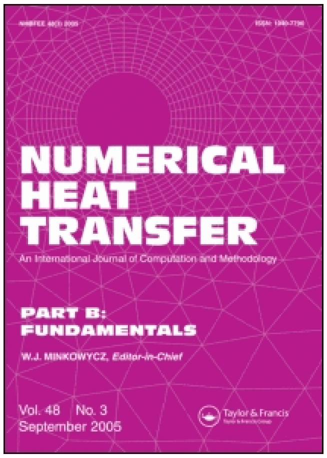

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DESIGN OF HIGH-ORDER DIFFERENCE SCHEME AND ANALYSIS OF SOLUTION CHARACTERISTICS—PART II: A KIND OF THIRD-ORDER DIFFERENCE SCHEME AND NEW SCHEME DESIGN THEORY

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In this Part II, on the basis of the general style design of second-order difference scheme and the analysis of the absolutely stable scheme proposed in Part I, the companion article, the general design method of any high-order difference scheme is proposed. Based on this method, a new kind of third-order difference scheme including 17 different variants is constructed, which uses the same grid points as existing second-order difference schemes but is different from them in that the grids are chosen symmetrically from two sides of the interface. Because they have the same matrix style created by the same grid plots of the discretization equation, these third-order schemes require the same CPU time and memory as the second-order schemes; however, this kind of symmetrical third-order difference scheme will keep the consistency between the false diffusion and the stability, and the stability of the scheme is better than that of the existing biased second-order scheme. Further research shows that under the conditions of matrix style and computer memory, the scheme constituted by symmetrically numbered grids from two sides of the interface with odd order of accuracy can maintain consistency between numerical accuracy and stability better than any kind of scheme designed according to the "upwind" idea. Based on this understanding, a new scheme design theory called symmetric and odd-order accuracy scheme design theory is proposed.

1. INTRODUCTION

As indicated in Part I [1] of this article, the discretization of the convective term is one of the most challenging tasks in numerical heat transfer, since the convective discretization schemes in the Navier-Stokes equation and scalar transport equation are related directly to the accuracy, efficiency, and convergence. Many studies have been conducted in order to improve the performance of the discretization scheme,

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especially to make an appropriate compromise between numerical accuracy and stability [2–21]. However, to the authors' knowledge, no integral and general theory has been established for the design of a scheme possessing required performance at least in the framework of incompressible fluid flow by the finite-volume method. In Part I of this article [1], through logical and mathematical deduction, a general design method and style of second-order difference scheme was proposed. Using this style, we could easily design new second-order difference schemes and study their solution characteristics in more detail. In addition, based on this general style of second-order difference scheme, we thoroughly studied the stability of the second-order scheme derived from its original definition. Through the analysis of the solution characteristics of 14 second-order schemes, it was observed that there is serious contradiction in the second-order difference scheme between the false diffusion (numerical accuracy) and the stability. In order to alleviate this contradiction and to further increase the solution accuracy, it is of great importance to design some high-order difference schemes with smaller critical a_{i0} values of absolutely stable scheme.

In this article, the design method of the general style of second-order scheme and the analysis method of the absolutely stable scheme presented in [1] are further extended. It will be shown that we can easily build the general style of any high-order difference scheme and thoroughly study its stability. In this regard, a new kind of third-order difference scheme is designed, which has the same number and sort of grids as the existing second-order difference scheme in the matrix created by the discretization equation, but the grids constituting the scheme are symmetric from two sides of the interface; through comparing its solution characteristics with the second-order difference scheme, we proposed a new design theory for a high-order-accuracy scheme, called, symmetric and odd-order scheme design theory.

2. GENERAL DESIGN METHOD AND STABILITY ANALYSIS OF ANY HIGH-ORDER DIFFERENCE SCHEME

2.1. General Design Method for High-Order Difference Scheme

In Sections 2.1 and 2.2 of Part I, the companion article [1], the deduction methods of a general style and a second-order scheme with absolutely stability have been presented. In fact, this deduction method can be further extended to the design of any high-order difference scheme. For example, under the condition $u > 0$, a scheme can be defined by the interpolated dependent variables at the east, e, and west, w, interfaces as follows:

$$\begin{aligned}
\phi_e &= \cdots + a_{i-n}\phi_{i-n} + \cdots + a_{i-3}\phi_{i-3} + a_{i-2}\phi_{i-2} + a_{i-1}\phi_{i-1} + a_i\phi_i \\
&\quad + a_{i+1}\phi_{i+1} + a_{i+2}\phi_{i+2} + a_{i+3}\phi_{i+3} + \cdots + a_{i+n+1}\phi_{i+n+1} + \cdots \\
\phi_w &= \cdots + a_{i-n}\phi_{i-n-1} + \cdots + a_{i-3}\phi_{i-4} + a_{i-2}\phi_{i-3} + a_{i-1}\phi_{i-2} + a_i\phi_{i-1} \\
&\quad + a_{i+1}\phi_i + a_{i+2}\phi_{i+1} + a_{i+3}\phi_{i+2} + \cdots + a_{i+n+1}\phi_{i+n} + \cdots
\end{aligned} \tag{1}$$

Then the following expression can also be obtained, similar to Eq. (6) of Part I [1]:

$$
\begin{aligned}
\left.\frac{\partial \phi}{\partial x}\right|_i = & \frac{\phi_e - \phi_w}{\Delta x} \\
& \frac{\cdots - a_{i-n}\phi_{i-n-1} + (a_{i-n} - a_{i-n-1})\phi_{i-n} + \cdots + (a_{i-4} - a_{i-3})\phi_{i-4}}{\Delta x} \\
& + \frac{(a_{i-3} - a_{i-2})\phi_{i-3} + (a_{i-2} - a_{i-1})\phi_{i-2} + (a_{i-1} - a_i)\phi_{i-1}}{\Delta x} \\
& + \frac{(a_i - a_{i+1})\phi_i + (a_{i+1} - a_{i+2})\phi_{i+1} + (a_{i+2} - a_{i+3})\phi_{i+2}}{\Delta x} \\
& + \frac{(a_{i+3} - a_{i+4})\phi_{i+3} + \cdots + (a_{i+n} - a_{i+n+1})\phi_{i+n} + a_{i+n+1}\phi_{i+n+1} + \cdots}{\Delta x}
\end{aligned} \tag{2}
$$

The terms on the right-hand side of the above equation can be extended by Taylor series expansion at the grid i, and we can obtain an expression for the first-order derivative as follows:

$$
\begin{aligned}
\left.\frac{\partial \phi}{\partial x}\right|_i = & [(n+1)a_{i-n} - n(a_{i-n} - a_{i-n+1}) - \cdots - 3(a_{i-3} - a_{i-2}) - 2(a_{i-2} - a_{i-1}) \\
& - 1(a_{i-1} - a_i) + 1(a_{i+1} - a_{i+2}) + 2(a_{i+2} - a_{i+3}) + 3(a_{i+3} - a_{i+4}) + \cdots \\
& + n(a_{i+n} - a_{i+n+1}) + (n+1)a_{i+n+1}]\left.\frac{\partial \phi}{\partial x}\right|_i + [-(n+1)^2 a_{i-n} + n^2(a_{i-n} - a_{i-n+1}) \\
& + \cdots + 3^2(a_{i-3} - a_{i-2}) + 2^2(a_{i-2} - a_{i-1}) + 1^2(a_{i-1} - a_i) + 1^2(a_{i+1} - a_{i+2}) \\
& + 2^2(a_{i+2} - a_{i+3}) + 3^2(a_{i+3} - a_{i+4}) + \cdots + n^2(a_{i+n} - a_{i+n+1}) \\
& + (n+1)^2 a_{i+n+1}]\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i \cdot \frac{\Delta x}{2!} + [(n+1)^3 a_{i-n} - n^3(a_{i-n} - a_{i-n+1}) - \cdots \\
& - 3^3(a_{i-3} - a_{i-2}) - 2^3(a_{i-2} - a_{i-1}) - (a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) \\
& + 2^3(a_{i+2} - a_{i+3}) + 3^3(a_{i+3} - a_{i+4}) + \cdots + n^3(a_{i+n} - a_{i+n+1}) \\
& + (n+1)^3 a_{i+n+1}]\left.\frac{\partial^3 \phi}{\partial x^3}\right|_i \cdot \frac{\Delta x^2}{3!} \cdots + [(n+1)^{(2n+1)} a_{i-n} - n^{(2n+1)}(a_{i-n} - a_{i-n+1}) \\
& - \cdots - 3^{(2n+1)}(a_{i-3} - a_{i-2}) - 2^{(2n+1)}(a_{i-2} - a_{i-1}) - (a_{i-1} - a_i) \\
& + (a_{i+1} - a_{i+2}) + 2^{(2n+1)}(a_{i+2} - a_{i+3}) + 3^{(2n+1)}(a_{i+3} - a_{i+4}) + \cdots \\
& + n^{(2n+1)}(a_{i+n} - a_{i+n+1}) + (n+1)^{(2n+1)} a_{i+n+1}]\left.\frac{\partial^{(2n+1)} \phi}{\partial x^{(2n+1)}}\right|_i \frac{\Delta x^{2n}}{(2n+1)!} \\
& + [-(n+1)^{(2n+2)} a_{i-n} + n^{(2n+2)}(a_{i-n} - a_{i-n+1}) + \cdots + 3^{(2n+2)}(a_{i-3} - a_{i-2}) \\
& + 2^{(2n+2)}(a_{i-2} - a_{i-1}) + (a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) + 2^{(2n+2)}(a_{i+2} - a_{i+3}) \\
& + 3^{(2n+2)}(a_{i+3} - a_{i+4}) + \cdots + n^{(2n+2)}(a_{i+n} - a_{i+n+1}) \\
& + (n+1)^{(2n+2)} a_{i+n+1}]\left.\frac{\partial^{(2n+2)} \phi}{\partial x^{(2n+2)}}\right|_i \frac{\Delta x^{2n+1}}{(2n+2)!} + \cdots
\end{aligned} \tag{3}
$$

For the derivative terms on the right-hand side of Eq. (3), if we make the coefficient before the term

$$\left.\frac{\partial \phi}{\partial x}\right|_i$$

equal 1 and the coefficients before the terms

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i, \qquad \left.\frac{\partial^3 \phi}{\partial x^3}\right|_i \cdots \left.\frac{\partial^{(2n+1)} \phi}{\partial x^{(2n+1)}}\right|_i \quad \text{and} \quad \left.\frac{\partial^{(2n+2)} \phi}{\partial x^{(2n+2)}}\right|_i$$

equal 0, we can obtain the condition by which schemes with $(2n+2)$-order of accuracy should abide, and the condition is as follows:

$$\begin{cases} (n+1)a_{i-n} - n(a_{i-n} - a_{i-n+1}) - \cdots - 3(a_{i-3} - a_{i-2}) \\ \quad -2(a_{i-2} - a_{i-1}) - 1(a_{i-1} - a_i) \\ +1(a_{i+1} - a_{i+2}) + 2(a_{i+2} - a_{i+3}) + 3(a_{i+3} - a_{i+4}) + \cdots + n(a_{i+n} - a_{i+n+1}) \\ \quad +(n+1)a_{i+n+1} = 1 \\ -(n+1)^2 a_{i-n} + n^2(a_{i-n} - a_{i-n+1}) + \cdots + 3^2(a_{i-3} - a_{i-2}) + 2^2(a_{i-2} - a_{i-1}) \\ \quad +1^2(a_{i-1} - a_i) \\ +1^2(a_{i+1} - a_{i+2}) + 2^2(a_{i+2} - a_{i+3}) + 3^2(a_{i+3} - a_{i+4}) + \cdots + n^2(a_{i+n} - a_{i+n+1}) \\ \quad +(n+1)^2 a_{i+n+1} = 0 \\ (n+1)^3 a_{i-n} - n^3(a_{i-n} - a_{i-n+1}) - \cdots - 3^3(a_{i-3} - a_{i-2}) - 2^3(a_{i-2} - a_{i-1}) \\ \quad -(a_{i-1} - a_i) \\ +(a_{i+1} - a_{i+2}) + 2^3(a_{i+2} - a_{i+3}) + 3^3(a_{i+3} - a_{i+4}) + \cdots + n^3(a_{i+n} - a_{i+n+1}) \\ \quad +(n+1)^3 a_{i+n+1} = 0 \ \cdots \\ (n+1)^{(2n+1)} a_{i-n} - n^{(2n+1)}(a_{i-n} - a_{i-n+1}) - \cdots - 3^{(2n+1)}(a_{i-3} - a_{i-2}) \\ \quad -2^{(2n+1)}(a_{i-2} - a_{i-1}) \\ -(a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) + 2^{(2n+1)}(a_{i+2} - a_{i+3}) + 3^{(2n+1)}(a_{i+3} - a_{i+4}) \\ \quad + \cdots + n^{(2n+1)}(a_{i+n} - a_{i+n+1}) \\ +(n+1)^{(2n+1)} a_{i+n+1} = 0 \\ -(n+1)^{(2n+2)} a_{i-n} + n^{(2n+2)}(a_{i-n} - a_{i-n+1}) + \cdots + 3^{(2n+2)}(a_{i-3} - a_{i-2}) \\ \quad +2^{(2n+2)}(a_{i-2} - a_{i-1}) \\ +(a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) + 2^{(2n+2)}(a_{i+2} - a_{i+3}) + 3^{(2n+2)}(a_{i+3} - a_{i+4}) \\ \quad + \cdots + n^{(2n+2)}(a_{i+n} - a_{i+n+1}) \\ +(n+1)^{(2n+2)} a_{i+n+1} = 0 \end{cases} \tag{4}$$

By solving the above equation, we can make certain the values of $a_{i-n}, a_{i-n+1}, \ldots, a_{i-2}, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+n}$, and a_{i+n+1}.

Similarly, if we make the coefficient before the term

$$\left.\frac{\partial \phi}{\partial x}\right|_i$$

equal 1, the coefficients before the terms from

$$\left.\frac{\partial^2\phi}{\partial x^2}\right|_i \quad \text{to} \quad \left.\frac{\partial^{(2n+1)}\phi}{\partial x^{(2n+1)}}\right|_i$$

equal 0 and the coefficient before the term

$$\left.\frac{\partial^{(2n+2)}\phi}{\partial x^{(2n+2)}}\right|_i$$

do not equal 0, we can obtain a general style of difference schemes with $(2n+1)$-order of accuracy, and the equation is as follows:

$$\begin{cases}
(n+1)a_{i-n} - n(a_{i-n} - a_{i-n+1}) - \cdots - 3(a_{i-3} - a_{i-2}) - 2(a_{i-2} - a_{i-1}) - a_{i-1} \\
+1(a_{i+1} - a_{i+2}) + 2(a_{i+2} - a_{i+3}) + 3(a_{i+3} - a_{i+4}) + \cdots \\
\quad +n(a_{i+n} - a_{i+n+1}) + (n+1)a_{i+n+1} = 1 - a_i \\
-(n+1)^2 a_{i-n} + n^2(a_{i-n} - a_{i-n+1}) + \cdots + 3^2(a_{i-3} - a_{i-2}) \\
\quad +2^2(a_{i-2} - a_{i-1}) + a_{i-1} \\
+1^2(a_{i+1} - a_{i+2}) + 2^2(a_{i+2} - a_{i+3}) + 3^2(a_{i+3} - a_{i+4}) + \cdots \\
\quad +n^2(a_{i+n} - a_{i+n+1}) + (n+1)^2 a_{i+n+1} = a_i \\
(n+1)^3 a_{i-n} - n^3(a_{i-n} - a_{i-n+1}) - \cdots - 3^3(a_{i-3} - a_{i-2}) \\
\quad -2^3(a_{i-2} - a_{i-1}) - a_{i-1} \\
+(a_{i+1} - a_{i+2}) + 2^3(a_{i+2} - a_{i+3}) + 3^3(a_{i+3} - a_{i+4}) + \cdots \\
\quad +n^3(a_{i+n} - a_{i+n+1}) + (n+1)^3 a_{i+n+1} = -a_i \\
\qquad\qquad\qquad \cdots\cdots \\
(n+1)^{(2n-1)} a_{i-n} - n^{(2n-1)}(a_{i-n} - a_{i-n+1}) - \cdots \\
\quad -3^{(2n-1)}(a_{i-3} - a_{i-2}) - 2^{(2n-1)}(a_{i-2} - a_{i-1}) \\
-a_{i-1} + (a_{i+1} - a_{1+2}) + 2^{(2n-1)}(a_{i+2} - a_{i+3}) + 3^{(2n-1)}(a_{i+3} - a_{i+4}) + \cdots \\
\quad +n^{(2n-1)}(a_{i+n} - a_{i+n+1}) \\
+(n+1)^{(2n-1)} a_{i+n+1} = -a_i \\
-(n+1)^{2n} a_{i-n} + n^{2n}(a_{i-n} - a_{i-n+1}) + \cdots + 3^{2n}(a_{i-3} - a_{i-2}) + 2^{2n}(a_{i-2} - a_{i-1}) \\
+(a_{i-1} - a_i) + (a_{i+1} - a_{i+2}) + 2^{2n}(a_{i+2} - a_{i+3}) \\
\quad +3^{2n}(a_{i+3} - a_{i+4}) + \cdots + n^{2n}(a_{i+n} - a_{i+n+1}) \\
+(n+1)^{2n} a_{i+n+1} \neq 0
\end{cases} \tag{5}$$

By solving above equation, we can determine the values of $a_{i-n}, a_{i-n+1}, \ldots, a_{i-2}$, a_{i-1}, $a_{i+1}, a_{i+2}, \ldots, a_{i+n}$, and a_{i+n+1} in terms of a_i. Then any higher-order difference scheme with $(2n+1)$-order of accuracy can be constructed.

2.2. Scheme Stability Analysis

Based on the general style of high-order difference schemes, we can analyze the scheme stability thoroughly and completely.

Taking the above $(2n+1)$-order accuracy difference scheme as an example, we analyze the stability via the one-dimensional unsteady convection-diffusion equation

$$\rho \frac{\partial \phi}{\partial t} + \rho u \frac{\partial \phi}{\partial x} = \Gamma \frac{\partial^2 \phi}{\partial x^2} \tag{6}$$

When the coefficient values $a_{i-n}, a_{i-n+1}, \ldots, a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}, \ldots, a_{i+n}$, and a_{i+n+1} have been expressed by a_i through Eq. (5), the explicit finite-difference form with $(2n+1)$-order accuracy of the convective scheme can be obtained as follows (where the superscript n is the time instant and differs from the subscript n which is the location indication):

$$\begin{aligned}
&\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + u \frac{\cdots - a_{i-n}\phi_{i-n-1}^n + (a_{i-n} - a_{i-n-1})\phi_{i-n}^n + \cdots + (a_{i-4} - a_{i-3})\phi_{i-4}^n + (a_{i-3} - a_{i-2})\phi_{i-3}^n}{\Delta x} \\
&\quad + u \frac{(a_{i-2} - a_{i-1})\phi_{i-2}^n + (a_{i-1} - a_i)\phi_{i-1}^n + (a_i - a_{i+1})\phi_i^n + (a_{i+1} - a_{i+2})\phi_{i+1}^n + (a_{i+2} - a_{i+3})\phi_{i+2}^n}{\Delta x} \\
&\quad + u \frac{(a_{i+3} - a_{i+4})\phi_{i+3}^n + \cdots + (a_{i+n} - a_{i+n+1})\phi_{i+n}^n + a_{i+n+1}\phi_{i+n+1}^n + \cdots}{\Delta x} \\
&\quad = \Gamma \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\rho \Delta x^2}
\end{aligned} \tag{7}$$

The discrete disturbance analysis method [22, 23] is used to analyze the stability. Suppose that originally the field is everywhere uniform and, without loss of generality, the field value is zero. At some time instant n and at grid point i there is a disturbance, denoted by ε_i^n. And at other grid points and at any other time instant, no disturbance is imposed on the field. Then, we can use the above equation to predict the transport of the disturbance at other grid points at subsequent time instants. Rewriting Eq. (7) without the diffusion term at grids $(i+1)$ and $(i-1)$, we obtain

$$\begin{aligned}
&\frac{\phi_{i+1}^{n+1} - \phi_{i+1}^n}{\Delta t} \\
&\quad = -u \frac{\cdots - a_{i-n}\phi_{i-n}^n + (a_{i-n} - a_{i-n-1})\phi_{i-n+1}^n + \cdots + (a_{i-4} - a_{i-3})\phi_{i-3}^n + (a_{i-3} - a_{i-2})\phi_{i-2}^n}{\Delta x} \\
&\quad\quad - u \frac{(a_{i-2} - a_{i-1})\phi_{i-1}^n + (a_{i-1} - a_i)\phi_i^n + (a_i - a_{i+1})\phi_{i+1}^n + (a_{i+1} - a_{i+2})\phi_{i+2}^n + (a_{i+2} - a_{i+3})\phi_{i+3}^n}{\Delta x} \\
&\quad\quad - u \frac{(a_{i+3} - a_{i+4})\phi_{i+4}^n + \cdots + (a_{i+n} - a_{i+n+1})\phi_{i+n+1}^n + a_{i+n+1}\phi_{i+n+2}^n + \cdots}{\Delta x}
\end{aligned} \tag{8a}$$

Then, according to the above assumption,

$$\begin{aligned}
&\frac{\phi_{i-1}^{n+1} - \phi_{i-1}^n}{\Delta t} \\
&\quad = -u \frac{\cdots - a_{i-n}\phi_{i-n-2}^n + (a_{i-n} - a_{i-n-1})\phi_{i-n-1}^n + \cdots + (a_{i-4} - a_{i-3})\phi_{i-5}^n + (a_{i-3} - a_{i-2})\phi_{i-4}^n}{\Delta x}
\end{aligned}$$

$$-u\frac{(a_{i-2}-a_{i-1})\phi_{i-3}^n+(a_{i-1}-a_i)\phi_{i-2}^n+(a_i-a_{i+1})\phi_{i-1}^n+(a_{i+1}-a_{i+2})\phi_i^n+(a_{i+2}-a_{i+3})\phi_{i+1}^n}{\Delta x}$$
$$-u\frac{(a_{i+3}-a_{i+4})\phi_{i+2}^n+\cdots+(a_{i+n}-a_{i+n+1})\phi_{i+n-1}^n+a_{i+n+1}\phi_{i+n}^n+\cdots}{\Delta x} \tag{8b}$$

Equation (8*b*) gives the transportation of the disturbance by convection. And the transportation of the disturbance by diffusion is $\rho\,\Delta t/\Gamma\,\Delta x^2$ [23, 24]. Then, according to the sign preservation rule [23, 24], we can obtain

$$\begin{cases} \dfrac{(a_i-a_{i-1})\left(\dfrac{u\,\Delta t}{\Delta x}\right)\varepsilon_i^n+\left(\dfrac{\Gamma\,\Delta t}{\rho\,\Delta x^2}\right)\varepsilon_i^n}{\varepsilon_i^n}\geq 0 \\ \dfrac{(a_{i+2}-a_{i+1})\left(\dfrac{u\,\Delta t}{\Delta x}\right)\varepsilon_i^n+\left(\dfrac{\Gamma\,\Delta t}{\rho\,\Delta x^2}\right)\varepsilon_i^n}{\varepsilon_i^n}\geq 0 \end{cases} \tag{9a}$$

From Eq. (9*a*) it can obviously be seen that when $u > 0$, for the scheme to be absolutely stable the following condition should be satisfied:

$$\begin{cases} a_i - a_{i-1} \geq 0 \\ a_{i+2} - a_{i+1} \geq 0 \end{cases} \tag{9b}$$

As indicated in the companion article [1], the stability condition determined by Eq. (9*b*) is actually the combined results from pure convection and diffusion.

If we substitute a_{i-1}, a_{i+2}, and a_{i+1} in terms of a_i into Eq. (9*b*), we can obtain the critical a_{i0} related to the absolutely stable scheme definition of Part I, the companion article [1]. When the a_i value defining the schemes by expression (1) and Eq. (5) is equal to or greater than the critical a_{i0}, the defined schemes are absolutely stable; that is, they maintain stability under any value of grid Pelect number.

On the other hand, from expression (9*a*), we can obtain the following equality:

$$\begin{cases} P_\Delta \geq \dfrac{1}{a_{i-1}-a_i},\; P_{\Delta c}\dfrac{1}{a_{i-1}-a_i} \\ P_\Delta \geq \dfrac{1}{a_{i+1}-a_{i+2}},\; P_{\Delta c}\dfrac{1}{a_{i+1}-a_{i+2}} \end{cases} \tag{10}$$

If we substitute a_{i-1}, a_{i+2}, and a_{i+1} in terms of a_i into Eq. (10), we can get the variability of the critical $P_{\Delta c}$ number with the parameter a_i for the scheme to be stable, just as we did in Section 2.2 of Part I in the companion article.

3. DESIGN OF SYMMETRIC THIRD-ORDER DIFFERENCE SCHEME AND SOLUTION ANALYSIS

3.1. Design of Symmetric Third-Order Difference Scheme

It is well known that, in the design of the upwind-type difference scheme, the "upwind" flow information should be taken into account. For example, when $u_e > 0$, the value of variable ϕ at the east interface, e, for a second-order difference scheme is interpolated by three grids, two grids ϕ_{i-1} and ϕ_i adopted from the up flow direction and one grid ϕ_{i+1} from the down flow. The scheme expression looks like

$$\phi_e = a_{i-1}\phi_{i-1} + a_i\phi_i + a_{i+1}\phi_{i+1} \tag{11a}$$

Similarly, when $u_e < 0$, according to the above idea the value of variable ϕ on the east interface, e, is interpolated by two grids adopted from the up flow direction (ϕ_{i+1} and ϕ_{i+2}) and one grid adopted from the down flow (ϕ_i). The scheme expression can be expressed as follows:

$$\phi_e = a_i\phi_i + a_{i+1}\phi_{i+1} + a_{i+2}\phi_{i+2} \tag{11b}$$

When we adopt the grids from the two sides of the interface unsymmetric to define a scheme, all the schemes expressed by Eqs. (11a) and (11b) have at most second-order accuracy, except the third-order scheme when $a_i = 5/6$. In such schemes, the total grid number used in defining the scheme is actually four, namely, $\phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}$. If we use these grids symmetrically to define the interface variable for whatever $u > 0$ or $u < 0$, namely symmetrically adopt the grids from two sides of the interface, we can easily obtain a kind of third-order accuracy schemes and one fourth-order accuracy scheme, which require the same computer memory as the existing second-order difference scheme. In another words, if we design the scheme using symmetric allocation of two sides of the interfaces, we can obtain higher-order accuracy difference schemes based on the same sort and number of grids as the existing second-order scheme. When $u > 0$, for this kind of symmetric third-order difference scheme, the values of variable ϕ at the east and west interfaces are interpolated as

$$\begin{cases} \phi_e = a_{i-1}\phi_{i-1} + a_i\phi_i + a_{i+1}\phi_{i+1} + a_{i+2}\phi_{i+2} \\ \phi_w = a_{i-1}\phi_{i-2} + a_i\phi_{i-1} + a_{i+1}\phi_i + a_{i+2}\phi_{i+1} \end{cases} \tag{12}$$

Then, further,

$$\left.\frac{\partial\phi}{\partial x}\right|_i = \frac{\phi_e - \phi_w}{\Delta x} = \frac{-a_{i-1}\phi_{i-2} + (a_{i-1} - a_i)\phi_{i-1} + (a_i - a_{i+1})\phi_i}{\Delta x} + \frac{(a_{i+1} - a_{i+2})\phi_{i+1} + a_{i+2}\phi_{i+2}}{\Delta x} \tag{13}$$

Expanding the terms on the right-hand side of the above equation by Taylor series expansion in the grid i, we can further obtain

$$\begin{aligned} \left.\frac{\partial\phi}{\partial x}\right|_i &= (a_{i-1}+a_i+a_{i+1}+a_{i+2})\left.\frac{\partial\phi}{\partial x}\right|_i - (3a_{i-1}+a_i-a_{i+1}-3a_{i+2})\left.\frac{\partial^2\phi}{\partial x^2}\right|_i\cdot\frac{\Delta x}{2!} \\ &+(7a_{i-1}+a_i+a_{i+1}+7a_{i+2})\left.\frac{\partial^3\phi}{\partial x^3}\right|_i\cdot\frac{\Delta x^2}{3!} \\ &-(15a_{i-1}+a_i-a_{i+1}-15a_{i+2})\left.\frac{\partial^4\phi}{\partial x^4}\right|_i\cdot\frac{\Delta x^3}{3!}\cdots \end{aligned} \tag{14}$$

In expression (12), there are four variables, a_{i-2}, a_{i-1}, a_i, and a_{i+1}. If we let

$$\begin{cases} a_{i-1}+a_i+a_{i+1}+a_{i+2}=1 \\ -3a_{i-1}-a_i+a_{i+1}+3a_{i+2}=0 \\ 7a_{i-1}+a_i+a_{i+1}+7a_{i+2}=0 \\ -15a_{i-1}-a_i+a_{i+1}+15a_{i+2}=0 \end{cases} \longrightarrow \begin{cases} a_{i-1}=-\frac{1}{12} \\ a_i=\frac{7}{12} \\ a_{i+1}=\frac{7}{12} \\ a_{i+2}=-\frac{1}{12} \end{cases}$$

we can obtain the only fourth-order difference scheme defined by expression (12).

$$\begin{cases} \phi_e=-\frac{1}{12}\phi_{i-1}+\frac{7}{12}\phi_i+\frac{7}{12}\phi_{i+1}-\frac{1}{12}\phi_{i+2} \\ \phi_w=-\frac{1}{12}\phi_{i-2}+\frac{7}{12}\phi_{i-1}+\frac{7}{12}\phi_i-\frac{1}{12}\phi_{i+1} \end{cases}$$

Further, if the following equations can be satisfied,

$$\begin{cases} a_{i-1}+a_i+a_{i+1}+a_{i+2}=1 \\ -3a_{i-1}-a_i+a_{i+1}+3a_{i+2}=0 \\ 7a_{i-1}+a_i+a_{i+1}+7a_{i+2}=0 \\ -15a_{i-1}-a_i+a_{i+1}+15a_{i+2}\neq 0 \end{cases} \longrightarrow \begin{cases} a_{i-1}=\frac{1}{9}-\frac{a_i}{3} \\ a_i\neq\frac{7}{12} \\ a_{i+1}=\frac{7}{6}-a_i \\ a_{i+2}=-\frac{5}{18}+\frac{a_i}{3} \end{cases}$$

we can obtain the general style of symmetric third-order difference scheme created by expression (12):

$$\begin{cases} \phi_e=\frac{(2-6a_i)}{18}\phi_{i-1}+a_i\phi_i+\frac{(21-18a_i)}{18}\phi_{i+1}+\frac{(6a_i-5)}{18}\phi_{i+2} \\ \phi_w=\frac{(2-6a_i)}{18}\phi_{i-2}+a_i\phi_{i-1}+\frac{(21-18a_i)}{18}\phi_i+\frac{(6a_i-5)}{18}\phi_{i+1} \\ a_i\neq\frac{7}{12} \end{cases} \tag{15}$$

Based on this general style, we calculate the critical a_{i0} value of the absolutely stable scheme by means of the analysis method of Part I, the companion article [1].

Here we apply the above scheme to the one-dimensional unsteady convection-diffusion equation:

$$\rho\frac{\partial\phi}{\partial t}+\rho u\frac{\partial\phi}{\partial x}=\Gamma\frac{\partial^2\phi}{\partial x^2} \tag{16}$$

Substituting expression (15) into the discretization form of Eq. (16), we obtain

$$\begin{aligned} &\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}+u\frac{(6a_i-2)\phi_{i-2}^n+(2-24a_i)\phi_{i-1}^n+(36a_i-21)\phi_i^n+(26-24a_i)\phi_{i+1}^n}{18\cdot\Delta x} \\ &\quad+u\frac{(6a_i-5)\phi_{i+2}^n}{18\cdot\Delta x}=\Gamma\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{\rho\Delta x^2} \end{aligned} \tag{17}$$

According to the discrete disturbance analysis method, rewriting Eq. (17) without the diffusion term for grids $(i+1)$ and $(i-1)$, we obtain

$$\begin{aligned}\frac{\phi_{i+1}^{n+1}-\phi_{i+1}^{n}}{\Delta t} &= -u\frac{(6a_i-2)\phi_{i-1}^n+(2-24a_i)\phi_i^n+(36a_i-21)\phi_{i+1}^n+(26-24a_i)\phi_{i+2}^n}{18\cdot\Delta x}\\ &\quad -u\frac{(6a_i-5)\phi_{i+3}^n}{18\cdot\Delta x}\\ \frac{\phi_{i-1}^{n+1}-\phi_{i-1}^{n}}{\Delta t} &= -u\frac{(6a_i-2)\phi_{i-3}^n+(2-24a_i)\phi_{i-2}^n+(36a_i-21)\phi_{i-1}^n+(26-24a_i)\phi_{i}^n}{18\cdot\Delta x}\\ &\quad -u\frac{(6a_i-5)\phi_{i+1}^n}{18\cdot\Delta x}\end{aligned} \tag{18a}$$

Implementing the discrete disturbance method, we get

$$\begin{cases}\phi_{i+1}^{n+1}=\dfrac{(24a_i-2)}{18}\left(\dfrac{u\Delta t}{\Delta x}\right)\varepsilon_i^n\\ \phi_{i-1}^{n+1}=\dfrac{(24a_i-26)}{18}\left(\dfrac{u\Delta t}{\Delta x}\right)\varepsilon_i^n\end{cases} \tag{18b}$$

According to the sign preservation rule, we have

$$\begin{cases}\dfrac{\frac{(24a_i-2)}{18}\left(\frac{u\,\Delta t}{\Delta x}\right)\varepsilon_i^n+\left(\frac{\Gamma\,\Delta t}{\rho\,\Delta x^2}\right)\varepsilon_i^n}{\varepsilon_i^n}\ge 0\\ \dfrac{\frac{(24a_i-26)}{24}\left(\frac{u\,\Delta t}{\Delta x}\right)\varepsilon_i^n+\left(\frac{\Gamma\,\Delta t}{\rho\,\Delta x^2}\right)\varepsilon_i^n}{\varepsilon_i^n}\ge 0\end{cases} \tag{19}$$

For $u > 0$, in order to satisfy the above two conditions simultaneously, the following results can be deduced:

$$\begin{cases}24a_i-2\ge 0\\ 24a_i-26\ge 0\end{cases}\longrightarrow \mathrm{a}_i\ge\frac{13}{12}$$

So, when $u > 0$, we can obtain the absolutely stable scheme of third-order accuracy constituted by the grids $i-2$, $i-1$, i, and $i+1$:

$$\begin{cases}\phi_e=\frac{(2-6a_i)}{18}\phi_{i-1}+a_i\phi_i+\frac{(21-18a_i)}{18}\phi_{i+1}+\frac{(6a_i-5)}{18}\phi_{i+2}\\ \phi_w=\frac{(2-6a_i)}{18}\phi_{i-2}+a_i\phi_{i-1}+\frac{(21-18a_i)}{18}\phi_{i}+\frac{(6a_i-5)}{18}\phi_{i+1}\\ a_i\ge\frac{13}{12}\end{cases} \tag{20}$$

where the critical a_{i0} value of the absolutely stable scheme is 13/12, which is only 72.2% of the existing second-order difference scheme. It is implied that this kind of symmetric third-order difference scheme can keep a good balance between numerical accuracy and stability.

3.2. Analysis of Solution Characteristics of Symmetric Third-Order Difference Scheme

3.2.1. Design of different symmetric third-order different schemes. As shown in Part I, the companion article [1], with variation of the sign and value of a_i in Eq. (12), the signs of a_{i-1}, a_{i+1}, and a_{i+1} are different, leading to different schemes. The solution characteristics of the coefficient matrix created by different schemes will be shown later quite differently. For $u_e > 0$, according to the different sign and value of a_i, 17 schemes defined by their east interface interpolations are shown in Table 1, divided into six groups. Their solution characteristics are analyzed through the following computation examples.

Table 1 Seventeen third-order difference scheme expressions ($u_e > 0$)

Region and scheme characteristics	No.	Origin definition
	0	$\phi_e = \phi_i$ (FUD)
I	1	$\phi_e = \frac{7}{9}\phi_{i-1} - 2\phi_i + \frac{19}{6}\phi_{i+1} - \frac{7}{18}\phi_{i+2}$ $(a_i = -2)$
$a_i < 0$	2	$\phi_e = \frac{4}{9}\phi_{i-1} - \phi_i + \frac{13}{6}\phi_{i+1} - \frac{11}{18}\phi_{i+2}$ $(a_i = -1)$
$a_i - 1, a_i + 1 > 0$	3	$\phi_e = \frac{7}{36}\phi_{i-1} - \frac{1}{4}\phi_i + \frac{17}{12}\phi_{i+1} - \frac{13}{36}\phi_{i+2}$ $(a_i = -\frac{1}{4})$
$a_i + 2 < 0$		
II	4	$\phi_e = \frac{1}{9}\phi_{i-1} + \frac{7}{6}\phi_{i+1} - \frac{5}{18}\phi_{i+2}$ $(a_i = 0)$
$0 \le a_i < \dfrac{1}{3}$	5	$\phi_e = \frac{1}{18}\phi_{i-1} + \frac{1}{6}\phi_i + \phi_{i+1} - \frac{2}{9}\phi_{i+2}$ $(a_i = \frac{1}{6})$
$a_{i-1}, a_{i+1} > 0$	6	$\phi_e = \frac{1}{36}\phi_{i-1} + \frac{1}{4}\phi_i + \frac{11}{12}\phi_{i+1} - \frac{7}{36}\phi_{i+2}$ $(a_i = \frac{1}{4})$
$a_{i+2} < 0$		
III	7	$\phi_e = \frac{1}{3}\phi_i + \frac{5}{6}\phi_{i+1} - \frac{1}{6}\phi_{i+2}$ $(a_i = \frac{1}{3})$
$\dfrac{1}{3} \le a_i < \dfrac{5}{6}$	8	$\phi_e = -\frac{1}{18}\phi_{i-1} + \frac{1}{2}\phi_i + \frac{2}{3}\phi_{i+1} - \frac{1}{9}\phi_{i+2}$ $(a_i = \frac{1}{2})$
	9	$\phi_e = -\frac{1}{12}\phi_{i-1} + \frac{7}{12}\phi_i + \frac{7}{12}\phi_{i+1} - \frac{1}{12}\phi_{i+2}$ $(a_i = \frac{7}{12})$
$a_{i-1} \le 0$	10	$\phi_e = -\frac{1}{9}\phi_{i-1} + \frac{2}{3}\phi_i + \frac{1}{2}\phi_{i+1} - \frac{1}{18}\phi_{i+2}$ $(a_i = \frac{2}{3})$
$a_{i+1} > 0$		
$a_{i+2} < 0$		
IV	11	$\phi_e = -\frac{1}{6}\phi_{i-1} + \frac{5}{6}\phi_i + \frac{1}{3}\phi_{i+1}$ $(a_i = \frac{5}{6})$
$\dfrac{5}{6} \le a_i < \dfrac{13}{12}$	12	$\phi_e = -\frac{7}{36}\phi_{i-1} + \frac{11}{12}\phi_i + \frac{1}{4}\phi_{i+1} + \frac{1}{36}\phi_{i+2}$ $(a_i = \frac{11}{12})$
	13	$\phi_e = -\frac{2}{9}\phi_{i-1} + \phi_i + \frac{1}{6}\phi_{i+1} + \frac{1}{18}\phi_{i+2}$ $(a_i = 1)$
$a_{i-1} \le 0$		
$a_{i+1}, a_{i+2} > 0$		
V	14	$\phi_e = -\frac{1}{4}\phi_{i-1} + \frac{13}{12}\phi_i + \frac{1}{12}\phi_{i+1} + \frac{1}{12}\phi_{i+2}$ $(a_i = \frac{13}{12})$
$\dfrac{13}{12} \le a_i < \dfrac{7}{6}$	15	$\phi_e = -\frac{19}{72}\phi_{i-1} + \frac{9}{8}\phi_i + \frac{1}{24}\phi_{i+1} + \frac{7}{72}\phi_{i+2}$ $(a_i = \frac{9}{8})$
$a_{i-1} < 0$		
$a_{i+1}, a_{i+2} > 0$		
VI	16	$\phi_e = -\frac{15}{18}\phi_{i-1} + \frac{7}{6}\phi_i + \frac{1}{9}\phi_{i+2}$ $(a_i = \frac{7}{6})$
$a_i \ge \dfrac{7}{6}$	17	$\phi_e = -\frac{4}{9}\phi_{i-1} + \frac{5}{3}\phi_i - \frac{1}{2}\phi_{i+1} + \frac{5}{18}\phi_{i+2}$ $(a_i = \frac{5}{3})$
$a_{i-1} < 0$		
$a_{i+1}, a_{i+2} > 0$		

3.2.2. Test calculations for two benchmark problems. In this section, in order to compare with the biased second-order difference scheme, two benchmark problems are examined as in Part I, the companion article [1]. First, numerical calculations are performed for the lid-driven cavity flow investigated by Ghia et al. [25]. Second, numerical computations are conducted for the flow over a backward-facing step [26] for Reynolds numbers 50 and 150, and its reattachment length is calculated by two kinds of grids under Reynolds number 100. The discretization equations are solved by the SIMPLER algorithm, in which the internal iterative method is the alternative direction implicit (ADI) method without the block-correction technique. The CPU time is also calculated under different underrelaxation factors α. For convenience of presentation, the time-step multiple E (ETIME) is used.

Three aspects of solution characteristics of the symmetric third-order difference schemes, namely, the convergence, the numerical accuracy and the stability, are studied through the two benchmark problems.

Lid-driven cavity flow. In order to compare with the results of the biased second-order difference scheme presented in Part I of this article [1], the same kinds of uniform grids, constituted by 42×42, 72×72, and 102×102 nodes and Re values of 50 and 1,000 are used. Each calculation is terminated by the same criterion as in Part I of this article, namely, the control-volume maximum relative residual of the discretized continuity equations below 3×10^{-8}.

The comparison of the convergence characteristics and CPU time (second) for 17 schemes are shown in Table 2. From the results, the following features may be noted. First, the CPU time for the 17 symmetric schemes is actually the same as those of the biased second-order schemes at the same grid and Re value. In fact, this is not difficult to understand, because the symmetric third-order difference scheme and the biased second-order scheme have the same matrix structure and use the same computer memory. In addition, the CPU time for the 17 different schemes under the same grids and Re is almost the same. Second, the schemes with smaller a_i can well get convergence under lower Re and fine grids, and when Re is increased and the grid number is decreased, their convergence gradually becomes worse. This once again shows that the schemes with smaller a_i are actually related to smaller critical grid Pelect number. The comparison of numerical accuracy will be presented later.

Flow over a backward-facing step. For this problem, the CPU time with different a_i values under the same kinds of uniform grids and Re values as for biased second-order difference schemes is listed in Table 3. From the results, we obtain similar conclusions to those for the lid-driven cavity flow problem, and for simplicity presentation, they are not repeated.

In Table 4, the reattachment lengths predicted by various schemes are presented. From the table, it can be easily observed that the difference from the results of the 17 symmetric third-order schemes is not evident.

For the symmetric third-order difference scheme, we propose that the design range of a_i should be between 0.5 and 2, and Schemes 14, 15, and 16 are the best.

3.2.3. Analysis of numerical accuracy. In order to demonstrate the higher numerical accuracy of the symmetrical third-order schemes, we calculate the

Table 2. Caparison of convergence characteristic and CPU time for 17 schemes in lid-driven cavity flow

α	0.1	0.3	0.5	0.7	0.9
ETIME	0.1111	0.4286	1	2.3333	9
Re = 50					
Grid 42×42					
0	10.9	5.7	3.1	1.6	1
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	11.8	6	3.3	1.7	1.1
II					
4	11.8	6.2	3.3	1.7	1.1
5	11.8	6	3.3	1.7	1
6	11.7	6.1	3.3	1.6	1.1
III					
7	11.8	6	3.3	1.7	1
8	11.7	6.2	3.3	1.7	1.1
9	11.8	6	3.3	1.7	1
10	11.8	5.9	3.2	1.7	1.1
IV					
11	11.8	6	3.3	1.7	1.1
12	11.9	6	3.3	1.7	1.1
13	11.7	6	3.3	1.7	1
V					
14	11.8	6	3.3	1.7	1
15	11.8	6	3.2	1.7	1.1
VI					
16	11.8	6	3.3	1.7	1.1
17	11.6	6	3.3	1.7	1
Grid 72×72					
0	67.8	46.9	28.4	15.2	9.4
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	70.8	49.3	29.7	16	9.6
II					
4	70.8	49	29.8	16.1	9.6
5	70.6	48.8	29.7	16.1	9.6
6	70.4	48.8	29.8	16.1	9.6
III					
7	70.6	48.7	29.6	16.2	9.5
8	70.4	48.8	29.6	16	9.4
9	70.3	48.8	29.4	16	9.4
10	70.4	48.8	29.6	16.2	9.6
IV					
11	70.1	48.5	29.4	16.1	9.6
12	69.8	49	29.6	16.1	9.4
13	70.3	48.8	29.8	16	9.4
V					
14	70	48.5	29.5	16	9.3
15	70.3	49	29.4	16.1	9.6

(*Continued*)

Table 2. Continued

α	0.1	0.3	0.5	0.7	0.9
ETIME	0.1111	0.4286	1	2.3333	9
VI					
16	70.1	48.7	29.6	16	9.6
17	69.9	48.5	29.4	16.1	9.5
Grid 102 × 102					
0	219.8	161.6	112.7	65	37.5
I					
1	—	—	—	—	—
2	227	166.7	116.5	65.5	38
3	225.8	167.5	116.2	65.6	38
II					
4	226	167.3	115.9	65.4	37.9
5	226	167	116	65.7	38
6	225.8	167.1	116.8	65.9	37.9
III					
7	225.3	167.5	115.9	65.3	38.1
8	225.3	166.7	115.8	65.4	38
9	224.6	166	115.8	65.6	37.9
10	224.5	166.6	115.8	65.4	38
IV					
11	224	165.7	115.6	65.2	37.9
12	224.4	165.5	115.8	65.5	38
13	224.3	165.8	115.6	65.5	38
V					
14	224.1	166.3	115.4	65.3	37.9
15	224.5	166.1	115.7	65.5	37.9
VI					
16	224.6	166	115.4	65.5	37.9
17	223.9	166.4	115	65.5	37.9
Re = 1,000					
Grid 42 × 42					
0	19.3	7.1	3.5	1.8	0.89
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	28.5	13.2	7	3.9	2.3
9	34.9	15.4	8.3	4.6	2.6
10	34.6	15.2	8.2	4.5	2.6
IV					
11	34.3	15	8.3	4.2	2.6
12	34.2	14.9	8.1	4.3	2.6

(*Continued*)

Table 2. Continued

α ETIME	0.1 0.1111	0.3 0.4286	0.5 1	0.7 2.3333	0.9 9
13	34.3	15.2	8.2	4.5	2.4
V					
14	34	15.1	8.1	4.5	2.6
15	33.8	15.2	8	4.5	2.6
VI					
16	34.2	15.1	8.2	4.5	2.5
17	34.7	14.9	8.1	—	—
Grid 72×72					
0	162	65.6	34.4	17.8	9.4
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	220.7	115	63.7	34.3	16
9	220.3	114.7	63.4	34.2	16.2
10	219.8	114.5	63.3	34.1	16.2
IV					
11	218.1	113.4	62.8	33.9	16
12	218	113.3	62.7	34	16.1
13	218	113.3	62.7	33.8	16
V					
14	216.7	112.7	62.7	33.9	16.1
15	217	112.6	62.8	33.9	16
VI					
16	217.2	112.9	62.5	34	16.1
17	218	111.7	62.3	—	—
Grid 102×102					
0	621	273.5	149.1	77.7	45.4
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	758.8	406	234.7	127	55.7
9	764.1	406.3	233.9	127.4	55.8
10	761.3	407.3	233.7	127.5	55.6
IV					
11	757.4	404	232.4	126.4	55.3

(*Continued*)

Table 2. Continued

α	0.1	0.3	0.5	0.7	0.9
ETIME	0.1111	0.4286	1	2.3333	9
12	758.4	404	232.5	126.7	55.4
13	759.7	405.7	233.1	126.7	55.3
V					
14	758	404.5	231.8	126.7	55.8
15	759	404.6	232	126.5	55.7
VI					
16	758.8	403.5	231.5	126.4	55.5
17	756.4	403.1	230.6	125.5	—

Missing numbers correspond to cases that did not converge.

lid-driven cavity flow problem with 24×24 grids under Re = 3,200, 5,000, 7,500, and 10,000, and analyze the relative errors of the present numerical results for seven systematic third-order schemes (schemes 11–17) and two second-order schemes, CD and QUICK, using with the results of Ghia et al. as the benchmark solutions. Comparisons are made for the centerline velocities of u(x direction) and v(y direction). Figure 1 shows that the relative errors of all seven third-order schemes are smaller than those of CD and QUICK for the four Re values compared. The relative error of the CD scheme can be higher by about 10–15% than those of the symmetric third-order schemes; the relative error of the QUICK scheme for u velocity is higher by about 5–7%, and that for v velocity is higher by about 2% than those for the seven third-order schemes under the most compared conditions.

4. SYMMETRIC AND ODD-ORDER SCHEME DESIGN THEORY

From the analysis and comparison of the previous sections, it can be easily seen that, when we design schemes by means of symmetric allocation of grids from two sides of the interface we can obtain a better scheme whose accuracy is higher and whose critical a_{i0} value for an absolutely stable scheme is smaller than with the existing biased second-order difference scheme, but they require the same CPU time because they actually use the same sorts and numbers of grids and have the same matrix structure. This conclusion further inspired us to consider whether a similar conclusion can be obtained for other higher-order accuracy difference schemes. The present authors have conducted such an analysis. For simplicity of presentation, the details are not shown here, and only the final analysis results are summarized in Table 5. The stencils for the schemes of four-grid, six-grid, and eight-grid are shown in Figures 2–4, respectively.

From Table 5, it is not difficult to see that, for the six-grid scheme (see its stencil in Figure 3), if we symmetrically use three grids from two sides of the interface, such as in case III in Table 5 and Figure 3 (III), we can obtain a kind of scheme whose accuracy is fifth-order and whose critical a_{i0} value for an absolutely stable scheme is 67/60. However, if we use three grids from the up flow direction of the interface and one grid from the down flow direction according to the "upwind" idea,

Table 3. Caparison of convergence characteristic and CPU time for 17 schemes in flow over a backward-facing step

α	0.1	0.3	0.5	0.7	0.9
ETIME	0.1111	0.4286	1	2.3333	9
Re = 10					
Grid 62 × 32					
0	5.6	4.1	3.5	2.6	
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	5.9	4.4	3.6	2.8	—
9	6	4.3	3.7	2.7	—
10	5.9	4.3	3.7	2.8	—
IV					
11	5.8	4.3	3.6	2.7	—
12	5.9	4.2	3.6	2.7	—
13	6	4.3	3.6	2.8	—
V					
14	5.8	4.2	3.5	2.8	—
15	6	4.3	3.6	2.8	—
VI					
16	6	4.3	3.6	2.8	—
17	5.8	4.3	3.6	2.8	—
Grid 102 × 52					
0	43.1	33.6	28.1	20.5	—
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	43.9	34.1	27.8	20.8	—
9	43.7	34	27.7	20.8	—
10	43.7	34.3	27.8	20.8	—
IV					
11	43.6	33.9	27.7	20.7	—
12	43.9	34	27.8	20.7	—
13	43.8	34.2	27.8	20.7	—
V					
14	43.8	34.3	27.7	20.7	—
15	43.8	34.1	27.8	20.8	—

(*Continued*)

Table 3. Continued

α ETIME	0.1 0.1111	0.3 0.4286	0.5 1	0.7 2.3333	0.9 9
VI					
16	43.9	34	27.7	20.7	—
17	43.9	34.3	27.8	20.7	—
Re = 150					
Grid 62 × 32					
0	7.3	4.1	3.4	2.6	1.5
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	—	—	—	—	—
9	—	—	—	—	—
10	—	—	—	—	—
IV					
11	10.8	5.3	3.7	2.7	2
12	10.8	5.3	3.7	2.7	1.5
13	11	5.1	3.7	2.7	1.6
V					
14	11	5.1	3.7	2.6	1.6
15	10.9	5.2	3.7	2.6	1.6
VI					
16	11	5.3	3.7	2.6	1.6
17	10.9	5.2	3.7	2.7	—
Grid 102 × 52					
0	57.4	33.5	26.5	20	—
I					
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
II					
4	—	—	—	—	—
5	—	—	—	—	—
6	—	—	—	—	—
III					
7	—	—	—	—	—
8	—	—	—	—	—
9	81.9	50	34.2	—	—
10	92.1	43.6	32.6	20.5	—
IV					
11	70.5	41.8	32	20.3	—
12	72.5	40	32	20.4	—
13	73.3	36.4	32	20.3	—

(*Continued*)

Table 3. Continued

α	0.1	0.3	0.5	0.7	0.9
ETIME	0.1111	0.4286	1	2.3333	9
V					
14	69.5	38.2	31.4	20.3	—
15	70.2	38.1	31.6	20.3	—
VI					
16	70.5	38.2	32	20.3	—
17	71.3	37.4	31.2	20.3	—

Missing numbers correspond to cases that did not converge.

that is, case I and Figure 3 (I), we can obtain another kind of scheme whose accuracy is just third-order and whose critical a_{i0} for an absolutely stable scheme is $11/6$, which is 1.6 times of $67/60$. If we use three grids from the up flow direction of the interface and two grids from the down flow direction, such as case II and Figure 3 (II), another kind of scheme can be obtained whose accuracy is fourth-order and whose critical a_{i0} for an absolutely stable scheme is $83/60$, which is 1.2 times $67/60$. From these results, we can easily see that in the design of a scheme, the fewer grids are taken from the down flow direction, the lower will be the accuracy and the larger will be the critical a_{i0} for an absolutely stable scheme. But all the six-grid schemes actually require the same computer memory because they have same sorts and numbers of grids. Moreover, for the eight-grid schemes of Table 5 and the stencil of Figure 4, we can obtain the same conclusion. The symmetric seventh-order difference scheme, i.e., case IV and Figure 4 (IV), has the highest accuracy and the smallest critical a_{i0} for an absolutely stable scheme; similarly, for cases I, II, and III with the eight-grid scheme and Figure 4 (I), (II), and (III), the more serious the unsymmetry of the scheme stencil, the worse is the solution accuracy and the larger is its critical value of a_{i0} for the absolutely stable scheme. However, they all use the same computer memory.

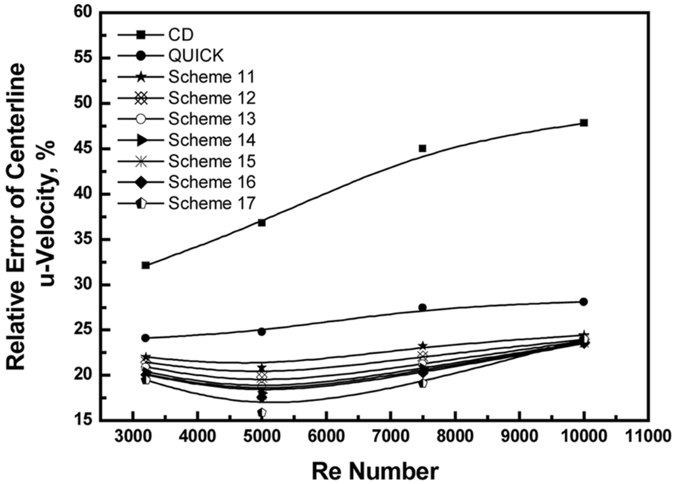
So, it is implied that, when we design a scheme, in order to obtain the best characteristics, we should use symmetric number grids from two sides of the interface and construct an odd-order difference scheme. This scheme design idea can be called symmetric and odd-order scheme design theory.

5. CONCLUSIONS

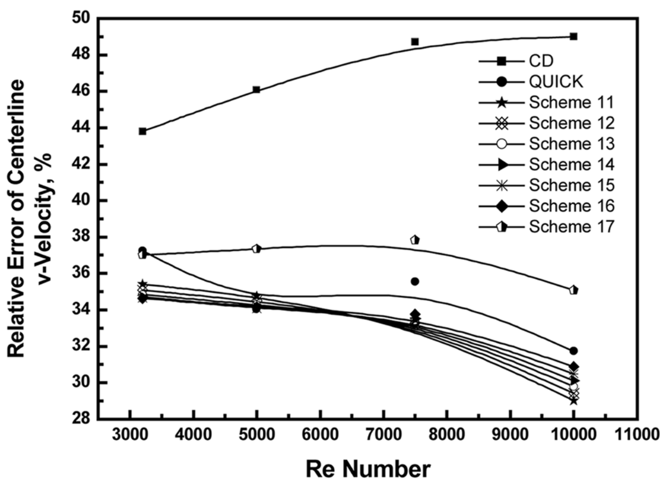
In this article, based on the research of second-order difference schemes of Part I, the companion article [1], we have further studied the design and solution

Table 4. Predicted reattachment lengths

Scheme	10	11	12	13	14	15	16	17
Grid 62×32	6.226	6.213	6.207	6.201	6.196	6.193	6.191	6.162
Grid 92×47	6.218	6.212	6.209	6.206	6.204	6.203	6.202	6.19



(a) **Relative Error of Centerline** u**-Velocity**



(b) **Relative Error of Centerline** v**-Velocity**

Figure 1. Comparison of relative error (%) of centerline u and v velocities obtained using uniform grid (24×24) under Re = 3,200–10,000.

characteristics of higher-order difference schemes. The major findings can be summarized as follows.

1. Based on the general style design concept of second-order difference schemes of Part I, a general design method of any higher-order difference scheme has been proposed.
2. Using the analysis method of the absolutely stable second-order difference scheme of Part I, we can thoroughly examine the stability characteristics of any higher-order difference scheme and design an absolutely stable scheme.

Table 5. Critical a_{i0} value for absolutely stable scheme

Scheme name	Four-grid scheme		Six-grid scheme			Eight-grid scheme			
Scheme order	Biased second-order	Third-order	Biased third-order	Biased fourth-order	Fifth-order	Biased fourth-order	Biased fifth-order	Biased sixth-order	Seventh-order
No.	I	II	I	II	III	I	II	III	IV
Interpolated grids on the east, e, surface ($u_e > 0$ and $u_e < 0$)	$\phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}$		$\phi_{i-2}, \phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}, \phi_{i+3}$			$\phi_{i-3}, \phi_{i-2}, \phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}, \phi_{i+3}, \phi_{i+4}$			
Critical a_{i0} value	3/2	13/12	11/6	83/60	67/60	125/60	97/60	559/420	953/840

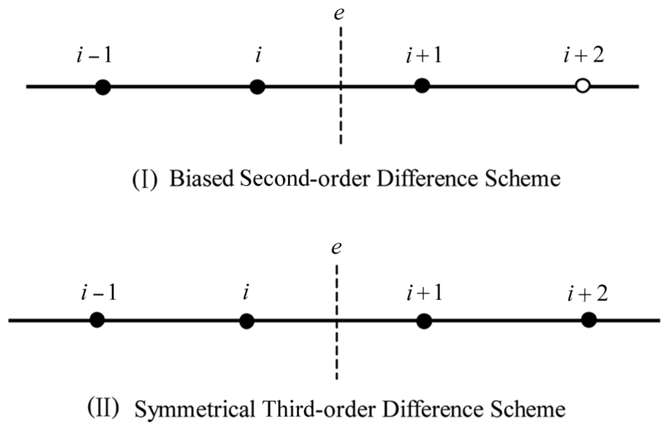


Figure 2. Scheme stencil for four-grid difference scheme ($u_e > 0$).

3. A new kind of third-order difference scheme has been constructed for which the scheme stencil is symmetric to the interface. The accuracy of this type of scheme is higher than that of the existing second-order difference scheme, and the critical a_{i0} of absolutely stable scheme is smaller than that of the existing second-order difference scheme. Moreover, the schemes require the same CPU time as the existing second-order schemes because the two kinds of schemes actually use the same kinds and numbers of grids and have the same matrix structure. Numerical examples have also shown that all the solution characteristics of this

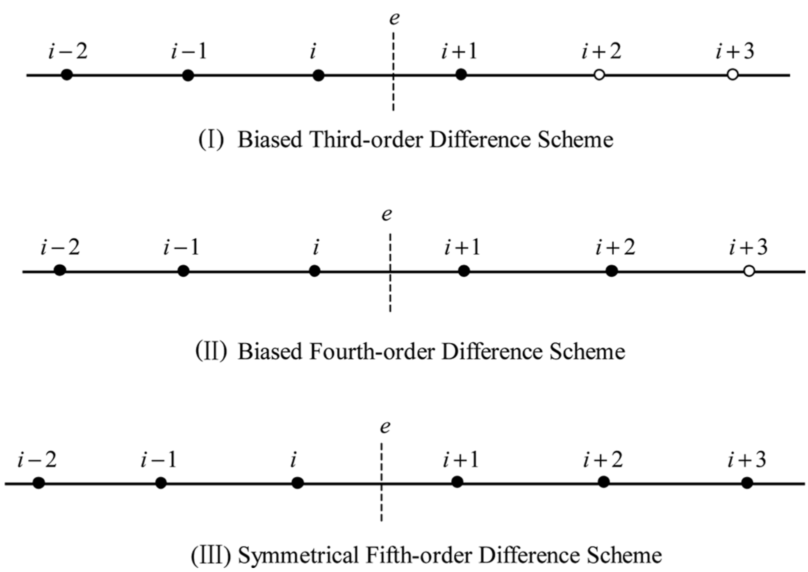


Figure 3. Scheme stencil for six-grid difference scheme ($u_e > 0$).

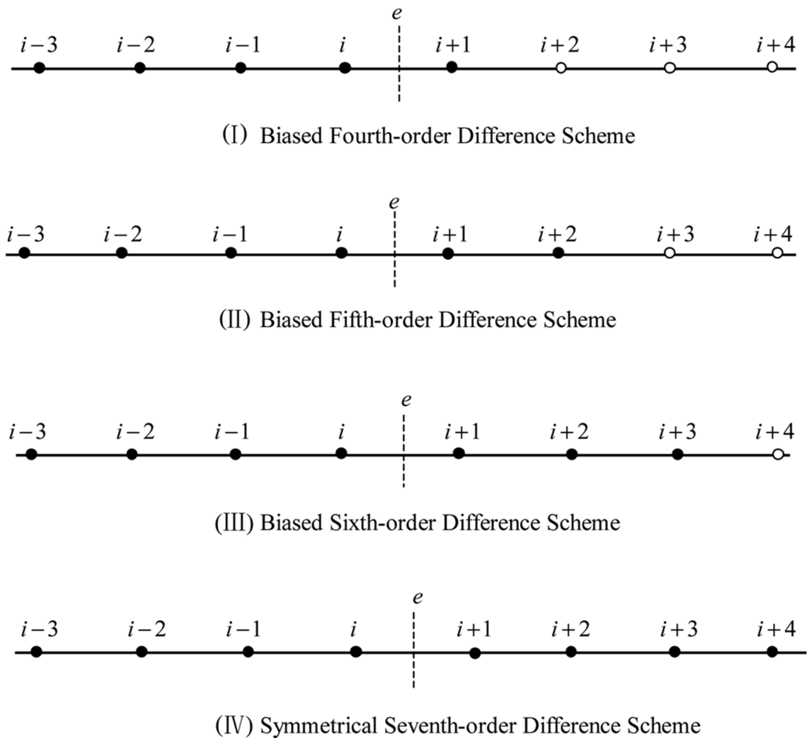


Figure 4. Scheme stencil for eight-grid difference scheme ($u_e > 0$).

kind of symmetric third-order difference scheme are better than those of the existing second-order difference scheme.

4. From the comparison and analysis, it can be stated that, under the condition of the same matrix style and computer memory, the scheme constituted by symmetric number grids from two sides of the interface with odd-order of accuracy has the highest accuracy and smallest critical a_{i0} for an absolutely stable scheme, and that this kind of scheme can maintain consistency between numerical accuracy and stability better than any kind of schemes designed according to the "upwind" idea. Based on this understanding, a new schemes design theory called symmetric position and odd-order accuracy (or simply symmetric and odd) scheme design theory is proposed.

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