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STABILITY ANALYSIS FOR DISCRETIZED STEADY CONVECTIVE-DIFFUSION EQUATION

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A comprehensive study is presented regarding the numerical stability of several common iteration numerical methods applied to the discretized steady convective-diffusion equations with some common finite-difference spatial discretizations. One-dimensional and multidimensional results are obtained using the classical Von Neumann method of stability analysis. The analysis results show that numerical stability in solving the resulting discretization equations depends on both the finite-difference scheme and the numerical method for solving the resulting algebraic equations.

1. INTRODUCTION

The steady convective-diffusion equation is usually adopted to investigate the numerical characteristics of different finite-difference schemes. It has long been recognized, especially the numerical heat transfer community, that the "convective stability" of the discretized convective-diffusion equation depends on the difference scheme, and the critical grid Peclet number is thought of as the stability criterion of a discretized convective-diffusion equation [1–3]. In this article we study the discretized convective-diffusion equations of one, two, and three dimensions in a Eulerian reference frame. Specifically, we examine the stability of the solution procedure of certain finite-difference approximations to the convective-diffusion equation:

$$\mathbf{U} \cdot \nabla\varphi = \nabla \cdot (\mathbf{K} \cdot \nabla\varphi) \tag{1}$$

Where φ is a general dependent variable, $\mathbf{U}$ is a (constant) convective velocity vector, and $\mathbf{K}$ is a (constant) diffusivity coefficient.

This study is motivated by our numerical practices. When we use some classical difference schemes and underrelaxation iteration methods for some 2-D or 3-D problems, we find that even for the central difference (CD) scheme, we can get a steady smooth solution under a large grid Peclet number if we use an appropriate underrelaxation factor. In investigating the reasons for the above

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situation, we found that we should distinguish two kinds of numerical stability, i.e., the stability of solving resulting discretized equations and the stability in simulation of the convective term. When we adopt a specific iterative method to solve a discretized convection-diffusion equation with a certain finite-difference scheme for the convective term, if the iterative procedure can obtain a stable solution (i.e., the iterative procedure converges) at any grid Peclet number, then the numerical method of iteration is considered absolutely stable. This stability will be referred to as solution stability. However, a numerically stable solution does not necessarily mean that the solution is smooth and physically realistic (not wiggled). If a discretized convection-diffusion equation can be solved to obtain a physically realistic, not wiggled solution at any grid Peclet number, then the scheme possesses absolute stability. This stability is called convective stability. In order to obtain a convectively stable solution, the necessary condition is that the iterative procedure method is stable. The critical Peclet number of solution stability differs from the critical Peclet number of convective stability.

In the following we analyze the solution stability of some iteration methods applied to solve discretized equations of several classical spatial difference schemes. The fundamental tool will be the Von Neumann method, which ignores the boundary-conditions effect yet usually can yield very useful results. In the analysis, the diffusion term is always discretized by the second-order CD scheme; hence the difference in schemes is from the convection term.

2. THE ONE-DIMENSIONAL CASE

Difference Schemes

The convective-diffusion Eq. (1) in the 1-D case can be written as follows:

$$u\frac{\partial \phi}{\partial x} = K\frac{\partial^2 \phi}{\partial x^2} \tag{2}$$

where the numerical diffusion coefficient $K(>0)$ and u are constants in analysis. Hereafter, we will assume that u is greater than zero, for simplicity. However, the results also can be applied to the case of $u<0$.

Equation (2) can be discretized in conservative form as

$$u_i\frac{\phi_{i+1/2} - \phi_{i-1/2}}{\Delta x} = K\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2} \tag{3}$$

The convective term is usually discretized by using central difference (CD), first-order upwind difference (FUD), second-order upwind difference (SUD), QUICK, and stability-controllable second-order difference (SCSD) schemes [4]. The interpolation formulas for these schemes for the interface value of ϕ are

FUD scheme: $$\phi_{i+1/2} = \phi_i \tag{4}$$

CD scheme:
$$\phi_{i+1/2} = \frac{\phi_i + \phi_{i+1}}{2} \tag{5}$$

SUD scheme:
$$\phi_{i+1/2} = \frac{3\phi_i - \phi_{i-1}}{2} \tag{6}$$

Leonard's QUICK Scheme
$$\phi_{i+1/2} = \frac{\phi_{i+1} + \phi_i}{2} + \frac{2\phi_i - \phi_{i-1} - \phi_{i+1}}{8} \tag{7}$$

SCSD scheme [4]:
$$\phi_{i+1/2} = \frac{(\phi_{i+1} - \phi_i)}{2} + (1-\beta)\frac{(2\phi_i - \phi_{i-1} - \phi_{i+1})}{2} \tag{8}$$

where β is a weighting factor ranging from 0 to 1. It has been shown [4] that the SCSD scheme becomes the CD scheme with $\beta = 1$, the QUICK scheme with $\beta = 3/4$, and the SUD scheme with $\beta = 0$.

Substituting Eqs. (4)–(8) into Eq. (3), we have the discretized formula for Eq. (2):

FUD:
$$(P_\Delta + 2)\phi_i - (P_\Delta + 1)\phi_{i-1} + \phi_{i+1} = 0 \tag{9}$$

CD:
$$4\phi_i - (2 + P_\Delta)\phi_{i-1} - (2 - P_\Delta)\phi_{i+1} = 0 \tag{10}$$

SUD:
$$(4 + 3P_\Delta)\phi_i - (2 + 4P_\Delta)\phi_{i-1} - 2\phi_{i+1} + P_\Delta\phi_{i-2} = 0 \tag{11}$$

QUICK:
$$(16 + 3P_\Delta)\phi_i - (8 + 7P_\Delta)\phi_{i-1} - (8 - 3P_\Delta)\phi_{i+1} + P_\Delta\phi_{i-2} = 0 \tag{12}$$

SCSD:
$$(4 + 3(1-\beta)P_\Delta)\phi_i - [2 + (4 - 3\beta)P_\Delta]\phi_{i-1} - (2 - \beta P_\Delta)\phi_{i+1} + (1-\beta)P_\Delta\phi_{1-2} = 0 \tag{13}$$

where $P_\Delta = u\,\Delta x/K$ is called the grid Peclet number.

The resulting algebraic equations for each scheme can be solved either by using a direct numerical method (such as TDMA) or by using iteration method, such as the simple Jacobi iteration method, the dominant diagonal-element Jacobi method, the underrelaxation method, etc. In the next part of this section, we give detailed analysis of these schemes using different numerical methods.

Von Neumann Analysis Method

Historically, two different methods of stability analysis have been applied to difference schemes. One, due to Von Neumann, is based on a Fourier mode analysis [5, 6]; the other is based on a spectral radius analysis of the amplification matrix. Article by Hindmarsh et al. [7] and Hirsch [8] are enlightening, and we believe that the Von Neumann method is more appropriate for this study.

The Von Neumann method consists of examining Fourier modes $\varphi_j^n = \xi^n e^{ijk\Delta x}$ for a wave number k and an associated wavelength $\lambda = 2\pi/|k|$. On substituting the above Fourier mode into the interior difference equations, a value for the complex amplification factor $\xi = \varphi_j^{n+1}/\phi_j^n$ is determined, which is a function of the phase angle $\theta = k\,\Delta x$. The Von Neumann stability condition requires that $|\xi| \leq 1$ [6–8]. The standard procedure is to impose the condition for all real values of k or, equivalently, for all $\theta \in [0, \pi]$; it is therefore a slightly stricter condition than that required for any given finite space point number N.

Solution Stability Analysis for Different Numerical Methods

Simple Jacobi iteration method. For Eqs. (9)–(13), we have the following formulas by using the simple Jacobi iteration method.

FUD–Jacobi:

$$\phi_i^{n+1} = \frac{P_\Delta + 1}{P_\Delta + 2}\phi_{i-1}^n + \frac{1}{P_\Delta + 2}\phi_{i+1}^n \tag{14}$$

CD—Jacobi:

$$\phi_i^{n+1} = \frac{P_\Delta + 2}{4}\phi_{i-1}^n - \frac{P_\Delta - 2}{4}\phi_{i+1}^n \tag{15}$$

SUD—Jacobi:

$$\phi_i^{n+1} = \frac{4P_\Delta + 2}{3P_\Delta + 4}\phi_{i-1}^n + \frac{2}{3P_\Delta + 4}\phi_{i+1}^n - \frac{P_\Delta}{3P_\Delta + 4}\phi_{i-2}^n \tag{16}$$

QUICK—Jacobi:

$$\phi_i^{n+1} = \frac{7P_\Delta + 8}{3P_\Delta + 16}\phi_{i-1}^n + \frac{8 - 3P_\Delta}{3P_\Delta + 16}\phi_{i+1}^n - \frac{P_\Delta}{3P_\Delta + 16}\phi_{i-2}^n \tag{17}$$

SCSD—Jacobi:

$$\phi_i^{n+1} = \frac{(4 - 3\beta)P_\Delta + 2}{3(1-\beta)P_\Delta + 4}\phi_{i-1}^n + \frac{2 - \beta P_\Delta}{3(1-\beta)P_\Delta + 4}\phi_{i+1}^n - \frac{(1-\beta)P_\Delta}{3(1-\beta)P_\Delta + 4}\phi_{i-2}^n \tag{18}$$

Using the Von Neumann analysis method, we have the amplification factor for the FUD—Jacobi as

$$\xi = \cos\theta - I\frac{P_\Delta}{P_\Delta + 2}\sin\theta \tag{19}$$

where I is the unit of imaginary number.

It is apparent that $|\xi|^2 = \cos^2\theta + [P_\Delta/(P_\Delta + 2)]^2 \sin^2\theta \le \cos^2\theta + \sin^2\theta = 1$ is valid for any grid Peclet number. Thus, we conclude that the FUD—Jacobi is absolutely stable. That means the solution stability criterion is $|P_\Delta| < \infty$.

For the CD—Jacobi, we have

$$\xi = \cos\theta - \tfrac{1}{2} I P_\Delta \sin\theta \qquad (20)$$

The Von Neumann stability condition is $|\xi|^2 = \cos^2\theta + (P_\Delta^2 \sin^2\theta)/4 \le 1$, so the solution stability criterion for the CD—Jacobi is $|P_\Delta| \le 2$.

For the SUD—Jacobi, we have the amplification factor as

$$\xi = \frac{1}{4 + 3P_\Delta}\{[(4 + 4P_\Delta)\cos\theta - P_\Delta \cos 2\theta] + I(P_\Delta \sin 2\theta - 4P_\Delta \sin\theta)\} \qquad (21)$$

When $\theta = \pi$, Eq. (21) can be reduced to $\xi = -(5P_\Delta + 4)/(3P_\Delta + 4)$. It is apparent that $|\xi|_{\theta=\pi} > 1$ for any grid Peclet number with $P_\Delta > 0$, so we can conclude that the SUD—Jacobi is unconditionally unstable.

For the SCSD—Jacobi, we have the amplification factor as

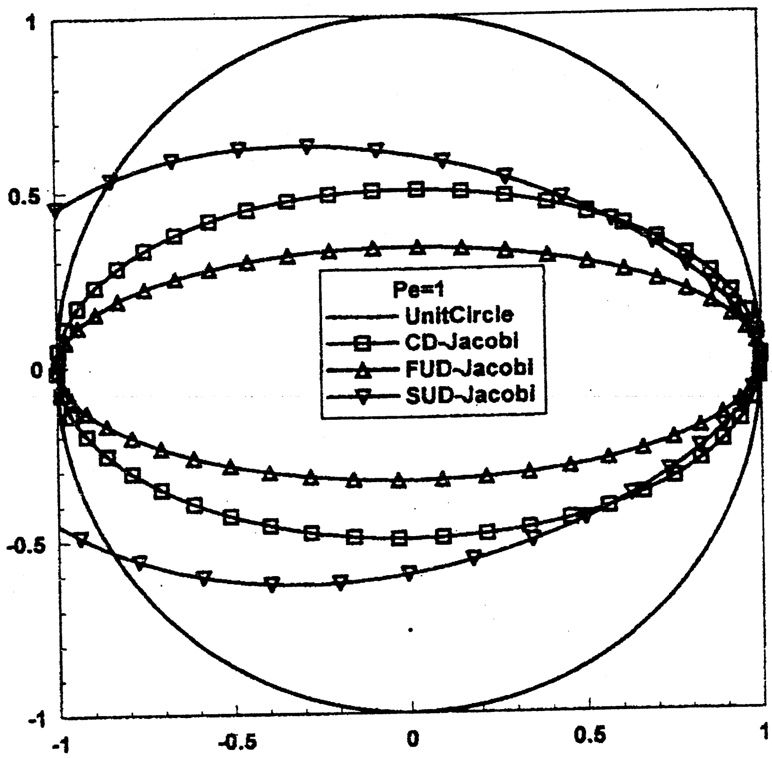
$$\xi = \frac{\{[(4 + 4(1 - \beta)P_\Delta)\cos\theta - 1(1 - \beta)P_\Delta \cos 2\theta] + I[(2\beta - 4)P_\Delta \sin\theta - (1 - \beta)P_\Delta \sin 2\theta]\}}{4 + 3(1 - \beta)P_\Delta} \qquad (22)$$

When $\theta = \pi$, we have $\xi = [5(1 - \beta)P_\Delta + 4]/[3(1 - \beta)P_\Delta + 4]$ from Eq. (22). When $\beta = 1$, i.e., for the CD—Jacobi $|\xi| = 1$, while $\beta < 1$, the amplification factor is always greater than 1. Thus we can conclude that the SCSD—Jacobi is unconditionally unstable with $\beta \in [0, 1)$.

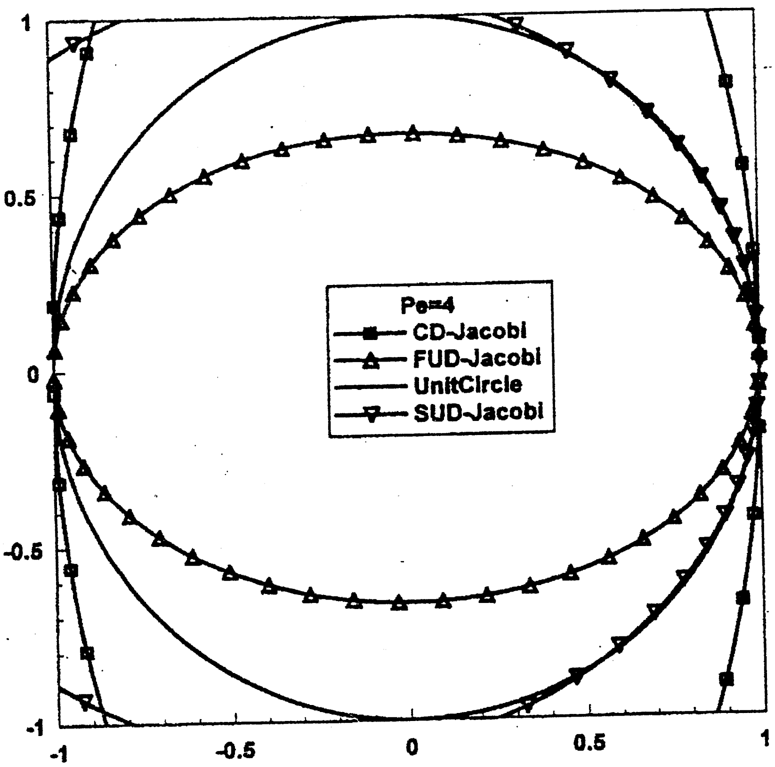
Figure 1 give the stability region for the Jacobi iteration method applied to the four schemes under different grid Peclet numbers. For the SUD—Jacobi, that there are always some points outside the unit circle under the three Peclet numbers tested, and for the CD—Jacobi we can see that the amplification factor is greater than 1 when the Peclet number is greater than 2.

Dominant diagonal-element Jacobi method. For a linear problem, the dominant diagonal-element Jacobi iteration method is stable [8], and we have shown that the Jacobi iteration method for the SUD scheme is unstable for any Peclet number. We can expect that the dominant diagonal-element Jacobi iteration will help to improve the stability of the solution procedure for different finite-difference schemes. For the above difference schemes, the dominant diagonal-element Jacobi (DDJ) methods can be written as

CD—DDJ:
$$\phi_i^{n-1} = \frac{P_\Delta + 2}{4}\phi_{i-1}^n - \frac{P_\Delta - 2}{4}\phi_{i+1}^n \qquad (23)$$



(a) $|P_\Delta| = 1$



(b) $|P_\Delta| = 4$

Figure 1. Von Neumann stability region for four schemes using the simple Jacobi iteration method: (a) $|P_\Delta| = 1$; (b) $|P_\Delta| = 4$.

SUD—DDJ:

$$\phi_i^{n+1} = \frac{4P_\Delta + 2}{4P_\Delta + 4}\phi_{i-1}^n + \frac{2}{4P_\Delta + 4}\phi_{i+1}^n + \frac{P_\Delta}{4P_\Delta + 4}(\phi_i^n - \phi_{i-2}^n) \tag{24}$$

SCSD—DDJ:

$$\phi_i^{n+1} = \frac{(4-3\beta)P_\Delta + 2}{4(1-\beta)P_\Delta + 4}\phi_{i-1}^n + \frac{2 - \beta P_\Delta}{4(1-\beta)P_\Delta + 4}\phi_{i+1}^n + \frac{(1-\beta)P_\Delta}{4(1-\beta)P_\Delta + 4}(\phi_i^n - \phi_{i-2}^n) \tag{25}$$

For the central difference scheme, the formula for the DDJ method is just the same as that for the simple Jacobi method, and the solution stability region for the CD—DDJ is also $|P_\Delta| \le 2$. For the SUD–DDJ, we can obtain the amplification factor as

$$\xi = \cos\theta + \frac{P_\Delta}{2 + 2P_\Delta}\sin^2\theta + I\frac{P_\Delta}{2 + 2P_\Delta}\sin\theta(\cos\theta - 2) \tag{26}$$

Von Neumann stability condition requires $|\xi| \le 1$ for $\theta \in [0, \pi]$. When $\theta = 0$, we should have $R \le 1$, where R is the curvature radius in the complex plane for the amplification factor. We can obtain the following expression for the curvature radius:

$$R|_{\theta=0} = \left.\frac{(\dot{x}^2 + \dot{y}^2)^{1/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}\right|_{\theta=0} = \frac{f^2}{1 - 2f} \le 1 \Rightarrow P_\Delta \le 2 - 2\sqrt{2} = 4.828 \tag{27}$$

where $x = \cos\theta + f\sin^2\theta$, $y = 0.5f\sin 2\theta - 2f\sin\theta$, and $f = P_\Delta/(2 + 2P_\Delta)$. This means the solution stability criterion for SUD–DDJ is $P_\Delta \le 2 + 2\sqrt{2} = 4.828$.

For the SCSD–DDJ, we can obtain the amplification factor as

$$\xi = \cos\theta + \frac{(1-\beta)P_\Delta}{2 + 2(1-\beta)P_\Delta}\sin^2\theta + I\frac{P_\Delta}{2 + 2(1-\beta)P_\Delta}\sin\theta[(1-\beta)\cos\theta - (2-\beta)] \tag{28}$$

By using the same analysis method as for the SUD—DDJ, we can get the solution stability criterion for the SCSD—DDJ:

$$P_\Delta \le 2\left[1 - \beta + \sqrt{1 + (1-\beta)^2}\right] \tag{29}$$

According to Eq. (29), we reach the following conclusions:

For the CD—DDJ, the solution stability region is $|P_\Delta| \le 2$ ($\beta = 1$).
For the SUD–DDJ, the solution stability region is $|P_\Delta| \le 2(1 + \sqrt{2})$ ($\beta = 0$).
For the QUICK—DDJ, the solution stability region is $|P_\Delta| \le (\sqrt{17} + 1)/2$ ($\beta = \frac{3}{4}$).

Figure 2 shows the numerical solution stability region in the complex plane for the SUD—DDJ. We can see that when the Peclet number is 16 (greater than 4.828), the amplification factor is outside the unit circle, thus it can be concluded that the SUD—DDJ is not unconditionally stable. As we indicated earlier, the critical Peclet number of solution stability is different from the critical Peclet number of convective stability. The former depends both the solution method and the scheme, while the later depends only on the scheme. Their values may also be quite different. For example, the critical Peclet number of convective stability for the SUD is infinite, while the Jacobi—SUD combination is unconditionally unstable, which means when the discretized equations with the SUD are solved by the Jacobi iteration method, the procedure will always diverge for whatever Peclet number. The convective stability is related to the important characteristics of finite-difference scheme boundedness [9], which will be discussed elsewhere.

Underrelaxation method. The underrelaxation method is widely used in numerical heat transfer. Here, we give the stability analysis of the dominant diagonal-element Jacobi iteration method with the underrelaxation method, and call it DDJ(α), where α is the underrelaxation factor. The formulas for the

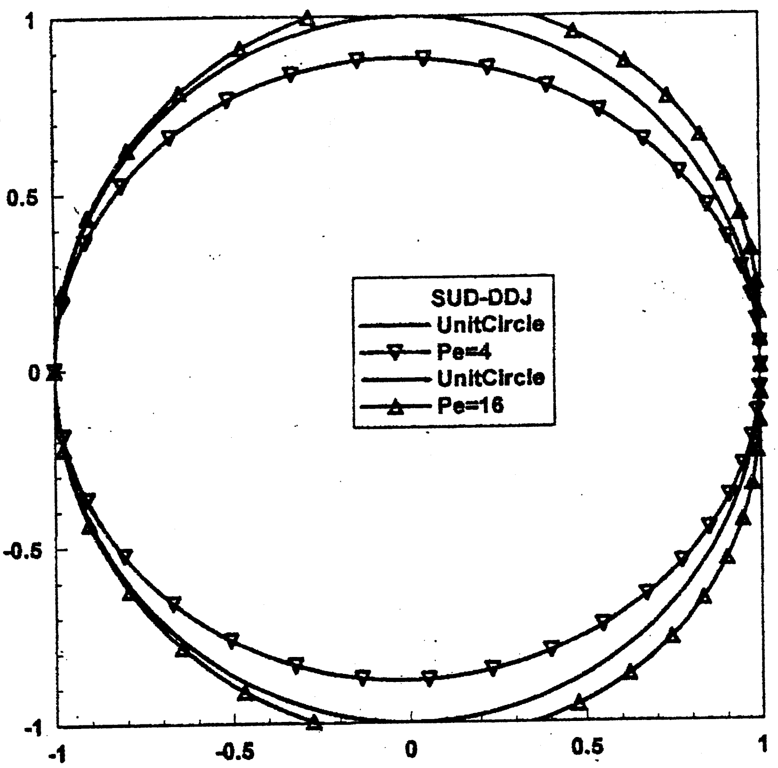


Figure 2. Von Neumann stability region for SUD—DDJ under different Peclet numbers.

different difference schemes can be written as

CD—DDJ(α):

$$\frac{1}{\alpha}\phi_i^{n+1} = \frac{P_\Delta + 2}{4}\phi_{i-1}^n - \frac{P_\Delta - 2}{4}\phi_{i+1}^n + \frac{1-\alpha}{\alpha}\phi_i^n \tag{30}$$

SUD—DDJ(α):

$$\frac{1}{\alpha}\phi_i^{n+1} = \frac{4P_\Delta + 2}{4P_\Delta + 4}\phi_{i-1}^{n+1} + \frac{2}{4P_\Delta + 4}\phi_{i+1}^n + \frac{P_\Delta}{4P_\Delta + 4}(\phi_i^n - \phi_{i-2}^n) + \frac{1-\alpha}{\alpha}\phi_i^n \tag{31}$$

SCSD—DDJ(α)

$$\frac{1}{\alpha}\phi_i^{n+1} = \frac{(4-3\beta)P_\Delta + 2}{4(1-\beta)P_\Delta + 4}\phi_{i-1}^{n+1} + \frac{2-\beta P_\Delta}{4(1-\beta)P_\Delta + 4}\phi_{i+1}^n + \frac{(1-\beta)P_\Delta}{4(1-\beta)P_\Delta + 4}(\phi_i^n - \phi_{i-2}^n) + \frac{1-\alpha}{\alpha}\phi_i^n \tag{32}$$

For the CD—DDJ(α), the amplification factor can be expressed as

$$\xi = \alpha\cos\theta + (1-\alpha) - I\frac{\alpha P_\Delta}{2}\sin\theta \tag{33}$$

The Von Neumann stability condition requires that the curvature radius at $\theta = 0$ be less than 1:

$$R|_{\theta=0} = \left.\frac{(\dot{x}^2 + \dot{y}^2)^{1/2}}{|\dot{x}\ddot{y} + \dot{y}\ddot{x}|}\right|_{\theta=0} = \frac{\alpha P_\Delta^2}{4} \le 1 \Rightarrow P_\Delta \le \frac{2}{\sqrt{\alpha}} \tag{34}$$

Furthermore, it can be easily proven that the solution stability region for the CD—DDJ(α) is $P_\Delta \le 2/\sqrt{\alpha}$.

For the SUD—DDJ(α) method the amplification factor is

$$\xi = \alpha\left(\cos\theta + \frac{P_\Delta}{2-2P_\Delta}\sin^2\theta + \frac{1-\alpha}{\alpha}\right) + I\frac{\alpha P_\Delta}{2+2P_\Delta}\sin\theta(\cos\theta - 2) \tag{35}$$

Similarly, we can show that the solution stability region for the SUD–DDJ(α) is

$$P_\Delta \le \frac{2(\sqrt{1+\alpha} + 1)}{\alpha} \tag{36}$$

The stability region for the SCSD—DDJ(α) is

$$P_\Delta \leq \frac{2\left[(1-\beta) + \sqrt{(1-\beta)^2 + \alpha}\right]}{\alpha} \tag{37}$$

For the SCSD scheme, at a given underrelaxation factor α, the stability region can be enlarged from $|P_\Delta| \leq 2/\sqrt{\alpha}$ to $|P_\Delta| \leq 2(\sqrt{1+\alpha}+1)/\alpha$ by adjusting the blended factor β. The enlarged ratio of the stability region for $|P_\Delta|$ is $(\sqrt{1+\alpha}+1)/\sqrt{\alpha}$. The less the value of α, the larger is the value of the enlarged ratio. Figure 3*a* shows a comparison of solution stability region for the CD scheme and the SUD scheme with an underrelaxation factor $\alpha = 0.25$, and Figure 3*b* shows a comparison for SUD—DDJ with or without an underrelaxation factor. It can be seen that the SUD—DDJ(α) has a bigger solution stability region than the CD—DDJ(α) [Eq. (34)] and the SUD—DDJ under the same underrelaxation factor.

A new iteration method. We have shown that the Jacobi iteration method with dominant diagonal elements can enlarge the solution stability region greatly. Here we propose a new iteration method, in which all the far-neighbor points take the values implicitly. Thus these terms must be moved from the right side of the algebraic equation to the left side, making all coefficients in the equation positive. We call this method the full positive coefficient (FPC) method. Thus we have

$$\text{SUD—FPC:}\quad (4+3P_\Delta)\phi_i^{n+1} + P_\Delta\phi_{i-2}^{n+1} = 2\phi_{i+1}^n + (2+4P_\Delta)\phi_{i-1}^n \tag{38}$$

SCSD—FPC:

$$\begin{aligned}(4+3(1-\beta)P_\Delta)\phi_i^{n+1} + (1-\beta)P_\Delta\phi_{i-2}^{n+1} \\ = (2-\beta P_\Delta)\phi_{i+1}^n + (2+(4-3\beta)P_\Delta)\phi_{i-1}^n\end{aligned} \tag{39}$$

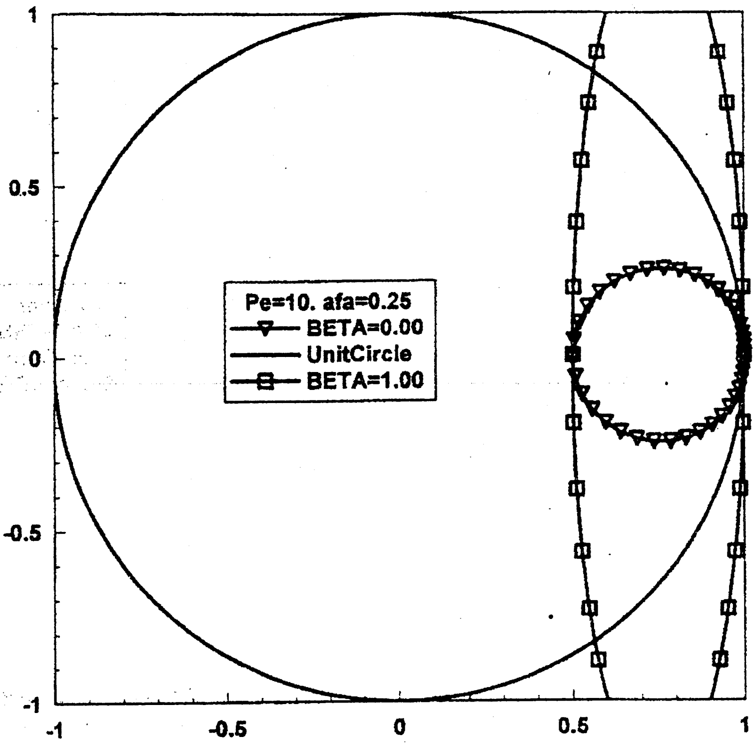
The amplification factor for Eq. (38) can be written as

$$\xi = \frac{4(1+P_\Delta\cos\theta) - I4P_\Delta\sin\theta}{4+3P_\Delta + P_\Delta\cos 2\theta - IP_\Delta\sin 2\theta} \tag{40}$$

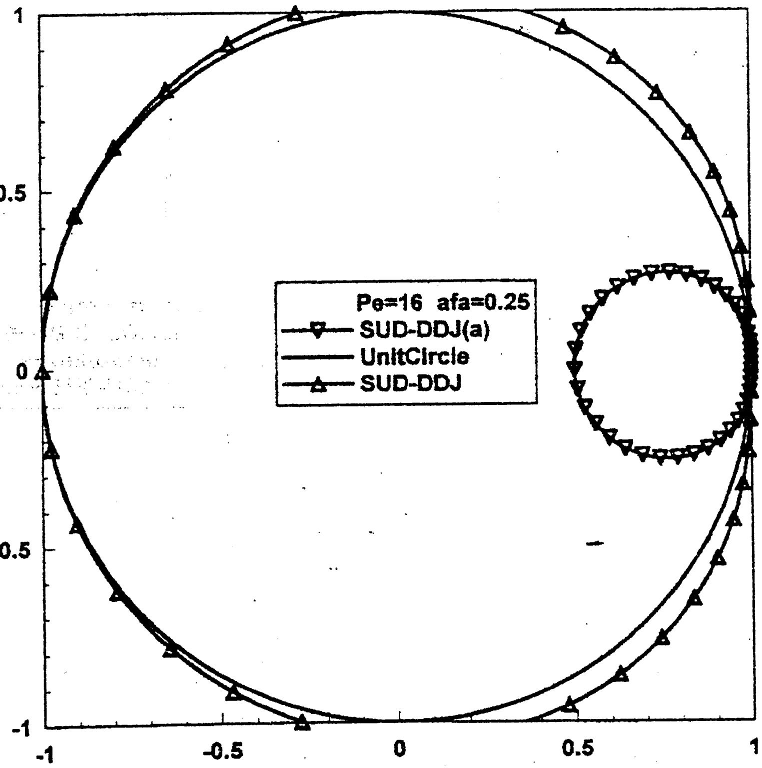
The Von Neumann stability condition requires $|\xi| \leq 1$ for all $\phi \in [0, \pi]$. At $\phi = \pi/2$, we have $(4P_\Delta)/(4+2P_\Delta) \leq 1 \Rightarrow |P_\Delta| \leq 2$, and we can prove that the solution stability condition for the SUD—FPC is $|P_\Delta| \leq 2$ according to $R|_{\phi=0} \leq 1$. Similarly, the solution stability condition for the SCSD—FPC is $|P_\Delta| \leq 2$. For the SUD scheme, the representation of the solution stability region for the FPC method is shown in Figure 4.

For the FUD and CD schemes, the FPC method is the same as the Jacobi method. The solution stability region for the FUD—FPC is $|P_\Delta| \leq \infty$; for the CD—Jacobi method it is $|P_\Delta| \leq 2$.

TDMA method. For a three-point scheme, the TDMA is one kind of direct algorithm method, but for a scheme with more than three points, the TDMA is one kind of implicit iteration method. The formula for the TDMA method can be



(*a*) For CD, SUD Scheme under $|P_\Delta| = 10$



(*b*) For SUD Scheme under $|P_\Delta| = 16$

Figure 3. Von Neumann stability region for CD and SUD with DDJ(α): (*a*) for CD, SUD under $|P_\Delta| = 10$; (*b*) for SUD under $|P_\Delta| = 16$.

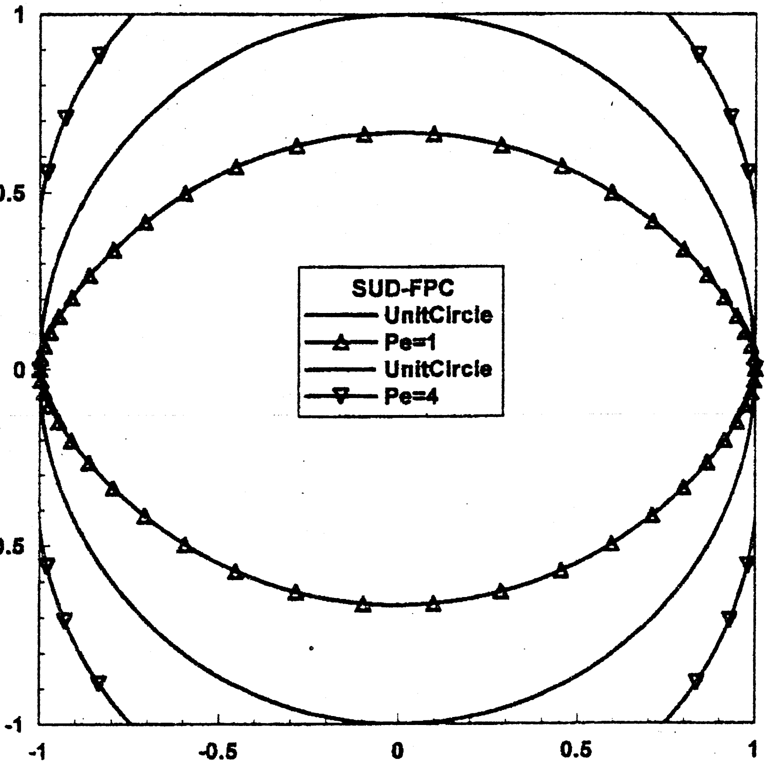


Figure 4. Representation of the stability region for the FPC method.

expressed as

$$\gamma_{-1}\phi_{i-1}^{n+1} + \gamma_0\phi_i^{n+1} + \gamma_{+1}\phi_{i+1}^{n+1} = s^n \tag{41}$$

where $\gamma_i (i = -1, 0, 1)$ and s depend on the scheme. For the CD, SUD, SCSD, and FUD schemes, Eq. (41) leads to:

$$\text{FUD—TDMA:} \quad (1 + P_\Delta)\phi_{i-1}^{n+1} - (2 + P_\Delta)\phi_i^{n+1} + \phi_{i+1}^{n+1} = 0 \tag{42}$$

$$\text{CD—TDMA:} \quad (2 + P_\Delta)\phi_{i-1}^{n+1} - 4\phi_i^{n+1}(2 - P_\Delta)\phi_{i+1}^{n+1} = 0 \tag{43}$$

SUD—TDMA:

$$(2 + 4P_\Delta)\phi_{i-1}^{n+1} - (4 + 3P_\Delta)\phi_i^{n+1} + 2\phi_{i+1}^{n+1} = P_\Delta\phi_{i-2}^n \tag{44}$$

SCSD—TDMA:

$$[2 + (4 - 3\beta)P_\Delta]\phi_{i-1}^{n+1} - [4 + 3(1 - \beta)P_\Delta]\phi_i^{n+1} + (2 - \beta P_\Delta)\phi_{i+1}^{n+1}$$
$$= (1 - \beta)P_\Delta\phi_{i-2}^n \tag{45}$$

For three-point schemes (FUD and CD), due to the linearization hypothesis for the Von Neumann stability analysis method, it is in fact a kind of direct algorithm method, and therefore it is unconditionally stable. For the SUD—TDMA method the amplification factor from Von Neumann analysis can be written as

$$\xi = \frac{P_\Delta \cos 2\theta - IP_\Delta \sin 2\theta}{4(1 + P_\Delta)\cos\theta - (4 + 3P_\Delta) - I4P_\Delta \sin\theta} \tag{46}$$

It is easy to prove that the requirement $|\xi| \leq 1$ is valid for any grid Peclet number, in other words, that the SUD—TDMA is unconditionally stable. Figure 5 shows that the points are all in the unit circle for the SUD—TDMA method, even when $|P_\Delta| = 10{,}000$.

By using similar analysis, we can prove that the SCSD–TDMA is also unconditionally stable.

Deferred-correction method [10, 11]. In the deferred-correction method, the interface value for the CD scheme is defined as

$$\phi_{i+1/2} = \phi_{i+1/2}^{\mathrm{FUD}} + \left(\phi_{i+1/2}^{\mathrm{CD}} - \phi_{i+1/2}^{\mathrm{FUD}}\right)^{\mathrm{OLD}} \tag{47}$$

Consider the one-dimensional convective-diffusion equation:

$$u\,\frac{\partial \phi}{\partial x} = \varepsilon\,\frac{\partial^2 \phi}{\partial x^2} \tag{2}$$

Using the deferred-correction method, we have

$$u_i \frac{\phi_i - \phi_{i-1}}{\Delta x} + u_i\left(\frac{\phi_{i+1} - \phi_{i-1}}{2\,\Delta x} - \frac{\phi_i - \phi_{i-1}}{\Delta x}\right)^{\mathrm{OLD}} = \varepsilon\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2} \tag{48}$$

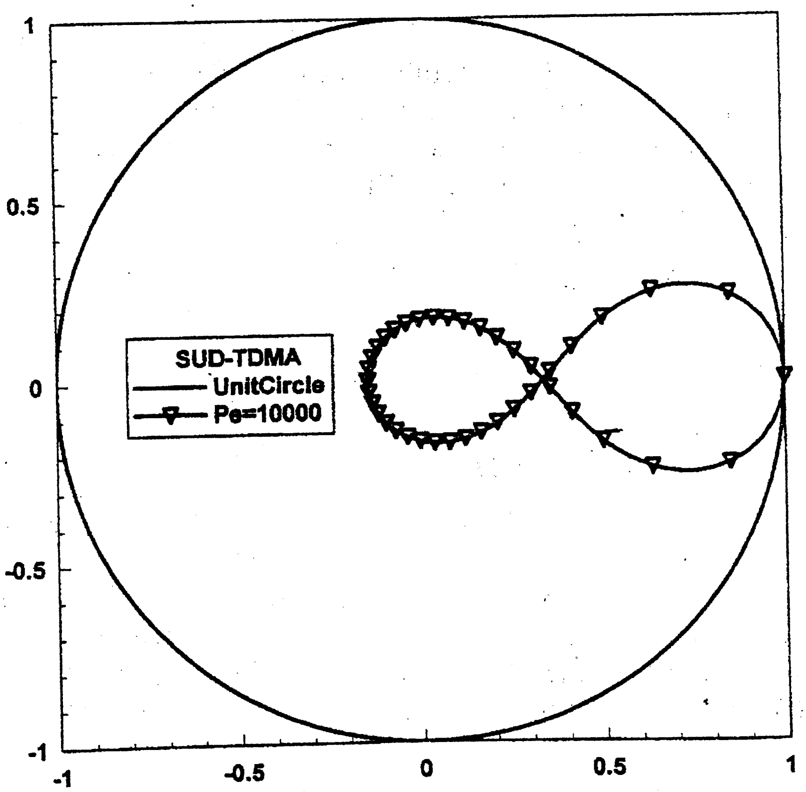


Figure 5. Representation for the TDMA method.

The iteration process for the deferred-correction method can be expressed as

$$\phi_i^{n+1} = \frac{P_\Delta + 1}{P_\Delta + 2}\phi_{i-1}^n + \frac{1}{P_\Delta + 2}\phi_{i+1}^n - \frac{P_\Delta}{P_\Delta + 2}\,\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{2} \tag{49}$$

The amplification factor for Eq. (49) can be found as

$$\xi = \cos\theta + \frac{2P_\Delta}{2 + 2P_\Delta}\sin^2\theta - I\frac{P_\Delta}{2 + P_\Delta}\sin\theta \tag{50}$$

According to the requirement for stability of the Von Neumann analysis,

$$R|_{\theta=0} = \left.\frac{(\dot{x}^2 + \dot{y}^2)^{1/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}\right|_{\theta=0} = \frac{\alpha P_\Delta^2}{2(P_\Delta + 2)} \leq 1 \Rightarrow P_\Delta \leq (1 + \sqrt{5}) \tag{51}$$

where

$$x = \cos\theta + \frac{2P_\Delta}{2 + 2P_\Delta}\sin^2\theta \qquad y = \frac{P_\Delta}{2 + P_\Delta}\sin\theta$$

This means that the solution stability region for the deferred-correction method applied to the CD scheme is $P_\Delta \leq (1 + \sqrt{5})$. When the numerical solution approaches a steady value, the above deferred-correction method can reach second-order accuracy results just like the CD scheme, and the deferred-correction approach can improve the solution stability region compared to the CD—Jacobi method. When connected with the underrelaxation method, the solution stability region can be improved even more. Lilek et al. [10] use the method to solve a two-dimensional flow field by using the CD scheme, and they get a converged solution with grid Peclet number less than 100.

Similar analysis may be conducted for other combinations, such as QUICK—deferred correction, and quantitatively the same conclusion may be obtained.

3. STABILITY ANALYSIS FOR MULTIDIMENSIONAL CONVECTIVE-DIFFUSION EQUATIONS

The solution stability analysis results for the one-dimensional case can give some instruction about multidimensional problem. Hindmarsh et al. [7] give the stability analysis of time-marching methods by using the FTCS scheme. In this section, for the multidimensional convective-diffusion problem, the Von Neumann analysis method is used to analyze the solution stability of three-point schemes, such as the CD scheme and the FUD scheme, and the necessary and sufficient stability condition is presented for the CD—Jacobi method, the FUD—Jacobi method, and the deferred-correction approach.

Consider a multidimensional convective-diffusion system:

$$\sum_{m=1}^{M} u_m \frac{\partial \varphi}{\partial x_m} = \sum_{m=1}^{M} K_m \frac{\partial^2 \varphi}{\partial x_m^2} \tag{52}$$

where M is the number of spatial dimensions. By using the central difference scheme, Eq. (52) can be discretized as

$$\sum_{m=1}^{M} u_m \frac{\Delta_m \varphi_j}{2\,\Delta x_m} = \sum_{m=1}^{M} K_m \frac{\delta_m^2 \varphi_j}{(\Delta x_m)^2} \tag{53}$$

where $\Delta_m \varphi_j/(2\,\Delta x_m)$ and $\delta_m^2 \varphi_j/(\Delta x_m)^2$ represent central difference schemes of first-order and second-order derivatives, respectively. If the Von Neumann stability analysis method is performed for the Jacobi iteration method of Eq. (53), we have

$$\varphi_j^n = \xi^n \exp\left(I \sum_{m=1}^{M} j_m \theta_m \right) \tag{54}$$

The amplification factor for the CD—Jacobi method can be written as

$$\xi = 1 - \sum_{m=1}^{M} \alpha_m (1 - \cos\theta_m) - I \sum_{m=1}^{M} c_m (1 - \sin\theta_m) \tag{55}$$

where

$$a_m = \frac{K_m/(\Delta x_m)^2}{\sum_{i=1}^{M} K_i/(\Delta x_i)^2} \qquad c_m = \frac{u_m/\Delta x_m}{\sum_{i=1}^{M} K_i/(\Delta x_i)^2}$$

Hindmarch et al. [7] have proved that, for Eq. (55), the necessary and sufficient condition for $|\xi| \le 1$ is

$$\sum_{m=1}^{M} \alpha_m \le 1 \qquad \sum_{m=1}^{M} \frac{c_m^2}{\alpha_m} \le 1 \tag{56}$$

In Eq. (56), the requirement of $\sum_{m=1}^{M} \alpha_m \le 1$ is met automatically. Thus the necessary and sufficient solution stability condition of the CD—Jacobi method for the multidimensional convective-diffusion equation is

$$\sum_{m=1}^{M} \frac{c_m^2}{\alpha_m} \le 1 \qquad \text{i.e.,} \qquad \sum_{m=1}^{M} \frac{u_m^2}{4K_m} \le \sum_{m=1}^{M} \frac{K_m}{(\Delta x_m)^2} \tag{57}$$

For Eq. (57), if $\Delta x_m = t_m \Delta x$, $k_m = K$ are met for all value of m, we have

$$\frac{\sum_{m=1}^{M} u_m^2}{M} \frac{(\Delta x)^2}{K^2} \le \frac{\sum_{m=1}^{M} t_m^{-2}}{M} \tag{58}$$

Let

$$\bar{u}^2 = \frac{\sum_{m=1}^{M} u_m^2}{M} \qquad \bar{P}_\Delta = \frac{\bar{u}\,\Delta x}{K}$$

Then we have

$$|\bar{P}_\Delta| \le 2\sqrt{M^{-1} \sum t_m^{-2}} \tag{59}$$

where $\bar{u}$ is an averaged velocity and $\bar{P}_\Delta$ is the averaged grid Peclet number for the multidimensional convective-diffusion equation. Equation (59) reflects the same essence as in the one-dimensional case.

For the deferred-correction method applied to the CD scheme, we have

$$\sum_{m=1}^{M} u_m \frac{\Delta_m \varphi_j}{2\,\Delta x_m} = \sum_{m=1}^{M} \left(K_m + \frac{|u_m|}{2} \Delta x_m \right) \frac{\delta_m^2 \varphi_j}{\Delta x_m^2} - \left[\sum_{m=1}^{M} \left(\frac{|u_m|}{2} \Delta x_m \right) \frac{\delta_m^2 \varphi_j}{\Delta x_m^2} \right]^{\mathrm{OLD}} \tag{60}$$

where the Jacobi iteration with dominant diagonal elements can be formed from the first two terms. By using the Von Neumann method, we have the amplification factor for the CD—deferred-correction combination in the multidimensional case as

$$\xi = 1 - \sum_{m=1}^{M} \alpha_m (1 - \cos\theta_m) - I \sum_{m=1}^{M} c_m (1 - \sin\theta_m) \tag{61}$$

where

$$\alpha_m = \frac{K_m/(\Delta x_m)^2}{\sum_{i=1}^{M}(K_i + |u_i|/2)/(\Delta x_i)^2} \qquad c_m = \frac{u_m/(2\,\Delta x_m)}{\sum_{i=1}^{M}(K_i + |u_i|/2)/(\Delta x_i)^2}$$

It can be seen that $\sum \alpha_m \le 1$ is also met automatically. In fact, we have $\sum \alpha_m \le 1, \forall u_i \ne 0$. Thus the necessary and sufficient solution stability condition for the deferred-correction method in the multidimensional case can be written as

$$\sum_{m=1}^{M} \frac{c_m^2}{\alpha_m} \le 1, \qquad \text{i.e.,} \qquad \sum_{m=1}^{M} \frac{u_m^2}{2K_m} \le \sum_{m=1}^{M} \frac{2K_m + |u_m|\,\Delta x_m}{(\Delta x_m)^2} \tag{62}$$

For the FUD scheme, we can get the solution stability for the FUD—Jacobi as

$$\sum_{m=1}^{M} \frac{u_m^2}{2K_m + |u_m| \Delta x_m} \leq \sum_{m=1}^{M} \frac{2K_m + |u_m| \Delta x_m}{(\Delta x_m)^2} \tag{63}$$

It is easy to find that the solution stability region for the FUD—Jacobi is bigger than for the CD—Jacobi and CD—deferred-correction combinations, while the CD—deferred-correction has a bigger region than the CD—Jacobi. If combined with the underrelaxation method, the solution stability region will be improved even more. The results of Lilek et al. [10] give strong support for the analytical result.

4. NUMERICAL EXAMPLE

Consider a one-dimensional linear convective-diffusion model equation:

$$\frac{\partial}{\partial x}(u\phi) = \frac{\partial}{\partial x}\left(K \frac{\partial \phi}{\partial x}\right) \tag{64a}$$

where u is a constant, and the boundary conditions can be expressed as

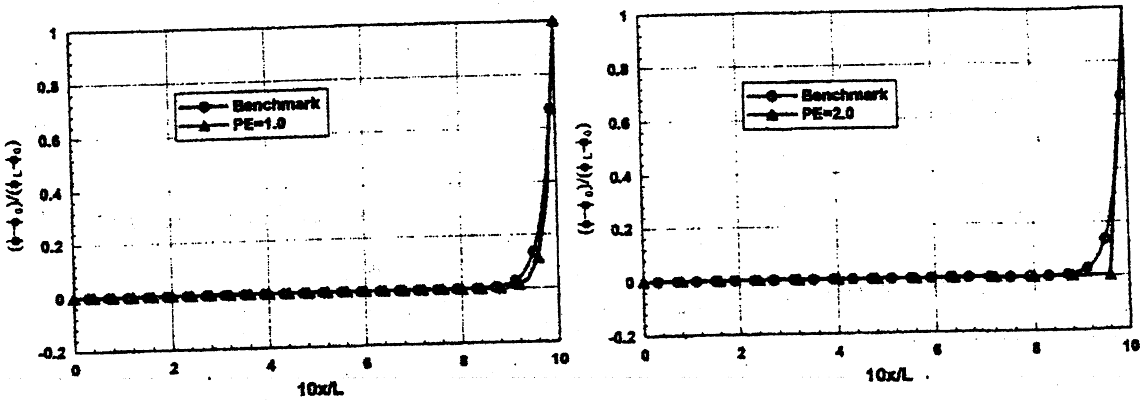
$$x = x_0 \qquad \phi = \phi_0 \qquad x = L \qquad \phi = \phi_L \tag{64b}$$

The analytical solution for Eq. (64) can be expressed as

$$\frac{\phi - \phi_0}{\phi_L - \phi_0} = \frac{\exp(ux/K) - 1}{\exp(uL/K) - 1} \tag{65}$$

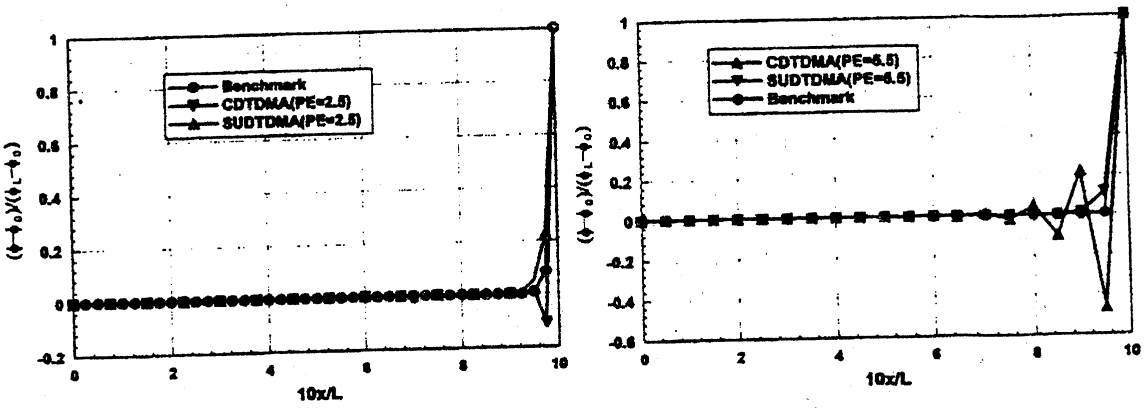
In this section, we use the CD—TDMA (tri-diagonal matrix algorithm), CD—DDJ, CD—DDJ(α), and SUD—DDJ algorithms to simulate the equation at different Peclet numbers.

Numerical results are shown in Figure 6. Figure 6a shows the results using the CD—DDJ algorithms with $|P_\Delta| = 1.0$ and $|P_\Delta| = 2.0$. Figure 6b shows the results using the CD—TDMA and SUD—TDMA with $|P_\Delta| = 2.5$ and $|P_\Delta| = 5.5$. Figure 6c shows the results using the CD—DDJ(α) ($\alpha = 0.1$ and $\alpha = 0.25$) with $|P_\Delta| = 2.5$ and $|P_\Delta| = 5.5$. We can find that the CD—DDJ can obtain a stable oscillation-free (convectively stable) solution when the Peclet number is less than 2. When the Peclet number is greater than 2, the CD—DDJ method cannot obtain a stable solution, while the CD—TDMA method and the CD—DDJ(α) with $\alpha = 0.1$ and $\alpha = 0.25$ can obtain a stable oscillatory solution (i.e., the iterative procedure does not diverge), and the SUD—TDMA and SUD—DDJ algorithms can obtain stable oscillation-free results. We can find also that the oscillation degree for $\alpha = 0.1$ is the same as that for $\alpha = 0.25$ by using the CD scheme, which means that the underrelaxation method can expand the solution stability region, but not the convective stability region. The results show clearly the difference between solution



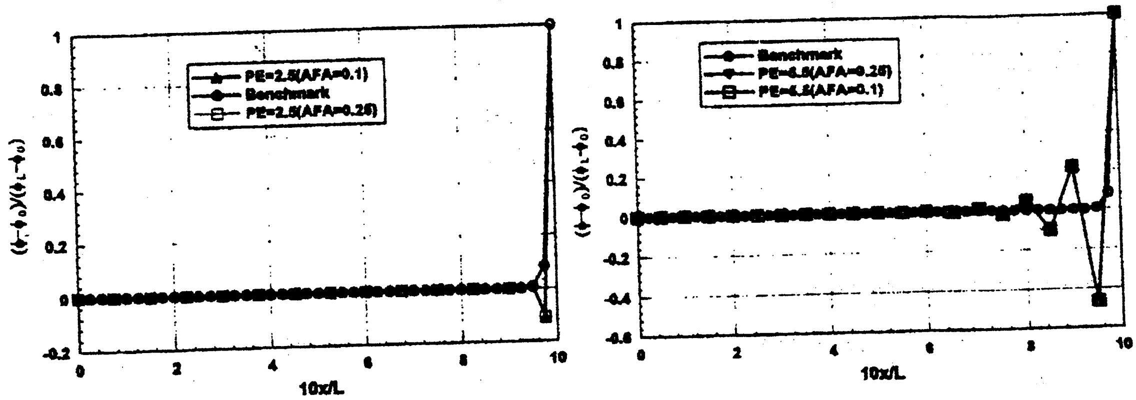
$P_\Delta = 1.0$ $P_\Delta = 2.0$

(*a*) Numerical Solution by Using CD-DDJ



$P_\Delta = 2.5$ $P_\Delta = 5.5$

(*b*) Numerical Solution by Using CD-TDMA & SUD-TDMA



$P_\Delta = 2.5$ $P_\Delta = 5.5$

(*c*) Numerical Solution by Using CD-DDJ(α) $(\alpha = 0.25 \ \& \ 0.1)$

Figure 6. Numerical results for the 1-D linear convective-diffusion equation: (*a*) numerical solutions using CD—TDMA, SUD—TDMA, and CD—DDJ; (*b*) numerical solutions using CD—TDMA and SUD—TDMA; (*c*) numerical solutions using CD—DDJ (α = 0.25 and 0.1).

stability and convective stability. Thus, the TDMA method, the Jacobi iteration method with dominant diagonal elements, the underrelaxation method, and the deferred-correction approach can expand the solution stability region greatly, but they cannot change the convective stability region of any discretization scheme of convective terms.

5. CONCLUSIONS

The Von Neumann stability analysis method has been used with a number of iterative solution methods applied to some finite-difference schemes. The numerical solution methods include Jacobi, dominant diagonal-element Jacobi, FPC, TDMA, underrelaxation, and deferred-correction methods. The major findings of this article can be summarized as follows:

1. The numerical stability of the solution procedure depends on both the difference scheme and the numerical solution method for the discretized equation. The stability of the solution procedure reflects the character of the numerical error propagation in the solution process. It gives the restriction on the space step for a given solution method applied to a given scheme, just like the initial stability for time-marching method provides the restriction on the time step. The regions of solution stability for different combinations of difference scheme and solution method analyzed in this article are summarized in Table 1.

2. For multidimensional linear convective-diffusion equations, the Von Neumann stability analysis has been performed for the CD—Jacobi, FUD—Jacobi, and CD—deferred-correction methods. The necessary and sufficient solution stability conditions for these methods are

CD—Jacobi:
$$\sum_{m=1}^{M} \frac{u_m^2}{2K_m} \leq \sum_{m=1}^{M} \frac{2K_m}{(\Delta x_m)^2}$$

FUD—Jacobi:
$$\sum_{m=1}^{M} \frac{u_m^2}{2K_m + |u_m|\Delta x_m} \leq \sum_{m=1}^{M} \frac{2K_m + |u_m|\Delta x_m}{(\Delta x_m)^2}$$

Table 1. Solution stability regions for different combination of method and scheme

No.	Combination of scheme and method	Solution stability region
1	FUD—Jacobi	$\lvert P_\Delta\rvert < \infty$
2	DS/FUD/SUD/QUICK/SCSD—TDMA	$\lvert P_\Delta\rvert < \infty$
3	CD—Jacobi, CD/FUD/QUICK/SCSD—FPC	$\lvert P_\Delta \leq 2$
4	SUD/QUICK/SCSD($\beta \leq 1$)—Jacobi	Unconditionally unstable
5	SUD—DDJ	$\lvert P_\Delta\rvert \leq 2(1+\sqrt{2})$
6	SCSD—DDJ with underrelaxation	$\lvert P_\Delta\rvert \leq \dfrac{2\left[1-\beta+\sqrt{(1-\beta)^2+\alpha}\right]}{\alpha}$
7	CD—deferred-correction	$\lvert P_\Delta\rvert \leq 1+\sqrt{5}$

CD—deferred-correction:

$$\sum_{m=1}^{M} \frac{u_m^2}{2K_m} \le \sum_{m=1}^{M} \frac{2K_m + |u_m| \Delta x_m}{(\Delta x_m)^2}$$

3. The solution stability is different from the convective stability of the convection term. The latter depends only on the discretization scheme itself and is related in some sense with the boundedness of a finite-difference scheme of the convection term.

The relationship between the convective stability and boundedness and the reasons why practical computations using conditionally stable difference schemes often can yield physically realistic (not wiggled) solutions with grid Peclet number much larger than their critical values obtained from the 1-D model equation are now underway in the author's group and will be reported elsewhere.

REFERENCES

1. S. V. Patankar, *Numerical Heat Transfer and Fluid Flow*, McGraw-Hill, New York, 1980.
2. T. M. Shih, *Numerical Heat Transfer*, Hemisphere, Washington, DC, 1984.
3. W. Q. Tao and E. M. Sparrow, The Transportive Property and Convective Numerical Stability of the Steady State Convection-Diffusion Equation, *Numer. Heat Transfer*, vol. 11, pp. 491–497, 1987.
4. M. J. Ni, W. Q. Tao, and S. J. Wang, Stability-Controllable Second-Order Difference Scheme for Convective Term, *J. Thermal Sci.*, in press.
5. R. Vichnevetsky and J. B. Bowles, *Fourier Analysis of Numerical Approximations of Hyperbolic Equations*, SIAM Publications, Philadelphia, 1982.
6. K. W. Morton, Initial Value Problems by Finite Difference and other Methods, in D. A. H. Jacobs (ed.), *The State of the Art in Numerical Analysis*, Academic Press, New York, pp. 699—756, 1977.
7. A. C. Hindmarsh and P. M. Grisho, The Stability of Explicit Euler Time-Integration for Certain Finite Difference Approximations of the Multi-Dimensional Advection-Diffusion Equations, *Int. J. Numer. Meth. Fluids*, vol. 4, pp. 853–897, 1984.
8. C. Hirsch, *Numerical Computation of Internal & External Flows, Volume 1, Fundamental of Numerical Discretization*, Wiley, New York, 1988.
9. P. H. Gaskell and A. K. C. Lau, Curvature-Compensated Convective Transport: SMART, A New Boundedness Preserving Transport Algorithm, *Int. J. Numer. Meth. Fluids*, vol. 8, pp. 617–641, 1988.
10. Z. Lilek, S. Muzaferija, and M. Peric, Efficiency and Accuracy Aspects of a Full-Multigrid SIMPLE Algorithm for Three-Dimensional Flows, *Numer. Heat Transfer B*, vol. 31, pp. 23–42, 1977.
11. P. K. Khosla and S. G. Rubin, A Diagonally Dominant Second-Order Accurate Implicit Scheme, *Comput. Fluids*, vol. 2, pp. 207–209, 1974.