The implementation of CLEAR algorithm for incompressible fluid flow and heat transfer on a non-staggered grid

Zhiguo Qu       Wenquan Tao      Yaling HE
State Key Laboratory of Multiphase Flow in Power Engineering
School of Energy & Power Engineering
Xi’an Jiaotong University, Xi’an, Shaanxi 710049, P.R.China
*Corresponding author: wqtao@mail.xjtu.edu.cn

ABSTRACT
In this paper, a detailed implementation procedure of CLEAR algorithm on a non-staggered grid system are formulated. Discussion is made on the method to predict velocity and pressure in a unique manner that is independent of relaxation factor. In order to verify the performance of CLEAR algorithm, four numerical examples on non-staggered grid of forced convective fluid flow and natural convection with available solutions are provided to compare the convergence performance between the CLEAR and SIMPLER. It is shown that on the collocated grid, the CLEAR algorithm can also greatly enhance the convergence rate based on the iteration number and the corresponding consumed CPU time compared with the SIMPLER algorithm with similar robustness.

INTRODUCTION
The pressure correction method that belongs to segregated algorithm is an approach widely employed for numerically solving the Navier-Stokes equations for fluid flow and heat transfer problems. The SIMPLE algorithm proposed by Patankar and Spalding (1972) is the first such algorithm widely used in literature. Since then several variants are reported to enhance its convergence rate, among whom are SIMPLE (Patankar, 1981), SIMPLEC (Van Doormal and Raithby, 1984), SIMPLEX (Van Doormal and Raithby, 1985, Raithby and Schneider, 1988), PISO (Issa, 1985), SIMPLET (Sheng and Shoukri, 1998), and MSIMPLER (Yu et al., 2001). In addition, Yen and Liu (1993) presented the explicit correction step method to accelerate the convergence rate. All the above-mentioned methods are usually named SIMPLE-like or SIMPLE-family algorithm. Moukalled and Darwish (2000) made a comprehensive review and given a unified reorganization expression form for all the pressure correction algorithms. The common characteristic of these algorithms is that a pressure correction term is introduced to the calculation procedure of each segregated solution step to improve the velocity that is solved from momentum-discretized equation as the intermediate value. The pressure correction term in the SIMPLE-family algorithms is used to improve the intermediate velocity solved from the momentum equation such that the modified velocity can satisfy the mass conservation condition for each control volume at each iteration level, that is critical for iteration convergence as indicated by Blosch and Shyy (1993). In the derivation of pressure correction equation, the effects of the pressure corrections of the neighboring grid points are neglected that will not affect the final solutions when the iterative process converges (Patankar, 1980), but does affect the convergence rate (Shyy and Mittal, 1998). Because of this neglecting, all the above-mentioned algorithms are of semi-implicit type.

Very recently, the present authors (Tao et al., 2004) proposed a fully implicit segregated algorithm CLEAR (Couple and Linked Equations Algorithm Revised) for incompressible fluid flow and heat transfer. The CLEAR algorithm avoids introducing the pressure correction term and velocity correction term to discard the basic assumption of the SIMPLE series algorithm. The new algorithm is tested with six examples on staggered grid system, and it is proved that it can enhance convergence rate to a great extent.

In this study, the CLEAR algorithm is extended to collocated grid system to test its feasibility. To fulfill the fundamental requirement for eliminating the decoupling problem between velocity and pressure and making the solution to be independent of under-relaxation factor, detailed investigation is made on the previous related methods. For some mildly irregular computation domain, the domain extension method (Tao, 2001) is applied, and is further formulated in this study to meet the special requirement on the collocated grid to treat the solid region in the computational domain. Based on the above preliminary research, the calculation procedures of SIMPLER and CLEAR on collocated grids are presented for comparison. Six 2-D numerical examples with available solution are computed to investigate the performance of the CLEAR and SIMPLER algorithm on collocated grid system.

Discussion on Momentum Interpolation
The staggered grid is widely employed in CFD/NHT literatures since it can efficiently guarantee the coupling between velocity and pressure. However it shows its inconvenience of complicity for the code
development in unstructured grid and curvilinear body-fitted grid, especially for 3-D computation. While on the non-staggered grids, such complicity can be greatly alleviated. The crucial issue in using non-staggered grids is how to eliminate the decoupling between pressure and velocity. In 1980’s the momentum interpolation method (MIM) on non-staggered grid is first presented by Rhie and Chow (1983) to prevent the decoupling problem. It is subsequently reformulated by Peric (1985) and Majumdar (1986). Later, Majumdar (1988) and Miller and Schmidt (1988) pointed out that the Rhie and Chow’s method has the weakness that the solution is under-relaxation factor dependent in some extent despite of removing the false pressure field. To make a remedy for this unpleasant deficiency, a few schemes are proposed lately. An easy technique (Kobayashi and Pereira, 1991) is to set the under-relaxation factor 1 before momentum interpolation is implemented, but it may decrease the robustness of the algorithm to some extent. Hence, we can see that in order to make a reliable and efficient computation on non-staggered grid, the following three aspects should be guaranteed. (1) The algorithm can avoid the checkerboard pressure distribution; (2) The converged resolution is independent of the under-relaxation factor; (3) The algorithm should possess required robustness. In order to develop a computation scheme on the collocated grid, which possesses the above-mentioned three features, we first make a brief review on the existing implementation procedures on the collocated grids, take advantage of some successful practice and then propose a new implementation scheme for this study.

In the following, a brief description of the governing equations and the discretization procedure will be presented. For simplicity of presentation, we take two-dimensional incompressible laminar steady fluid flow in Cartesian coordinates as an example. The system for non-staggered grid is depicted in Figure 1.

![Figure 1. Control volumes of non-staggered grids in 2-D Cartesian coordinates](image)

The governing equations are as follows:

Continuity equation:

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$$  
(1)

Momentum equation:

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = -\frac{\partial p}{\partial x} + \rho \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + S_x$$  
(2)

$$\frac{\partial (\rho u v)}{\partial x} + \frac{\partial (\rho v^2)}{\partial y} = -\frac{\partial p}{\partial y} + \rho \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + S_y$$  
(3)

Energy equation:

$$\frac{\partial (\rho u T)}{\partial x} + \frac{\partial (\rho v T)}{\partial y} = \frac{\partial}{\partial x} (\lambda \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y} (\lambda \frac{\partial T}{\partial y}) + S_r$$  
(4)

The above four equations can be expressed in a general form:

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = \frac{\partial}{\partial x} (\Gamma_1 \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y} (\Gamma_1 \frac{\partial \phi}{\partial y}) + S_{\phi}$$  
(5)

where $u_i$ and $v_i$ refer to the interface velocity whose interpolation scheme is the major issue on the non-staggered grid. Equation (5) is discretized with finite volume method (FVM) on non-staggered grid system and the source term $S_{\phi}$ is linearized as:

$$S_{\phi} = S_{r_0} + S_{r_\phi} \phi_f$$  
(with $S_r \leq 0$)  
(6)

By taking out the pressure gradient from the $S_{\phi}$ in Eq.(5) for u component, the final discretization equation of u-component takes the following form, into which under-relaxation is incorporated:

$$\alpha_u u_p = a_u u_p + a_u u_n + a_u u_s + a_u u_i + b_r + \Delta y(p_n - p_r) + \frac{1 - \alpha_u}{\alpha_u} a_u^p$$  
(7)

$$= a_u u_p + a_u u_n + a_u u_s + a_u u_i + B_r + \Delta y(p_n - p_r)$$

where

$$B_r = b_r + \frac{1 - \alpha_u}{\alpha_u} a_u^p$$  
(8)

$$b_r = S_{r_0} \Delta x \Delta y$$  
(9)

The two terms of $(p_{i-1})_p$ and $(p_{i+1})_p$ are linearly interpolated from the neighboring nodes: From Eq.(7), we can get:

$$u_p = \alpha_u \left[ \sum a_u u_{n+1} + B_r \right] + \alpha_u \Delta y(p_n - p_r)(a_u)_p$$  
(10)

$$u_p = \alpha_u \left[ \sum a_u u_{i+1} + B_r \right] + \alpha_u \Delta y(p_n - p_r)(a_u)_p$$  
(11)

To eliminate the checkerboard pressure field, Rhie and Chow (1983) proposed the MIM (Momentum interpolation method) to get the interface velocity, which is expressed as:

$$u_p = \alpha_u \left[ \sum a_u u_{n+1} + B_r \right] + \alpha_u \Delta y(p_n - p_r)(a_u)_p$$  
(12)

Majumdar (1988) pointed that the resolution using this scheme is independent of under-relaxation factor. To develop an appropriate interpolation scheme to get the interface velocity, which can discard this weakness of Rhie and Chow’s, we reformulate Eqs.(10),(11) as follows:

$$u_p = \alpha_u \left[ \sum a_u u_{n+1} + b_r \right] + \alpha_u \Delta y(p_n - p_r)(a_u)_p + (1 - \alpha_u) u^p$$  
(13)
\[ u_x = a_x \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} + \frac{\alpha \Delta \psi (p_x - p_e)}{(a_x)_{x}} + (1 - \alpha_x) u_x^0 \]  

Hence the interface value \( u_x \) can be expressed as

\[ u_x = a_x \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} + \frac{\alpha \Delta \psi (p_x - p_e)}{(a_x)_{x}} + (1 - \alpha_x) u_x^0 \]  

We define:

\[ u_{int} = \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} + \frac{\alpha \Delta \psi (p_x - p_e)}{(a_x)_{x}} \]  

where

\[ \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} = f_x \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} + (1 - f_x) \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} \]  

\[ \frac{1}{(a_x)_{x}} = f_x \left( \frac{1}{(a_x)_{x}} \right) + (1 - f_x) \left( \frac{1}{(a_x)_{x}} \right) \]

The Eq.(15) can be rewritten as:

\[ u_x = a_x u_{int} + (1 - \alpha_x) u_x^0 \]  

It is worth noting that when iteration converges, \( u_x \) and \( u_x^0 \) approach the same value, and Eq.(15) is equivalent to \( u_x = u_{int} \), which is independent of the under-relaxation factors. Equations (15) is the modified momentum interpolation method (MMIM) by Majumdar (1988), which is later re-provided by Choi (1999) for unsteady flow. Yu et al. (2002) made a detailed discussion on the role of interface velocity, and recommended that all the interface velocity should be gained with momentum interpolation method in the whole calculation process of \( u_f \) and \( v_f \), and Eq.(15) will be used in this study for the interpolation.

**Mathematical Formulation of CLEAR Algorithm**

As indicated by Tao et al.(2004), the main difference between SIMPLER and CLEAR is the way to obtain the improved velocity, which satisfies the mass conservation condition. In the SIMPLER algorithm, this improved velocity is obtained by adding a correction term to the intermediate velocity, while in the CLEAR algorithm, the improved velocity is directly obtained from an improved pressure field.

Similar to the expression for the revised velocity on staggered grid of CLEAR algorithm, the improved velocities on collocated grid are expressed as:

\[ u_x = \beta_x \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} + (1 - \beta_x) u_x^* + \beta_x \Delta \psi (p_x - p_e) \]  

\[ v_x = \beta_x \left( \sum a_{a \alpha} v_{a \alpha} + b_{a \alpha} \right)_{x} + (1 - \beta_x) v_x^* + \beta_x \Delta \psi (p_x - p_e) \]

The terms of \( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \) in the interface are interpolated from the neighboring main-grid points. The parameter \( \beta \) is named the second relaxation factor defined in , and its value is recommended as:

\[ \beta = \begin{cases} 0.5 & 0 < \alpha \leq 0.5 \\ 1 & 0.5 < \alpha \leq 1 \end{cases} \]  

It is noted that Eqs.(19a),(19b) have the same form as the MMIM, Eq.(15), so the interface velocities defined by Eqs.(19a),(19b) are also independent of relaxation factor \( \beta \) when iteration converges.

In order to derive the equation for improved pressure on collocated grid, we define:

\[ \hat{u}_x = \beta_x \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} + (1 - \beta_x) u_x \]

\[ = \beta_x \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} + (1 - \beta_x) \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} \]  

\[ = \beta_x \left( \sum a_{a \alpha} u_{a \alpha} + b_{a \alpha} \right)_{x} + (1 - \beta_x) u_x \]

\[ \beta_x \Delta \psi (p_x - p_e) \]  

Equations (19a),(19b) are rewritten as for simplicity:

\[ u_x = \hat{u}_x + d_{\beta} (p_p - p_e) \]  

\[ v_x = \hat{v}_x + d_{\beta} (p_p - p_e) \]

Equations (23), (24) are substituted into the continuous equation. We get:

\[ a_p p = \sum a_{a \alpha} p_{a \alpha} + b \]

where

\[ a_p = a_E + a_W + a_N + a_S \]  

\[ a_E = \left( \rho Ad_f \right)_{ef} \]  

\[ a_w = \left( \rho Ad_f \right)_{ef} \]  

\[ a_N = \left( \rho Ad_f \right)_{ef} \]  

\[ b = \left( \rho u \hat{A}_n \right) - \left( \rho u \hat{A}_n \right) \]  

The parameter \( \beta \) is the relaxation factor for the updated interface pseudo-velocity. The larger the value of \( \beta \), the larger the variation of the interface pseudo-velocity, and the smaller the relative change between the improved and the previous pressure. And for a case with high value of \( \alpha \), the improved pressure should be highly under-relaxed, to guarantee the convergence of the solution procedure, and this can be done by setting a larger value of \( \beta \), sometimes even greater than one. Hence due to the good coupling of velocity and pressure in the CLEAR algorithm, the improved pressure should be under-relaxed. There are two approaches to under-relax the improved pressure, one is to give a larger second
relaxation factor $\alpha$, the other is to incorporate a pressure under-relaxation factor $\alpha_p$ to the improved pressure equation. Our practices show that the value $\alpha_p$ may be 
taken within the range of 0.7-1.0.

For the improved velocity on the main-grid point, 
Eq.(13) is rewritten with intermediate value as:
\[
\tilde{u}_p = \beta \left( \sum a_{\beta} u_{\beta} + b_{\beta} \right) + \frac{\beta_\Delta \psi (p_u - p_f)}{a_p} + (1 - \beta) u_p^* 
\]
(30)

Defining:
\[
\tilde{u}_p = \beta \left( \sum a_{\beta} u_{\beta} + b_{\beta} \right) + (1 - \beta) u_p^* 
\]
(31)

d^*_v = \frac{\beta_\Delta \psi (p_u - p_f)}{a_p} 
\]
(32)

Then Eq.(37) can be recast into
\[
u_p = \n\tilde{u}_p + d^*_v (p_u - p_f) 
\]
(33)

Similarly, we can get the improved main-grid point velocity for $v$ component:
\[
\tilde{v}_p = \beta \left( \sum a_{\beta} v_{\beta} + b_{\beta} \right) + (1 - \beta) v_p^* 
\]
(35)

d^*_v = \frac{\beta_\Delta \psi (p_u - p_f)}{a_p} 
\]
(36)

The solution procedure of the CLEAR algorithm on 
collocated grid is summarized as follows:

Step1: Assuming the initial field of main node and 
interface velocity $u_i^*, v_i^*, u_j^*, v_j^*$;

Step2: Calculating the discretization coefficient of 
momentum equation and pseudo-velocity $\tilde{u}_i$ (Eq.(21a)) 
and $\tilde{v}_i$ (Eq.(22a)) to determine the source term for the 
pressure equation, based on the previous interface and 
main node velocity $u_i^*, v_i^*, u_j^*, v_j^*$ using $\beta$ replacing $\alpha$;

Step3: Calculating the coefficient $d_u$ (Eq.(21b)) 
and $d_v$ (Eq.(22b)) of the pressure equation with $u_i^*, v_i^*$ using 
$\beta$ replacing $\alpha$;

Step4: Solving the pressure equation and obtaining 
pressure field $P$;

Step5: Based on $P$, solving the discretized forms 
of momentum equation, obtaining the intermediate 
velocity field $u_i^*, v_i^*$;

Step6: Calculating the interface velocity with 
MMIM based on $u_i^*, v_i^*, P$; and the previous 
discretized momentum equation coefficient, which is derived from 
$u_i^*, v_i^*$, obtaining the intermediate interface velocity $u_i^*, v_i^*$;

Step7: Based on the intermediate interface velocity 
$u_i^*, v_i^*$, recalculating the momentum equation 
coefficients, and obtaining the improved pseudo-velocity 
$\tilde{u}_i$ (Eq.(21a)), $\tilde{v}_i$ (Eq.(22a)) by the intermediate main 
node velocity to determine the source term of the 
improved pressure equation;

Step8: Calculating the coefficient $d_u$ (Eq.(21b)) $d_v$ 
(Eq.(30b)) of improved pressure equation, Eq.(22b) with $u_i^*, v_i^*$;

Step9: Solving the improved pressure equation and 
obtaining the updated pressure field $P$;

Step10: Updating the interface velocity with 
Eqs.(31), (32), correcting the main-grid point velocity 
with Eqs.(41), (42);

Step11: Returning to step 2 and repeating until 
convergence is reached.

From the above solution procedure, we can see that 
the first six steps are the same for the two algorithms. In 
the CLEAR algorithm the pressure equation is solved 
twice, in the first time it is solved to determine the source 
term of momentum equation, and in the second time, it is 
solved to gain the improved pressure to correct the 
intermediate velocity, so the number of the solved 
solutions is equal to SIMPLER, while in the step 7, the 
coefficient of discretized moment equation is 
recalculated without solving that leads to a bit longer 
CPU time for the new algorithm in one iteration.

**Application Examples**

To verify the feasibility of the CLEAR algorithm on 
collocated grid, four typical numerical examples with 
available solution are computed. The ratio of iteration 
number and consumed CPU time are compared between 
SIMPLER and CLEAR algorithms with variation of time 
step multiple. The time step multiple is defined by 
Equation (Patankar, 1980)
\[
E = \frac{\alpha}{1 - \alpha} 
(0 < \alpha < 1) 
\]
(37)

The convergence criteria is
\[
R_s = MAX \left[ \frac{(p_u'A)_{\text{low}} - (p_u'A)_{\text{char}}} {\text{flow}_{\text{ch}}} \right] \leq \varepsilon 
\]
(38)

where $R_s$ is the maximum mass residual of each 
control volume, $\text{flow}_{\text{ch}}$ is the characteristic flow rate 
defined in reference (Tao et al, 2004) The value of $\varepsilon$ 
is individually defined in each example. The pressure 
under-relaxation factor is 0.85 for the former four 
examples and 0.95 for the later two ones as the default 
value. For cases where a larger value of $\beta$ and or a 
smaller $\alpha_p$ is used, special description will be provided..

**Example 1 lid driven cavity flow**:

The calculations is carried out for Re =100. A 
uniform grid of 52 x 52 is applied with $x = 5.0 \times 10^5$. The 
Reynolds number is defined by
\[
Re = \frac{UL}{v} 
\]
(39)

In Figure 2, the ratios of iteration number and CPU 
time of CLEAR over that of SIMPLER varied with time 
step multiple are presented. In the figure, we take taking 
$\beta = 1.3$, $\alpha_p = 0.8$ for velocity under-relaxation factor of 
0.9. Here under-relaxation of the improved pressure is 
treated simultaneously with two ways: adding the second 
relaxation factor and decreasing the pressure under-
relaxation factor It can be seen that the ratio of iteration
number ranges from 0.16 to 0.65 and that of CPU time ranges from 0.19 to 0.77. The saving of iteration number and CPU time is appreciable.

Example 2 Lid-driven cavity flow in a polar cavity

Figure 3 shows the configuration of the polar cavity \((\theta = 1 \text{ radian}, \delta/R_\text{in}=1)\). The Reynolds number is defined as:

\[
Re = \frac{U_{\text{lid}} \delta}{\nu}
\]

where \(U_{\text{lid}}\) is the circumferential velocity of the moving lid, \(U_{\text{lid}} = R_\text{in} \omega\). A uniform grid system \(52 \times 52\) is used in our computation \((\varepsilon = 5.0 \times 10^{-6})\). In Figure 4, the comparisons are presented. Again in the high under-relaxation factor (0.9), the \(\beta\) value is greater than 1 (here is 1.3), and the corresponding pressure under-relaxation factor is 0.75 presented by the dashed lines. It can be found that the new algorithm perform superiorly to SIMPLER. The iteration number and CPU time of the CLEAR are 14-67% and 19-90%, respectively of that of the SIMPLER for \(Re = 350\).

\[R_a = 10^4\ \epsilon = 2.0 \times 10^{-7}\] based on the Boussinesq assumption. The second relaxation factor \(\beta = 1.2\) is for \(\alpha = 0.9\). The Rayleigh number is defined by

\[
R_a = \frac{\rho g \beta \varepsilon T \Delta T}{\alpha \mu}
\]

(41)

Figure 5 shows the variation of the iteration number and CPU time with the time step multiple of the two algorithms. The ratio of the iteration number ranges from 0.31 to 0.39 and the value of the CPU time is from 0.37 to 0.45.

Problem 4: Natural convection in an annular enclosure

The diagram of the laminar natural convection between two horizontal concentric cylinders is depicted in Figure 6. The case studied is for \(Ra = 10^4\), where Rayleigh number is defined by:

\[
R_a = \frac{\rho g \beta \varepsilon T \Delta T}{\alpha \mu}
\]

(42)

Computations were conducted on a uniform grid system with \(42 \times 32\) meshes with \(\varepsilon = 2.0 \times 10^{-7}\). The performance comparison result is displayed in Figures 7. The ratio of the iteration number varies from 0.1 to 0.3 and the CPU time ratio varies from 0.12 to 0.38. Obviously, the saving in CPU time is enormous (\(\beta = 1.3\) for \(\alpha = 0.9\)).
Conclusion

In this paper, the implementation of fully implicit algorithm CLEAR is proposed on the collocated grid system. Six typical numerical examples with available solutions are applied to compare the CLEAR and SIMPLER algorithm. It is revealed that the CLEAR algorithm possesses good convergent performance on collocated grids with similar robustness compared to SIMPLER algorithm for various applications, showing the feasibility of the new algorithm.

For conjugated problems solved by the domain extension method, special care must be taken on the collocated gird to guarantee the zero velocity in the solid region and at the boundary of the solid part.

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