Numerical Heat Transfer

(数值传热学)

Chapter 8 Numerical Simulation for Turbulent Flow and Heat Transfer



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8.1 Introduction to turbulence

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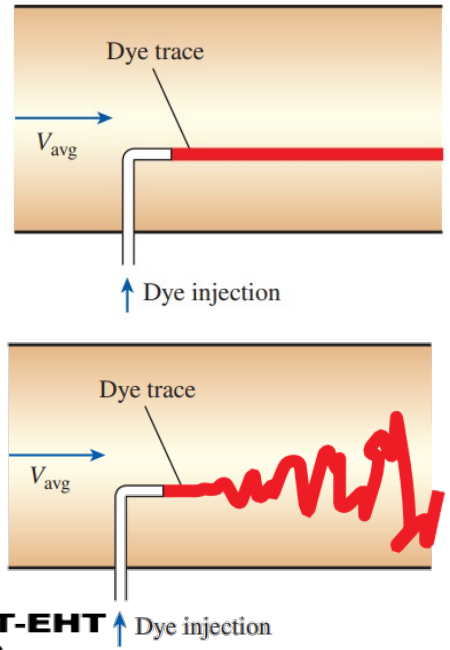
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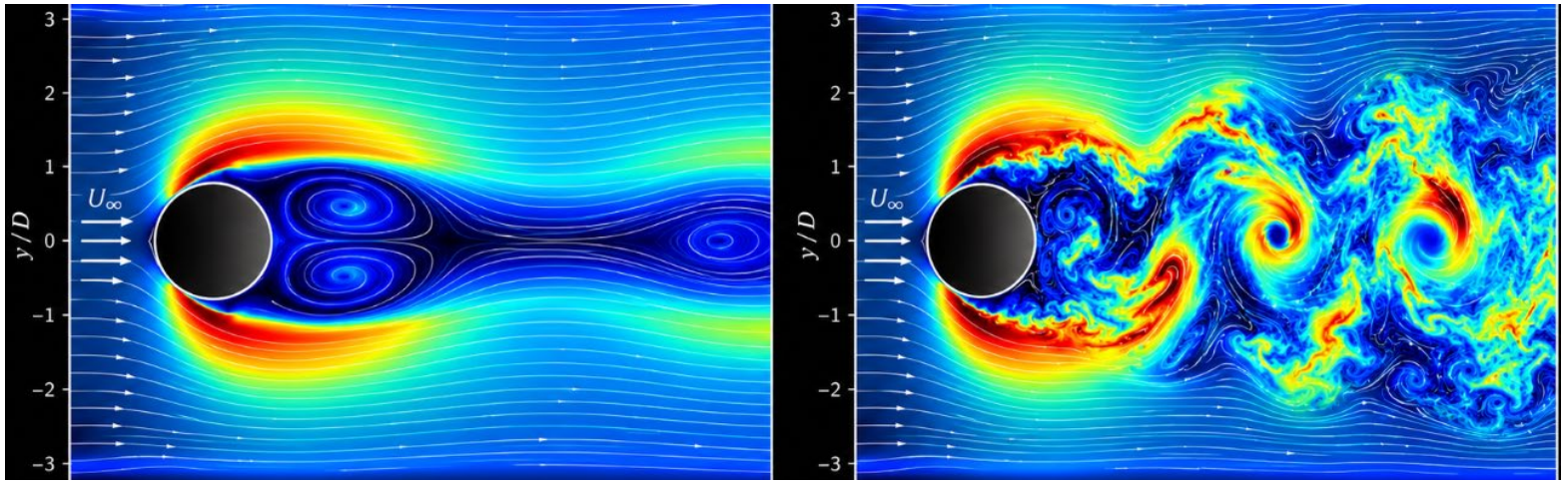
8.1 Introduction to turbulence

8.1.1 Present understanding of turbulence

1. **Two flow regimes**: Laminar flow & Turbulent flow $Re = \dfrac{\rho u d}{\eta}$

2. Turbulence is a highly complicated unsteady flow, within which all kinds of physical quantities are randomly varying with both time and space;





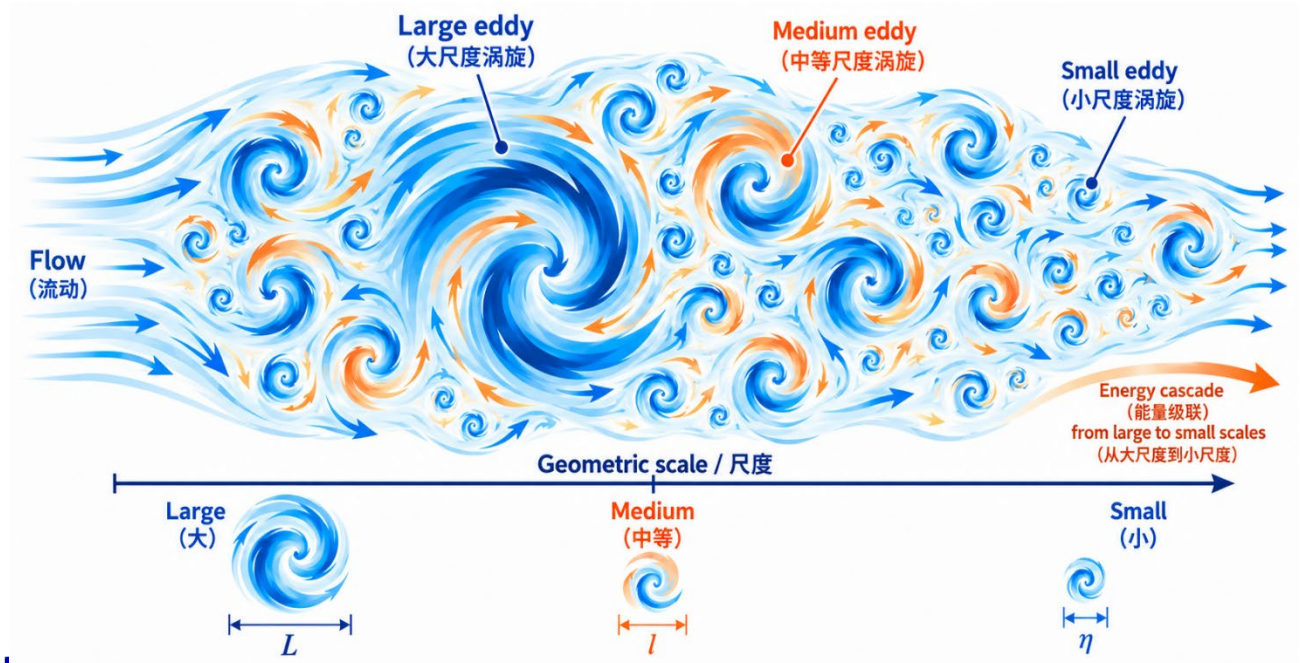
$Re = 100$ $Re = 100000$

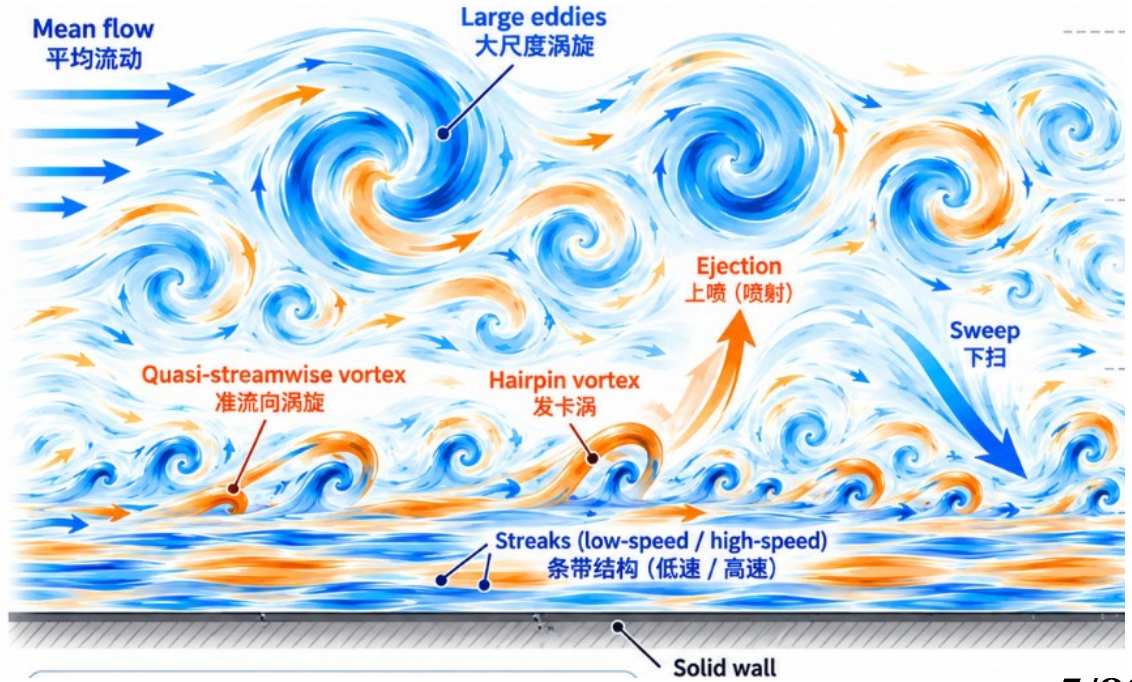
2. Transient Naiver-Stokes are valid for turbulent flows;

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right] \xrightarrow{\mu=\text{const}} \rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j^{\,2}}$$

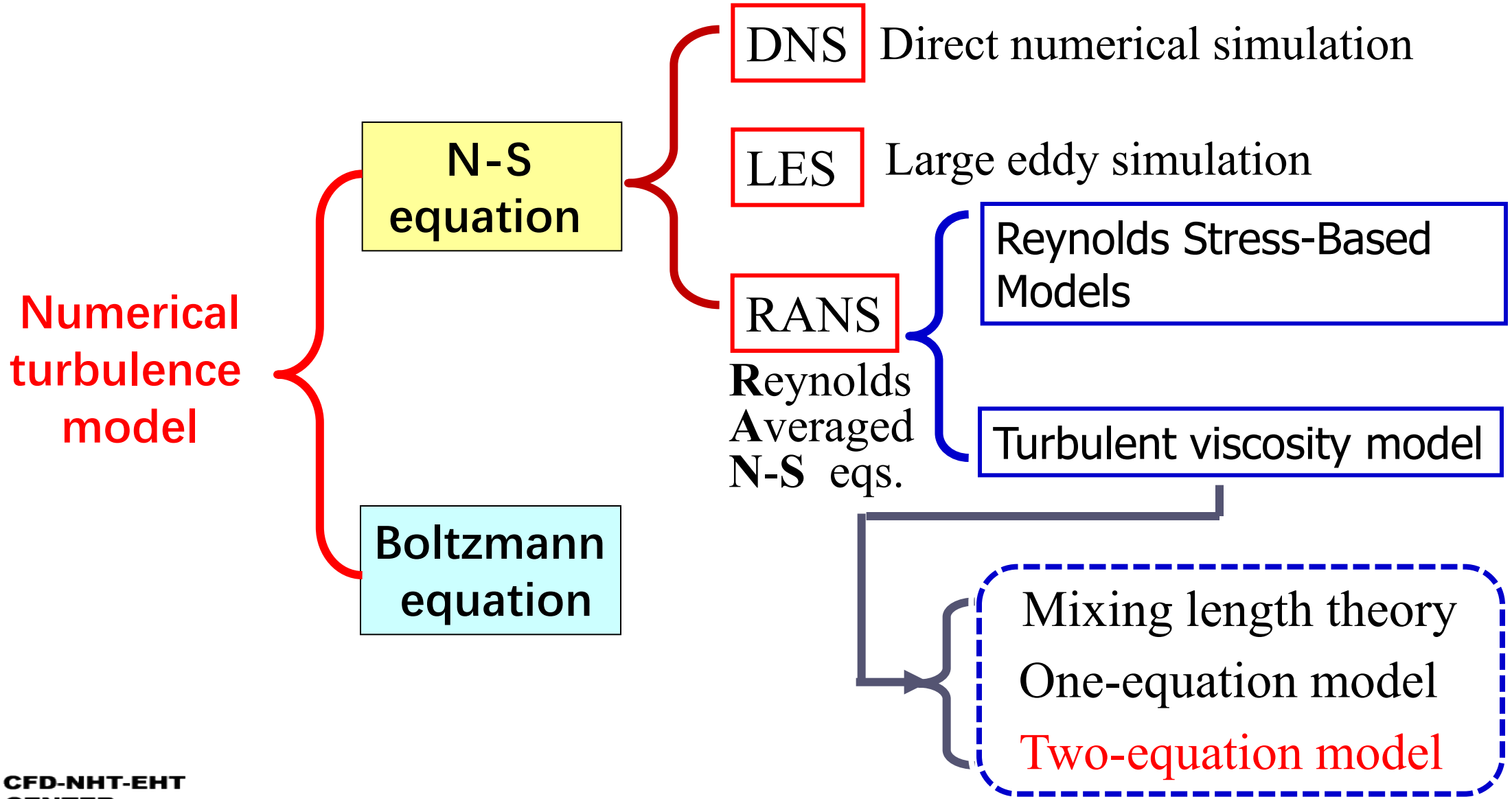
$$\mu\left[\nabla^2 u_i+\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{u})\right]$$

3. Turbulent flow field can be regarded as a collection of eddies（涡漩）with different geometric scales（尺度）.





8.1.2 Classifications of turbulence simulation methods



1.DNS (Direct numerical simulation)

In DNS, very small time step and space step are needed to resolve the evolutions (演化) of eddies with different scales. Required computer resource is very high. Often high-performance computers are needed.

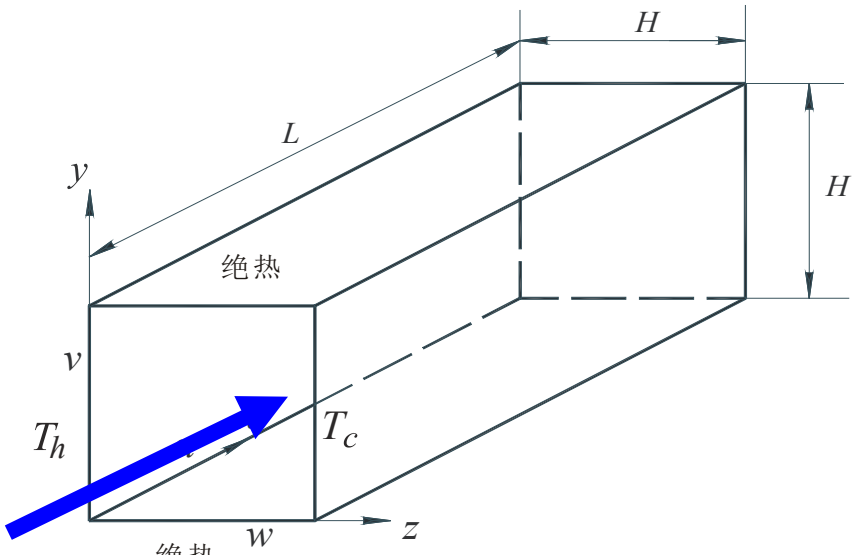
Kolmogorov theory:

Kolmogorov Length Scale

$$\ell_K = \left(\frac{\nu_m^3}{\varepsilon}\right)^{1/4} \sim LRe_L^{-3/4}$$

Kolmogorov Time Scale

$$\tau_K = \left(\frac{\nu_m}{\varepsilon}\right)^{1/2} \sim \frac{L}{U}Re_L^{-1/2}$$



For a fully developed mixed convection in a square duct (L=6.4H), when Re= 6400, Gr = 10^4~10^7, DNS is conducted with 4.194×10^6 nodes (=256×128×128), and 8×10^5 time steps are needed for statistical average.

2. LES (Large eddy simulation)

Basic idea: Turbulent fluctuations are mainly generated by large scale eddies, which are non-isotropic(各向异性) and vary with flow situation； Small scale eddies dissipate(耗散) kinetic energy and are almost isotropic (各向同性).

The N-S eqs. are used to simulate the large scale eddies and the behavior of small scale eddies is simulated by simplified model.

LES requires less computer resource than that of DNS, even though still quite high, and has been used for some engineering problems

For the above problem, when simulated by LES, only 128×80×80=819200 grids are needed (compared with 4.194×10^6).

3. RANS (Reynolds time average N-S Eqs. Methods)

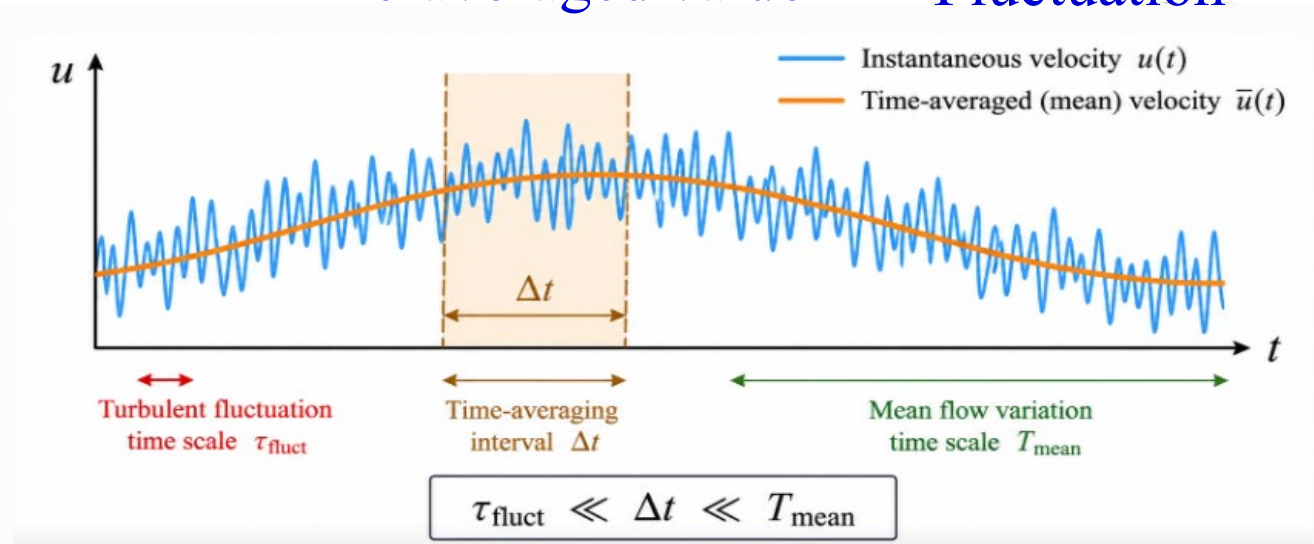
Expressing a transient term as the sum of average term and fluctuation (脉动) term. Time average is conducted for transient N-S equations. The unknown terms arising from turbulent fluctuations are modeled in terms of mean flow quantities to close the Reynolds-averaged equations.

8.1.3 Reynolds time averages and their characteristics

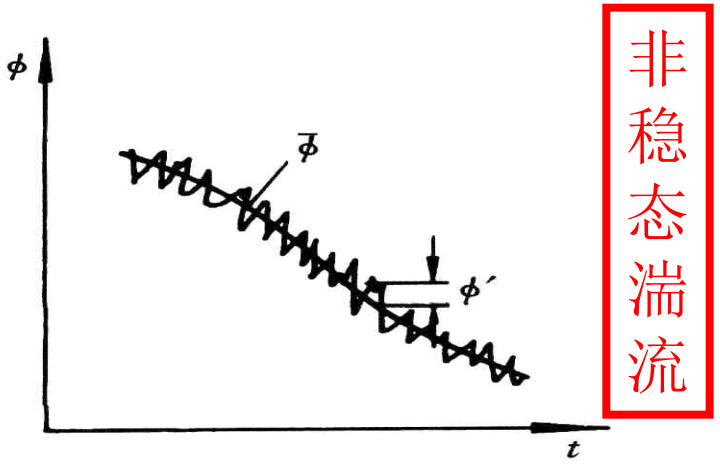
Instantaneous value $\phi = \overline{\phi} + \phi'$

Time-averaged value $\overline{\phi}$ Fluctuation ϕ'

$$\overline{\phi} = \frac{1}{\Delta t}\int_{t}^{t+\Delta t} \phi(t)dt$$

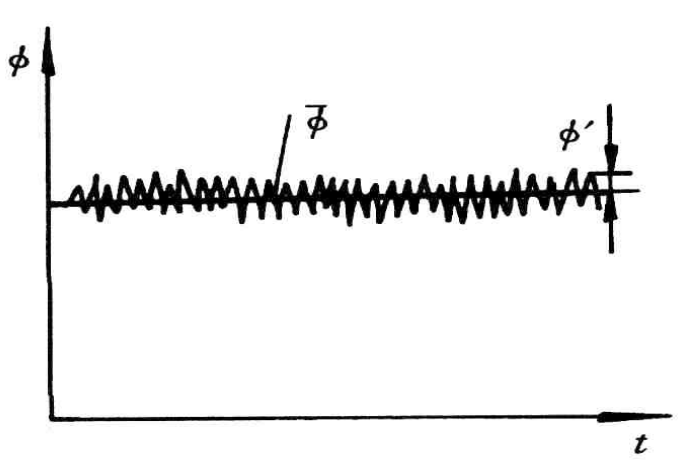


Δt should be large enough relative to fluctuation, but small enough relative to the mean-flow variation time scale.



非稳态湍流

(a) Unsteady



准稳态湍流

(b) Quasi-steady

Characteristics of time average operations

1. $\overline{\phi'} \equiv 0;$

2. $\overline{\overline{\phi}} = \overline{\phi};$

3. $\overline{\phi f} = \overline{(\overline{\phi} + \phi')(\overline{f} + f')} = \overline{\phi}\,\overline{f} + \overline{\phi' f'}$

4. $\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \overline{\phi}}{\partial x};$

5. $\overline{\frac{\partial \phi'}{\partial x}} = \frac{\partial \overline{\phi'}}{\partial x} = 0$

8.2 Time-averaged governing equation for incompressible fluid convective heat transfer

8.2.1 New formulation of general governing equation

8.2.2 Time average governing equation

8.2.3 Ways for determining additional terms

8.2.4 Governing equations with turbulent viscosity

8.2 Time-averaged governing equation for incompressible fluid convective heat transfer

8.2.1 New formulation of general governing equation

The frame work of the previous G.G.E. is retained (保留), but the diffusion coefficient is resumed to（恢复到） its original value by introducing a nominal density as follows:

$$\frac{\partial(\rho^{*}\phi)}{\partial t}+div(\rho^{*}\phi\vec{U})=div(\Gamma_{\phi}grad\phi)+S_{\phi}^{*}$$

Li W, Yu B, Wang Y , et al. Communications in Computational Physics, 2012, 12(5):1482-1494

The new form of G.E. are:

Equation	ρ^*	ϕ	Γ_ϕ	$S_\phi{}^*$
Continuity equation	ρ	1	0	0
Momentum eqn. (x direction)	ρ	u	μ	$\rho f_x - \frac{\partial p}{\partial x}$
Momentum eqn. (y direction)	ρ	v	μ	$\rho f_y - \frac{\partial p}{\partial y}$
Energy equation	ρc_p	T	λ	S_T

That is, now we regard ρc_p in the energy equation as a general density:

$$\frac{\partial(\rho c_p T)}{\partial t}+\frac{\partial(\rho c_p u T)}{\partial x}+\frac{\partial(\rho c_p v T)}{\partial y}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+S_T.$$

Such a treatment is much better than taking Γ / c_p as a nominal diffusion coefficient and S_T / c_p as a nominal source term .

8.2.2 Time average governing equation

1. Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad u=\overline{u}+u' \longrightarrow \qquad \left(\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}\right)+\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)=0$$

$$\overline{\frac{\partial(\overline{u}+u')}{\partial x}}+\overline{\frac{\partial(\overline{v}+v')}{\partial y}}+\overline{\frac{\partial(\overline{w}+w')}{\partial z}}=\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}+\overline{\frac{\partial u'}{\partial x}}+\overline{\frac{\partial v'}{\partial y}}+\overline{\frac{\partial w'}{\partial z}}=0$$

$$\overline{\frac{\partial u'}{\partial x}}+\overline{\frac{\partial v'}{\partial y}}+\overline{\frac{\partial w'}{\partial z}}=0$$

$$\Rightarrow \qquad \frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}=0 \qquad \frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Both time average velocity and time average fluctuation velocity satisfy continuity condition.

2. Momentum equation

Taking x-direction as an example:

$$\overline{\phi f}=\overline{(\bar{\phi}+\phi')(\bar{f}+f')}=\overline{\bar{\phi}\bar{f}}+\overline{\phi' f'}$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\overline{\frac{\partial(\bar{u}+u')}{\partial t}}+\overline{\frac{\partial(\bar{u}+u')^2}{\partial x}}+\overline{\frac{\partial(\bar{u}+u')(\bar{v}+v')}{\partial y}}+\overline{\frac{\partial(\bar{u}+u')(\bar{w}+w')}{\partial z}}=-\frac{1}{\rho}\overline{\frac{\partial(\bar{p}+p')}{\partial x}}+$$

$$\nu[\overline{\frac{\partial^2(\bar{u}+u')}{\partial x^2}}+\overline{\frac{\partial^2(\bar{u}+u')}{\partial y^2}}+\overline{\frac{\partial^2(\bar{u}+u')}{\partial z^2}}]$$

$$\frac{\partial\bar{u}}{\partial t}+\frac{\partial(\bar{u}^2)}{\partial x}+\frac{\partial\overline{uv}}{\partial y}+\frac{\partial\overline{uw}}{\partial z}+\frac{\partial\overline{(u')^2}}{\partial x}+\frac{\partial\overline{(u'v')}}{\partial y}+\frac{\partial\overline{(u'w')}}{\partial z}=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}+\nu\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial^2\bar{u}}{\partial z^2}\right)$$

Moved to right hand side and combined with the corresponding viscous term

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial(\bar{u}^2)}{\partial x}+\frac{\partial \bar{u}\bar{v}}{\partial y}+\frac{\partial \bar{u}\bar{w}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\frac{\partial}{\partial x}[\nu\frac{\partial \bar{u}}{\partial x}-\overline{(u')^2}]+\frac{\partial}{\partial y}[\nu\frac{\partial \bar{u}}{\partial y}-\overline{(u'v')}]+\frac{\partial}{\partial z}[\nu\frac{\partial \bar{u}}{\partial z}-\overline{(u'w')}]$$

Rewritten in a tensor (张量) form in Cartesian coordinate:

$$\frac{\partial(\rho\bar{u}_i)}{\partial t}+\frac{\partial(\rho\bar{u}_i\bar{u}_j)}{\partial x_j}=-\frac{\partial \bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}(\eta\frac{\partial \bar{u}_i}{\partial x_j}-\rho\overline{u_i'u_j'})\ (i=1,2,3)$$

Double subscript means summation over the subscript.

3. Other scalar (标量) variables

$$\frac{\partial(\rho\bar{\phi})}{\partial t}+\frac{\partial(\rho\bar{u}_j\bar{\phi})}{\partial x_j}=\frac{\partial}{\partial x_j}(\Gamma\frac{\partial \bar{\phi}}{\partial x_j}-\rho\overline{u_j'\phi'})+S$$

4. Discussion on the time-averaged procedure

(1) Linear term remains unchanged during time average, while product term (乘积项) generates product of two fluctuations, representing the additional transport caused by the fluctuation.

$$\overline{u+v}=\overline{u}+\overline{v} \qquad \overline{uv}=\overline{u}\,\overline{v}+\overline{u'v'}$$

(2) The resulted equations are not closed: for 3-D problem, there are five equations with 14 unknown variables：

Five time average variables — $\overline{u},\overline{v},\overline{w},\overline{p},\overline{\phi},$

Nine products of fluctuations

$\overline{u_i'u_j'}(i,j=1,2,3)$（6 terms）; $\overline{u_i'\phi'}(i=1,2,3)$（3 terms）

$$\frac{\partial(\rho\bar{u}_i)}{\partial t}+\frac{\partial(\rho\bar{u}_i\bar{u}_j)}{\partial x_j}=-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}(\eta\frac{\partial\bar{u}_i}{\partial x_j}-\rho\overline{u_i'u_j'})\ (i=1,2,3)$$

$$\frac{\partial(\rho\bar{\phi})}{\partial t}+\frac{\partial(\rho\bar{u}_j\bar{\phi})}{\partial x_j}=\frac{\partial}{\partial x_j}(\Gamma\frac{\partial\bar{\phi}}{\partial x_j}-\rho\overline{u_j'\phi'})+S$$

In order to close the above set of equations, additional relations must be added. **Such additional relations are called turbulence model, or closure model (封闭模型).**

Prof. P. Y. Chou (周培源) made pioneering contributions to turbulence closure theory in 1945.

Chou, P. Y. (1945). On velocity correlations and the solutions of the equations of turbulent fluctuation. Quarterly of Applied Mathematics, 3(1), 38–54.

8.2.3 Ways of determining additional terms

1. Reynolds stress method

Nine additional governing equations are required to close the system for $\overline{u_i'u_j'}(i,j=1,2,3)$（6 terms）; $\overline{u_i'\phi'}(i=1,2,3)$（3 terms）

However, deriving transport equations for these additional unknowns introduces higher-order correlations, such as triple velocity correlations, which in turn require further equations. Therefore, it must be truncated at a certain level, and the remaining unknown terms must be modeled to achieve closure. Historically, some complicated models with more than 20 equations have been derived.

relation $\overline{w_i w'_k}$ were known. But unfortunately the equation of continuity (4.4) and the general dynamical equation of double correlation (7.6) are insufficient to yield a definite solution for $\overline{w_i w'_k}$, because of the presence of the triple correlation $\overline{w_i w_j w'_k}$ in (7.6).

Copied From the original paper of Prof. P. Y. Chou.

In the Reynolds stress models, the second moment model (model for $R_{ij}=\overline{u_i'u_j'}$) is quite famous and has been applied in some engineering problems.

In the second moment model, for the product terms with two fluctuations, their equations are derived, while for the terms with three or more fluctuations models are used to relate such terms with time average variables.

Prof. L X Zhou (周力行) in Tsinghua university contributed a lot in this regard.

2. Turbulent viscosity method

$$\frac{\partial(\rho\bar{u}_i)}{\partial t}+\frac{\partial(\rho\bar{u}_i\bar{u}_j)}{\partial x_j}=-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}(\eta\frac{\partial\bar{u}_i}{\partial x_j}-\rho\overline{u_i'u_j'})$$

(1) Definition of the turbulent viscosity (η_t)

Its main idea is using η_t to express $\overline{u_i'u_j'}$

In 1877, Boussinesq proposed the eddy viscosity hypothesis by analogy with the constitutive equation for Newtonian fluids.:

Viscous Stress Tensor $\tau_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$(\tau_{i,j})_t = -\overline{\rho u_i' u_j'} = (-p_t \delta_{i,j}) + \eta_t\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) - \frac{2}{3}\eta_t \delta_{i,j} div\vec{U} \Rightarrow 0$$

$$p_t = \frac{1}{3}\rho[\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2}] = \frac{2}{3}\rho k \qquad k = \frac{1}{2}[\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2}]$$

The unknown $\overline{u_i' u_j'}$ is expressed via turbulent viscosity to close model.

(2) Definition of turbulent diffusivity of other scalar variables

$$-\overline{\rho u_i' \phi'} = \Gamma_t \frac{\partial \overline{\phi}}{\partial x_i} \qquad \Gamma_t = \frac{\eta_t}{\mathrm{Pr}_t}$$

Pr_t ---turbulent Prandtl number, usually treated as a constant.

Please note:

Here we take

$\Gamma = \lambda,$

Rather than

$\Gamma = \lambda / c_P$

For laminar heat transfer we have

$$\Gamma_l = \lambda = \frac{\lambda}{c_p}\frac{\eta_l}{\eta_l}c_p = (\frac{\lambda}{c_p\eta_l})\eta_l c_p = \frac{\eta_l c_p}{(\frac{c_p\eta_l}{\lambda})} = \frac{\eta_l c_p}{\mathrm{Pr}_l}$$

Similarly: $\Gamma_t = \lambda_t = \eta_t c_p / \mathrm{Pr}_t$

Therefore, for turbulent viscosity model, its major task is to find η_t, Pr_t.

The name of turbulence models (zero/one/two equations model) comes from the number of Partial Differential Equation (PDE) included in the model to determine turbulence viscosity.

8.2.4 Governing equations of turbulent viscosity models

1. Governing equations

For simplicity of presentation, the symbol(符号) of time average "bar" is omitted hereafter:

$$\begin{cases} \dfrac{\partial u_k}{\partial x_k} = 0 \\ \dfrac{\partial u_i}{\partial t} + \dfrac{\partial(\rho u_k u_i)}{\partial x_k} = -\dfrac{\partial p_{eff}}{\partial x_i} + \dfrac{\partial}{\partial x_k}\underbrace{[(\eta_l + \eta_t)}_{\eta_{\text{eff}}}\dfrac{\partial u_i}{\partial x_k}] + S_i \;;\; p_{eff} = p + p_t \\ \dfrac{\partial(\rho^* \phi)}{\partial t} + \dfrac{\partial(\rho^* u_k \phi)}{\partial x_k} = \dfrac{\partial}{\partial x_k}\underbrace{[(\Gamma_l + \Gamma_t)}_{\Gamma_{eff}}\dfrac{\partial \phi}{\partial x_k}] + S_\phi \end{cases}$$

2. Differences from laminar governing equations:

(1) u_i, p, ϕ -Time average; (2) Replacing Γ by $\Gamma_{eff} = \Gamma + \Gamma_t$

(3) Replacing p by p_{eff} (4) In the source term S_i of u_i the additional terms caused by time averaging are included.

$\Gamma_t = c_p \eta_t / \mathrm{Pr}_t$ For temperature ρ^* ---ρc_p

For Cartesian coordinates, the source terms of three velocities:

$$u:\ S=\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial x}\right)$$

$$v:\ S=\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial y}\right)$$

$$w:\ S=\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial z}\right)$$

In laminar flow of constant properties, all source terms are zero, but for turbulent flow they are not zero.

$$S_u=\eta\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$ Laminar flow

3. Turbulent Prandtl number

It is usually taken as a constant, and $\Gamma_t=c_p\eta_t/\mathrm{Pr}_t$

For the turbulent viscosity models, Eq. for determining η_{eff} should be added.

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_k)}{\partial x_k}=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)\right]$$ Original NS Eq.

⇩ Reynold averaged

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_k)}{\partial x_k}=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)\right]-\frac{\partial}{\partial x_k}\left(\rho\overline{u_i'u_k'}\right)$$

Boussinesq assumption ⇩ $$-\rho\overline{u_i'u_k'}=\eta_t\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)-\frac{2}{3}\rho k\delta_{ik}$$

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_k)}{\partial x_k}=-\frac{\partial p_{\text{eff}}}{\partial x_i}+\frac{\partial}{\partial x_k}\left[\eta_{\text{eff}}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)\right] \qquad \eta_{\text{eff}}=\eta+\eta_t,\ p_{\text{eff}}=p+\frac{2}{3}\rho k$$

(η_{eff} is varied) ⇩ Rewritten into the general form

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_k)}{\partial x_k}=-\frac{\partial p_{\text{eff}}}{\partial x_i}+\frac{\partial}{\partial x_k}\left(\eta_{\text{eff}}\frac{\partial u_i}{\partial x_k}\right)+S_i \qquad S_i=\frac{\partial}{\partial x_k}\left(\eta_{\text{eff}}\frac{\partial u_k}{\partial x_i}\right)$$

8.3 Zero equation model and one equation model

8.3.1 Zero equation model

1.Turbulent additional stress of zero equation model

2.Equations for mixing length

3.Application range of zero eq. model

8.3.2 One equation model

1.Turbulent fluctuation kinetic energy as dependent variable

2.Prandtl-Kolmogorov equation

3.Governing equation of turbulent fluctuation kinetic energy

4.Boundary condition

8.3 Zero Equation Model and One Equation Model

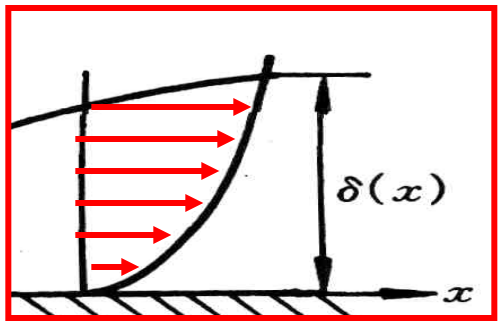
8.3.1 Zero equation model

1. Zero equation model for turbulent additional stress

In zero eq. model, no PDE is involved to determine turbulent viscosity.

The turbulent stress is expressed as:

$$\tau_t = -\rho\overline{u_i' u_j'} = -\rho\overline{u'v'} = \rho\nu_t\left(\frac{du}{dy}\right)$$



Velocity difference: $\Delta u \approx l_m \frac{du}{dy}$ l_m: mixing length

Turbulent velocity scale: $u_t \approx l_m \left|\frac{du}{dy}\right|$ Turbulent kinematic viscosity: $\nu_t \approx l_m u_t$

$$\tau_t = -\rho\overline{u_i' u_j'} = -\rho\overline{u'v'} = \rho\nu_t\left(\frac{du}{dy}\right) = \rho\, l_m^2 \left|\frac{du}{dy}\right|\left(\frac{du}{dy}\right)$$

$$\nu_t = l_m^2 \left|\frac{du}{dy}\right|$$

The determination of l_m is the key of zero-eq. model.

2. Equations for mixing length l_m

(1) Flow and HT over a plate

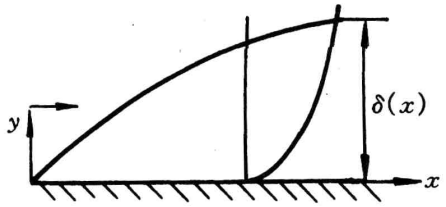
l_m/δ vs. y/δ is a slope

function (斜坡函数) :

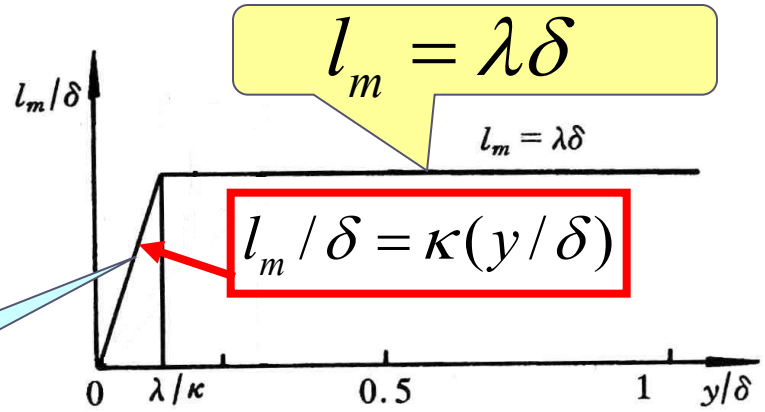
At $y/\delta < \lambda/\kappa$, $l_m = \kappa y$

At $y/\delta \geq \lambda/\kappa$, $l_m = \lambda\delta$

Authors	κ	λ
Cebeci	0.41	0.08
P-S	0.435	0.09



(a)

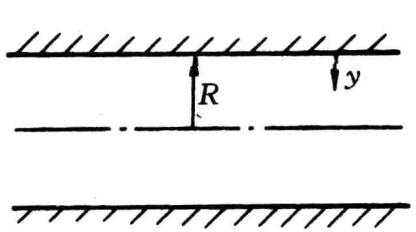


$l_m = \kappa y$

δ ---thickness of B.L.

(2) Turbulent HT in a circular tube--- Nikurads eq.

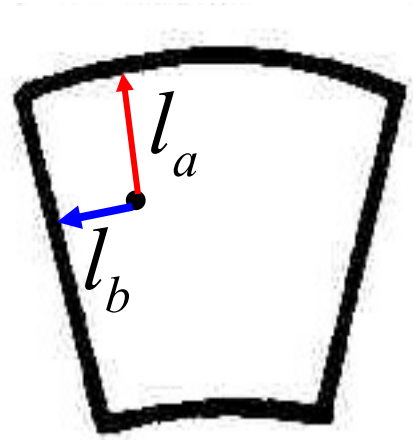
$$l_m / R = 0.14 - 0.08(1 - y/R)^2 - 0.06(1 - y/R)^4$$



Application range: $\mathrm{Re} = 1.1\times10^5 \sim 3.2\times10^6$

(3) Fluid in a duct corner

$$\frac{1}{l_m} = \frac{1}{l_a} + \frac{1}{l_b};\ l_a, l_b \text{ from above eqs.}$$



(4) Modification caused by molecular viscosity — van Driest eq.

$$l_m = \kappa y[1 - \exp(-\frac{y(\tau_m/\rho)^{1/2}}{A\nu})] = \kappa y[1 - \exp(-\frac{y^+}{A})],\quad A = 26$$

Correction caused by molecular viscosity

For $\frac{y^+}{A} = 6$, its value $= 0.997$

3. Application range of zero eq. model

(1) Boundary layer flow & HT（Flow over a wing before separation)

(2) FF & HT in straight ducts;

(3) Boundary layer type flow with weak recirculation.

Drawbacks of zero eq. model：

(1) At duct center line, velocity gradient equals zero, and according to this model, turbulent viscosity is zero, but actually turbulent viscosity still exists.

$$\nu_t = l_m^2 \left| \frac{du}{dy} \right|$$

(2) Effects of oncoming flow turbulence is not considered.

(3) Effects of turbulent flow itself is not considered.

Li ZY, Hung TC, Tao WQ. Numerical simulation of fully developed turbulent flow and heat transfer in annular-sector ducts. Heat Mass Transfer,2002, 38 (4-5): 369-377

8.3.2 One-equation model

1.Turbulent fluctuation kinetic energy k is taken as a dependent variable to be solved by a PDE.

Definition: $$k=\frac{1}{2}\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right)$$

The most important feature of turbulence is fluctuation. Fluctuation kinetic energy k is an appropriate quantity to indicate fluctuation intensity(脉动强度). It is taken as a dependent variable for reflecting the effects of turbulence itself.

2.Prandtl-Kolmogorov equation

Mimicking (模仿) the molecular viscosity caused by the random motion of molecules, which is:

Molecular viscosity $\eta_l \propto \rho \overline{u} \overline{\lambda}$

Then the viscosity caused by turbulent fluctuation (turbulent viscosity) can be expressed by

$$\eta_t \propto \rho k^{1/2} l \longrightarrow \eta_t = C_\mu' \rho k^{1/2} l$$

where l is the fluctuation scale, usually different from mixing length;

—Prandtl-Kolmogorov equation

Coefficient C_μ' is within the range from 0.2 to 1.0;

In order to get the distribution of k, a related PDE is required.

3.Governing equation of turbulent kinetic energy k

Derivation of the Turbulent Kinetic Energy Transport Equation:

Instantaneous N-S Eq. $\rightarrow$ Reynolds Decomposition ($u_i = U_i + u_{i'}$) $\rightarrow$ Fluctuation Velocity Eq. $\rightarrow$

$u_{i'} \times$ (Fluctuation Eq.) + Time Averaging $\rightarrow$ Turbulent Kinetic Energy Eq. ($k = \frac{1}{2}\overline{u_{i'}u_{i'}}$)

$$\rho\frac{\partial k}{\partial t} + \rho u_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}[(\eta_l + \frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}] + \eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}) - \rho(C_D\frac{k^{3/2}}{l})$$

source

transient | convection | diffusion | production | dissipation

where σ_k is called turbulent Prandtl number of k, and its introduction can increase the application range of the model.

4. Boundary condition treatment: wall function method

8.4 Two-Equation Model

8.4.1 Second variables related to l

8.4.2 $k-\varepsilon$ governing equations

8.4.3 General governing equation for $k-\varepsilon$ model

8.4.4 Remarks

8.4 Two-Equation model

8.4.1 Second variables related to fluctuation scale l

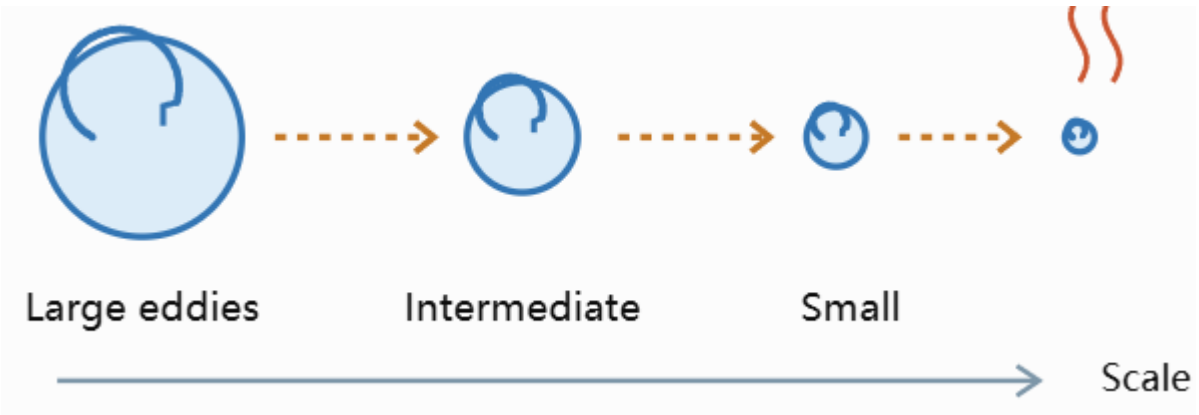
1. Five physical variables related to l

Z-variables	$k^{1/2}/l$	$k^{3/2}/l$	kl	k/l^2	ε/k
Proposed by	Kolmogorov [32]	Chou (周培源)[19]	Rodi, Spalding [38]	Spalding[39]	Wilox(1988)
Symbol	f	ε	kl	W	ω
Physical meaning	Eddy frequency	Energy dissipation	Product of energy and length scale	Mean square root of vorticity fluctuation	Energy dissipation per unit energy

2. Two definitions of dissipation rate

(1) Strict definition given by Chou P.Y.

$$\varepsilon = \nu_l \overline{\left(\frac{\partial u_i'}{\partial x_k}\right)\left(\frac{\partial u_i'}{\partial x_k}\right)}$$



It represents dissipation rate per unit mass of isotropic small eddies, and is used in the derivation of its governing equation.

(2) Modeling definition The dissipation term is defined by

$$\varepsilon = C_D k^{3/2} / l$$

Estimates ε using macroscopic turbulence scales

This is called modeling definition (模拟定义). It is used in the derivation process of ε-equation to simplify some complicated terms.; C_D is a dimensionless constant.

Understanding of its meaning： energy transit rate from large eddies to small eddies for unit volume is proportional to ρk , and $1/t$, where the transit time t is proportional to $l / k^{1/2}$, thus

$$\rho\varepsilon \sim \rho k /(\frac{l}{k^{1/2}}) \sim \rho\frac{k^{3/2}}{l} = C_D\rho\frac{k^{3/2}}{l}$$

8.4.2 $k-\varepsilon$ governing equations

(1) ε equation

Starting from strict definition, $\varepsilon = \nu_l\overline{(\frac{\partial u_i^{'}}{\partial x_k})(\frac{\partial u_i^{'}}{\partial x_k})}$ conducting time average operation for N-S equation, and adopting some assumptions (including modeling definition), yielding

$$\frac{\partial(\rho\varepsilon)}{\partial t}+\frac{\partial(\rho u_j\varepsilon)}{\partial x_j}=\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_\varepsilon})\frac{\partial\varepsilon}{\partial x_j}]+C_1\frac{\varepsilon}{k}\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})-C_2\rho\frac{\varepsilon^2}{k}$$

transient | convection | diffusion | source

σ_ε Prandtl number of ε; C_1, C_2 are empirical (经验的) coefficients

(2) k equation After introducing ε

k equation can be re-written as

$$-\rho(C_D\frac{k^{3/2}}{l})$$

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}]+\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})-\rho\varepsilon$$

Source term

Introducing: $G=\frac{\eta_t}{\rho}\frac{\partial u_i}{\partial x_j}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$ called as unit mass production function

The source term of k eq. $\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})-\rho\varepsilon$ ⇨ $\rho G-\rho\varepsilon$

(3)Determination of turbulent viscosity of $k-\varepsilon$ model

$$\eta_t = C_\mu' \rho k^{1/2} l = C_\mu' C_D \rho k^{1/2+3/2} \frac{l}{C_D k^{3/2}} = C_\mu \rho k^2 / \varepsilon$$

$$C_\mu' C_D \to C_\mu$$

$$\varepsilon = C_D \frac{k^{3/2}}{l}$$

8.4.3 General gov. eq. of $k-\varepsilon$ model

$$\frac{\partial(\rho^* \phi)}{\partial t} + div(\rho^* \vec{u} \phi) = div(\Gamma_\phi grad \phi) + S_\phi$$

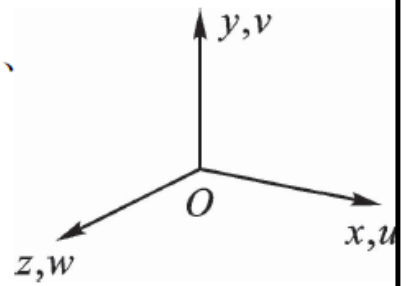
ϕ represents: $u, v, w, T, k, \varepsilon$

Most widely accepted values of model constants

C_1	C_2	C_μ	σ_k	σ_ε	σ_T
1.44	1.92	0.09	1.0	1.3	0.9-1.0

表 8-4 三维直角坐标系中 $k-\varepsilon$ 模型的稳态控制方程

类别	详细方程
控制方程	$\frac{\partial(\rho^* u\phi)}{\partial x}+\frac{\partial(\rho^* v\phi)}{\partial y}+\frac{\partial(\rho^* w\phi)}{\partial z}=\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\Gamma\frac{\partial\phi}{\partial z}\right)+S$ 对 u、v、w、k、ε、T，广义扩散系数 Γ 为 u，v，w：$\Gamma=\eta_{\text{eff}}=\eta+\eta_t$　k：$\Gamma=\eta+\frac{\eta_t}{\sigma_k}$　ε：$\Gamma=\eta+\frac{\eta_t}{\sigma_\varepsilon}$　T：$\Gamma=\lambda+\frac{\eta_t c_p}{\sigma_T}$
源项	u：$S=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial x}\right)$
	v：$S=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial y}\right)$
	w：$S=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial z}\right)$
	k：$S=\rho G_k-\rho\varepsilon$
	ε：$S=\frac{\varepsilon}{k}(c_1\rho G_k-c_2\rho\varepsilon)$
	$G_k=\frac{\eta_t}{\rho}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right\}$
	T：S 按实际问题而定



Diffusion coefficients

Source term

Γ_ϕ, S_ϕ depend on variable and coordinate： $u, v, w, T, k, \varepsilon$

8.4.4 Remarks

(1) Expansion of G term for 2D case

$$G=\frac{\eta_t}{\rho}\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=\frac{\eta_t}{\rho}\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}+\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right)=$$

$$\frac{\eta_t}{\rho}\left[\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)+\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)\right]$$

$G=\frac{\eta_t}{\rho}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\}$ There are 18 terms for 3D case.

(2) The above model is called standard $k-\varepsilon$ model. It can be applied to vigorously developed (旺盛发展) turbulent flow, also called as high-Re $k-\varepsilon$ model. Here, *Re* is not the conventional Reynolds number defined by average velocity. $\mathrm{Re}_t=\rho k^2/\eta\varepsilon$

Kunming

38

ON VELOCITY CORRELATIONS AND THE SOLUTIONS OF THE EQUATIONS OF TURBULENT FLUCTUATION*

BY

P. Y. CHOU**
National Tsing Hua University, Kunming, China

1. Introduction. The theory of turbulence, as developed from Reynolds' point of view, is based upon the equations of turbulent fluctuation [1] and has been applied to the solutions of various special problems [2, 3, 4, 5, 6, 7]. Owing to present circumstances, these papers either have not been submitted to scientific journals for publication or are already printed but have failed to appear before the scientific public. The theory in its original form and its applications has three apparent difficulties: first, the equations of correlation of the second, third or even higher orders constructed out of the equations of turbulent fluctuation contain the unknown terms of correlation between the pressure and velocity fluctuations; secondly, there exist in these equations the terms of decay of turbulence the values of which have to be determined; thirdly, when the differential equations of the velocity correlations of a given order are derived from the equations of turbulent fluctuation, the presence of the inertia terms causes the appearance of the velocity correlations of the next higher order, which are also unknown. This has been pointed out by von Kármán and Howarth [8] in their theory of homogeneous isotropic turbulence.

In the present paper we shall show that the pressure fluctuation can be derived from the equations of turbulent fluctuation, and is expressible as a function of the velocity fluctuation, the mean velocity inside the fluid volume, and the pressure fluctuation on the boundary. We shall also show that the decay terms can be put into simpler and more familiar forms by kinematic considerations. A general equation of vorticity decay will be derived for the determination of Taylor's scale of the micro-turbulence which appears in the decay term; in the case of homogeneous isotropic turbulence, this equation was given first by von Kármán [8]. To get over the third difficulty we shall compare the orders of magnitudes of the different terms in the equations of triple correlation. We shall find that the term involving the divergence of the quadruple correlation is actually smaller than the correlation between the pressure gradient and the two components of velocity fluctuation, and can therefore be neglected as a first approximation. From this we can also understand why, for the flows in channels and pipes in which the mean velocity profile is comparatively steep, particularly in the neighborhood of the walls, all the equations of mean motion and the equations of double and triple correlation are necessary to describe the phenomena of turbulent motions of fluids. On the other hand, as a consequence of the approximation based on the fact that the divergence of the quadruple correlation is smaller than the correlation between the pressure gradient and the two components of velocity fluctuation, we can stop at the equations of triple correlation instead of building equations of higher orders. As a matter of fact, for the flows in jets [3] and wakes [4]

* Received Aug. 21, 1944.
** Now at California Institute of Technology.



（1902-1993）

被国际学术界认同的20世纪流体力学四位巨人(jiant)是：美国的冯卡门 (von Karman) ， 前苏联的柯尔莫哥洛夫(Kolmogorov)，英国的泰勒(G.I.Taylor)， 和中国的周培源。

门生也是巨人：包括胡宁、彭桓武、 何泽慧、 钱三强、 张宗燧、 钱伟长、 王竹溪、林家翘、 于光远等著名学者。

在1945年的这篇文章中，他给出了二阶耗散项的下列表达式：

$$2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}} = -\frac{2\nu}{3\lambda^2}(k-5)q^2 g_{ik} + \frac{2\nu k}{\lambda^2}\overline{u_i' u_k'}$$

8.5 Wall Function Method

8.5.1 Two ways for grid settlement near wall in turbulence simulation

8.5.2 Fundamentals of wall function method

8.5.3 Boundary conditions for standard k, ε model

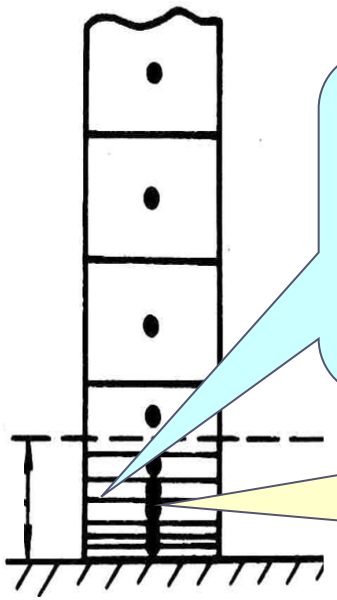
8.5.4 Cautions in implementing wall function method

8.5 Wall Function Method (壁面函数法)

8.5.1 Two ways for grid settlement (节点设置) near a wall in turbulence simulation

1.Setting enough number of grids in the viscous sublayer（>10 grids）

For this treatment, k equation can be used from vigorous turbulent flow to the wall, and k_w=0 for its boundary condition. This treatment is used in low Reynolds number $k-\varepsilon$ model. Here Reynolds number is defined by $\mathrm{Re}_t = \rho k^2 / \eta\varepsilon$.



Enough number of nodes should be set in the viscous sub-layer.

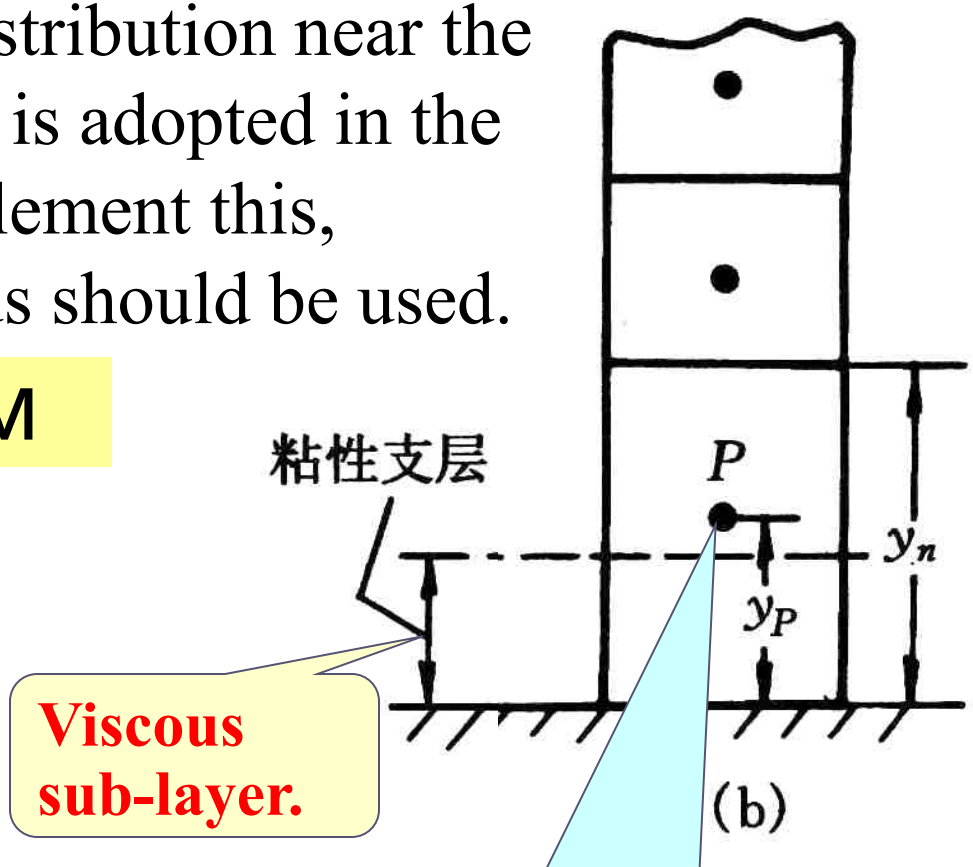
Viscous sublayer（粘性支层）

2. Set the 1st inner node outside the viscous sublayer

In this treatment, velocity distribution near the wall should be assumed, and it is adopted in the high Re $k-\varepsilon$ model. To implement this, practice Wall Function Methods should be used.

8.5.2 Fundamentals of WFM

1) Assuming that the dimensionless velocity and temp. distributions outside the viscous sub-layer are of logarithmic law(对数律) type.



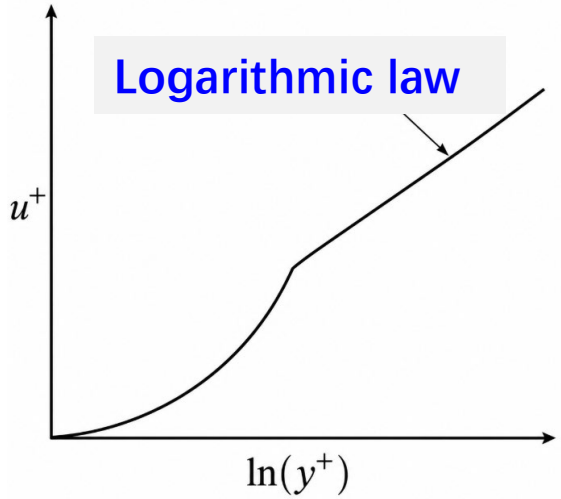
Viscous sub-layer.

The 1st inner node is set outside the viscous sub-layer

(1) Logarithmic law of velocity distribution in boundary layer

$$u^+ = \frac{u}{v^*} = \frac{1}{\kappa}\ln(\frac{yv^*}{\nu}) + B = \frac{1}{\kappa}\ln(y^+) + B = \frac{1}{\kappa}\ln(Ey^+)$$

$$v^* = \sqrt{\tau_w / \rho}, \quad \kappa = 0.4 \sim 0.42, \quad B = 5.0 \sim 5.5$$



(2) New definitions of y^+, u^+

In order that the logarithmic law can reflect some characteristics of turbulence, the law is reformed in two aspects as follows:

- Replacing v^* by $C_\mu^{1/4}k^{1/2}$ to define y^+:
- Introducing $C_\mu^{1/4}k^{1/2}$ into u^+ definition

$$\frac{yv^*}{\nu} \Rightarrow y^+ = \frac{y(C_\mu^{1/4}k^{1/2})}{\nu}$$

$$u^+ = \frac{u}{v^*} = \frac{u}{v^*}\frac{C_\mu^{1/4}k^{1/2}}{v^*} = \frac{u(C_\mu^{1/4}k^{1/2})}{\tau_w / \rho}$$

When dissipation and production of fluctuation kinetic energy are balanced, the above definitions are identical to conventional definition in fluid mechanics.

(3) Definition of dimensionless temperature：mimicking (模仿) the definition of u^+:

$$u^+ = \frac{u(C_\mu^{1/4} k^{1/2})}{\tau_w / \rho}$$

$$T^+ = \frac{(T - T_w)(C_\mu^{1/4} k^{1/2})}{(q_w / \rho c_p)}$$

Mimicking velocity

Mimicking stress

Required by dimension consistency

(4) Logarithmic laws of u & T in turbulence model

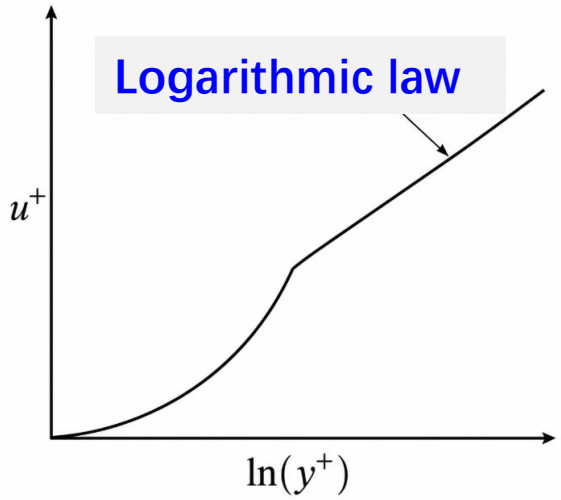
For $y^+ > 11.0$ following distributions are adopted:

$$u^+ = \frac{1}{\kappa}\ln(Ey^+), \qquad \frac{1}{\kappa}\ln(E) = 5.0 \sim 5.5$$

$$T^+ = \frac{\sigma_t}{\kappa}\ln(Ey^+) + P\sigma_t \qquad P = 8.96\left(\frac{\sigma_l}{\sigma_t} - 1\right)\left(\frac{\sigma_l}{\sigma_t}\right)^{-1/4}$$

$\sigma_l = \mathrm{Pr}_l; \sigma_t = \mathrm{Pr}_t$ If $\mathrm{Pr}_l = \mathrm{Pr}_t = 1$ then $T^+ = u^+$

Then this is Reynolds analogy（雷诺比拟）.



For $y^+ < 11.0$, it is regarded as laminar sublayer.

2) Placing the 1st inner node P outside the viscous sub-layer, where logarithmic law valid ($y_P^+ > 11.0$)

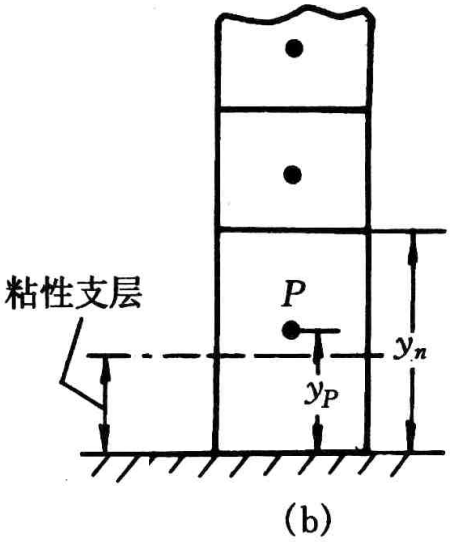
3) The effective turbulent viscosity and thermal conductivity between the 1st inner node and the wall should satisfy：

$$\tau_w = \eta_B \frac{u_P - u_W}{y_P}, \quad q_w = \lambda_B \frac{T_P - T_W}{y_P}$$

The equations of effective viscosity and thermal conductivity between the 1st inner node and the wall can be derived as follows:

(1) Equation for η_B: At point P, u^+ satisfy:

$$\frac{u_P(C_\mu^{1/4}k_P^{1/2})}{\tau_w/\rho}=\frac{1}{\kappa}\ln[Ey_P(\frac{C_\mu^{1/4}k_P^{1/2}}{\nu})]$$



This equation can be re-written as follows:

$$\tau_w=\frac{\rho u_P(C_\mu^{1/4}k_P^{1/2})}{\frac{1}{\kappa}\ln[Ey_P\frac{C_\mu^{1/4}k_P^{1/2}}{\nu}]}$$

$$\tau_w=\eta_B\frac{u_P-u_W}{y_P}$$

$$=\eta_B\frac{u_P-0}{y_P}$$

η_B equation can be derived from this equation.

$$\frac{\rho u_P (C_\mu^{1/4} k_P^{1/2})}{\frac{1}{\kappa}\ln[E y_P \frac{C_\mu^{1/4} k_P^{1/2}}{\nu}]} = \eta_B \frac{u_P}{y_P}$$

$$\eta_B = [\frac{y_P (C_\mu^{1/4} k_P^{1/2})}{\nu}](\rho\nu)\frac{1}{\frac{1}{\kappa}\ln(E y_P^+)} = (\frac{y_P^+}{u_P^+})\eta_l$$

y_P^+ η_l u_P^+

In the turbulent vigorous region $y_P^+ >> u_P^+$, above equation shows:

turbulent viscosity is y_P^+ / u_P^+ times of laminar viscosity.

For example $y_P^+ = 100, u_P^+ = \frac{1}{\kappa}\ln(100) + B = \frac{1}{0.4}4.605 + 5.0 = 16.5$

Then: $\eta_B = (100/16.5)\eta_l = 6.06\eta_l$

(2) Equation for λ_B: At point P, T^+ satisfy:

$$\frac{(T_P - T_W)(C_\mu^{1/4} k_P^{1/2})}{q_w / \rho c_p} = \frac{\sigma_t}{\kappa} \ln(E y_P^+) + \sigma_t P$$

From which:

$$q_w = \frac{\rho c_p (T_P - T_W)(C_\mu^{1/4} k_P^{1/2})}{\frac{\sigma_t}{\kappa} \ln(E y_P^+) + \sigma_t P} \quad \boxed{q_w = \lambda_B \frac{T_P - T_W}{y_P}} \quad = \lambda_B \frac{(T_P - T_W)}{y_P}$$

$$\lambda_B = \frac{(C_\mu^{1/4} k_P^{1/2}) y_P}{\nu} \frac{\rho c_p \nu}{\lambda_l} \frac{\lambda_l}{\frac{\sigma_t}{\kappa} \ln(E y_P^+) + \sigma_t P} = (\frac{y_P^+}{T_P^+}) \mathrm{Pr}_l \lambda_l$$

y_P^+

$\frac{\eta c_p}{\lambda} = \mathrm{Pr}_l$

T_P^+

This is equivalent to magnify the molecular conductivity by $(\frac{y_P^+}{T_P^+}) \mathrm{Pr}_l$ times.

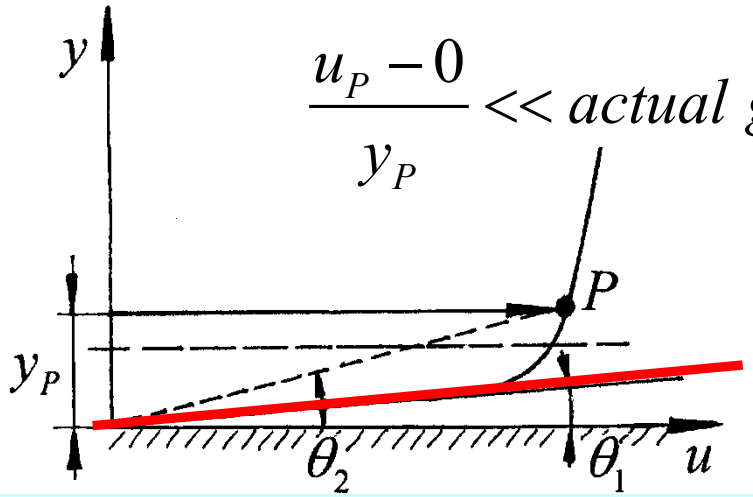
For $\mathrm{Pr}_l = 5.0, \mathrm{Pr}_t = 1.0, y_P^+ = 100,$

yielding $T_P^+ = 40.5, \dfrac{y_P^+}{T_P^+}\mathrm{Pr}_l = \dfrac{100}{40.5} \times 5.0 = 12.3$

The molecular conductivity is magnified by 12.3 times !

Why wall viscosity and conductivity η_B, λ_B should be magnified? This is because the 1st inner node is far from wall, leading to reduced wall gradient determined by FD method.

In WFM, the magnified transport properties compensate (弥补) the reduced gradients near the wall, so that their products will be approximately equal to the true values.



Wall functions refer to the expressions of η_B, λ_B

Fundamentals of Wall Function Method

1) Assuming that the dimensionless velocity and temp. distributions outside the viscous sub-layer are of logarithmic law(对数律) type.

$$\text{For } y^+ > 11.0 \begin{cases} u^+ = \dfrac{1}{\kappa}\ln(Ey^+), & \dfrac{1}{\kappa}\ln(E) = 5.0 \sim 5.5 \\ T^+ = \dfrac{\sigma_t}{\kappa}\ln(Ey^+) + P\sigma_t; & P = 8.96\left(\dfrac{\sigma_l}{\sigma_t} - 1\right)\left(\dfrac{\sigma_l}{\sigma_t}\right)^{-1/4} \end{cases}$$

2) Placing the 1st inner node P outside the viscous sub-layer, where logarithmic law valid ($y_P^+ > 11.0$)

3) The effective turbulent viscosity and thermal conductivity between the 1st inner node and the wall can be determined by following equations:

$$\eta_B = \left(\frac{y_P^+}{u_P^+}\right)\eta_l; \quad \lambda_B = \left(\frac{y_P^+}{T_P^+}\right)\mathrm{Pr}_l\lambda_l; \quad y_P^+ = \frac{y_P(C_\mu^{1/4}k_P^{1/2})}{\nu}$$

4) The boundary condition of k equation: $\left.\frac{\partial k}{\partial n}\right)_w = 0$

$$\rho u_j \frac{\partial k}{\partial x_j} = \underbrace{\frac{\partial}{\partial x_j}\left[\left(\eta + \frac{\eta_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]}_{\text{Diffusion}} + \underbrace{P_k}_{\text{Production}} - \underbrace{\rho\varepsilon}_{\text{Dissipation}} \qquad \frac{d}{dy}\left(\eta_{\text{eff}}\frac{dk}{dy}\right) \approx 0$$

In the equilibrium logarithmic region, turbulent kinetic energy production approximately balances dissipation. By further neglecting the wall-normal diffusive flux, a zero-gradient boundary condition for k is adopted.

5) The dissipation of fluctuation kinetic energy at 1st inner node is determined by the model equation:

$$\varepsilon_P = \frac{C_D k^{3/2}}{l} = \frac{C_\mu^{3/4} k_P^{3/2}}{\kappa y_P}$$

Ozoe, H., Mouri, A., Hiramitsu, M., Churchill, S. W., & Lior, N. (1986). ASME Journal of Heat Transfer, 108(4), 806–813.

For the 1st inner node, dissipation rate is specified by above equation, and computation is limited within the region surrounded by the 1st inner nodes.

8.5.3 Boundary conditions of k, ε for standard $k-\varepsilon$ model

1. Inlet boundary

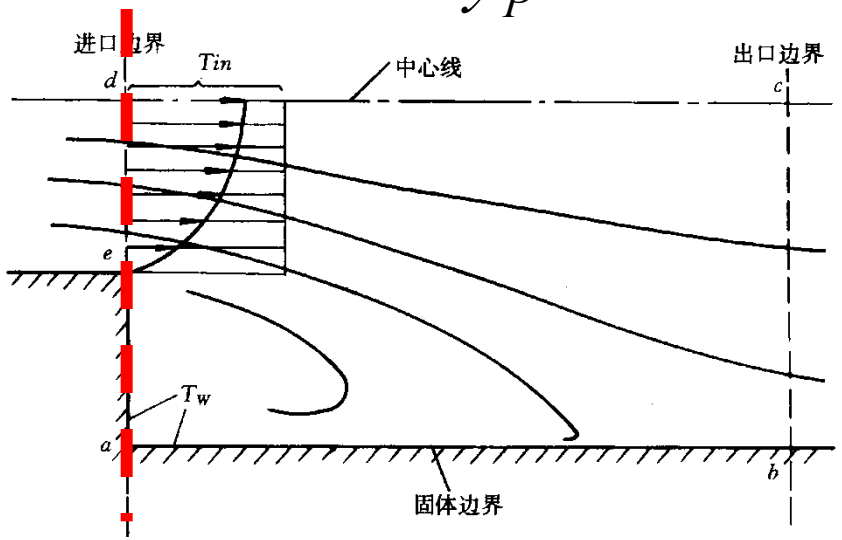
1) k: (1)Adopting test data; (2)Taking a percentage of kinetic energy of oncoming flow. For fully developed flow in ducts:0.5~1.5%;

2) ε: (1) Using model equation: $\varepsilon = \dfrac{C_\mu^{3/4} k^{3/2}}{\kappa y_P}$

(2) Using $\eta_t = C_\mu \rho k^2 / \varepsilon$

assuming $\dfrac{\rho u L}{\eta_t} = 100 \sim 1000$

yielding η_t with inlet u and L.



2. At central line: $\frac{\partial k}{\partial n}=\frac{\partial \varepsilon}{\partial n}=0$

3. Outlet: Adopting local one way assumption

3. Solid wall: Adopting wall function method

(1)Velocity— Velocity normal to wall $\partial\phi/\partial n)_w=0$

Velocity parallel to wall $\phi_w=0,$

And wall viscosity determined by WFM.

Remarks: here, velocity is the dependent variable to be solved, not the one in the nonlinear part of convection term, for which wall velocities always equal zero: u=v=0.

(2) k— Adopting $\frac{\partial k}{\partial n}=0$, implemented via setting $\Gamma_B=0$

(3) ε — Specifying the 1st inner node by $\varepsilon_P = \dfrac{C_\mu^{3/4} k^{3/2}}{\kappa y_P}$

Then cutting connection with boundary.

8.5.4 Cautions in implementing wall function method

1) Approximate range of y_P^+, x_P^+

$$11.5 \sim 30 \le (y_P^+, x_P^+) \le 200 \sim 400$$

Logarithmic law is valid in this range

2) Under-relaxation

In the iteration process, η_t, k, ε must be under-relaxed.

And it is organized within the solution process.

3) ε_P can be specified by large coefficient method

4) Source term treatment of k, ε

$$S_k = \rho G - \rho\varepsilon = \rho G - (\rho\varepsilon / k^*)k$$

S_C S_P

$$S_\varepsilon = \frac{C_1\rho\varepsilon G}{k} - \frac{C_2\rho\varepsilon^2}{k} = \frac{C_1\rho\varepsilon G}{k} - \frac{C_2\rho\varepsilon^*}{k}\varepsilon$$

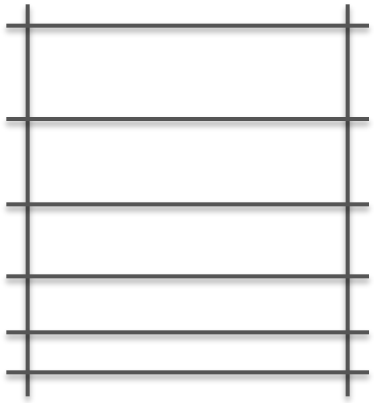
S_C S_P

8.5.5 Other wall treatment method

Why Near-Wall Treatment Depends on y^+

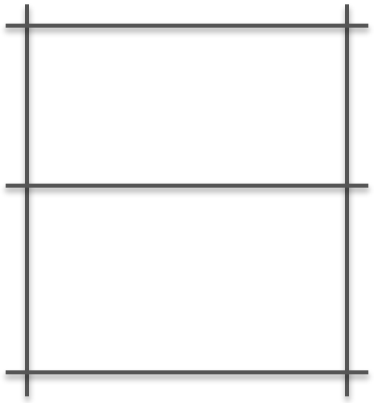
A standard wall function assumes a first cell in the log layer: $u^+ = \frac{\ln(y^+)}{\kappa} + B$.

Fine mesh



$y^+ \approx 1$

Coarse mesh



$y^+ > 30$

Why can refinement cause errors?

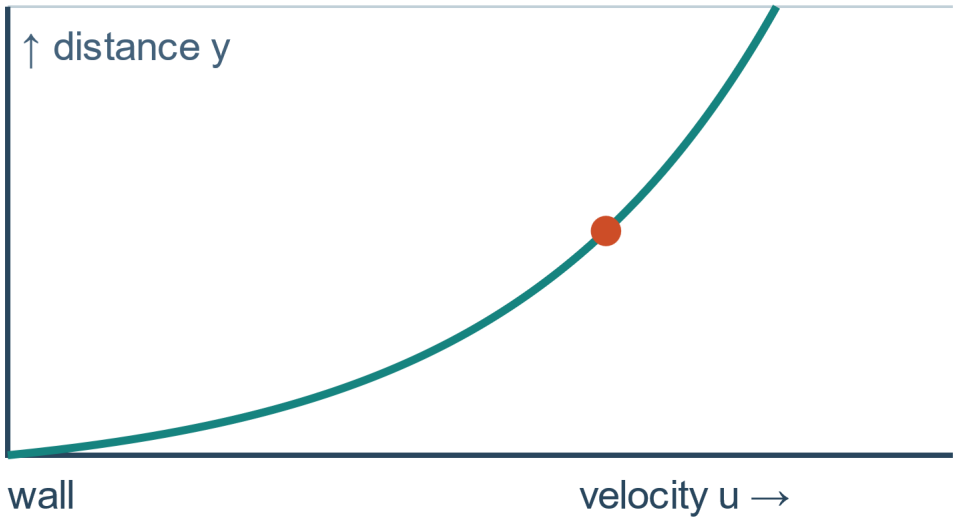
Refinement moves the first cell into the buffer or viscous layer. The log law no longer describes that point; near the wall, $u^+ = y^+$.

Goal: reduce changes in wall shear and heat flux when the first-cell y^+ changes.

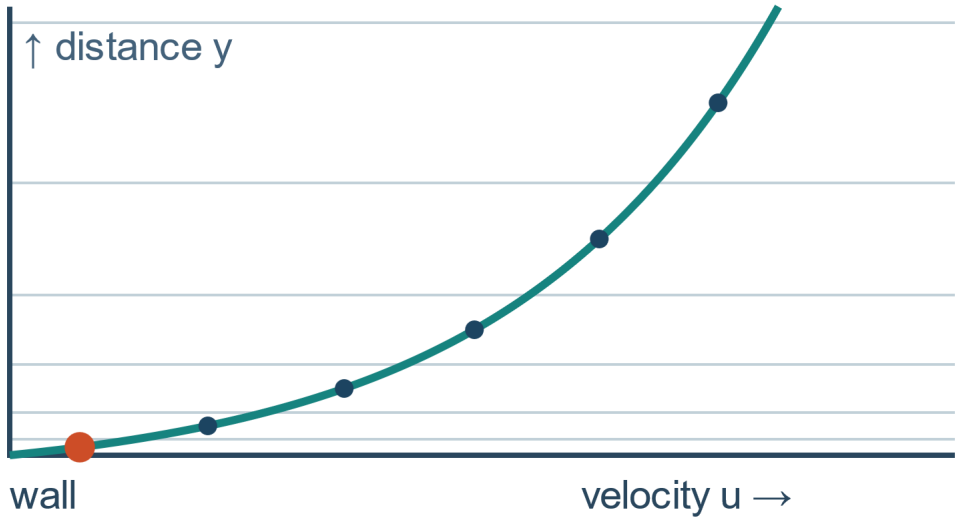
Wall Shear Stress Prediction: Viscous and Logarithmic Limits

Wall shear stress evaluation depends on near-wall mesh resolution: coarse meshes use the log law, while fine meshes use the resolved velocity gradient.

Coarse mesh · $y^+ \gtrsim 30$



Fine mesh · $y^+ \approx 1$



Log-layer limit: solve $\frac{U_p}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y_p u_\tau}{\nu}\right) + B$ for u_τ, then $\tau_w = \rho u_\tau^2$.

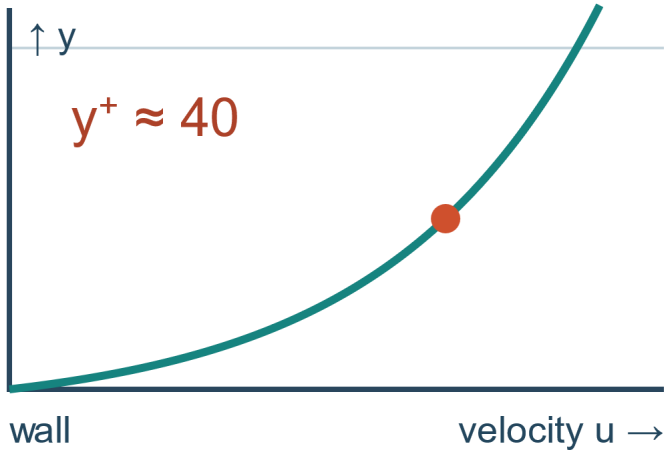
Viscous limit: $\frac{U_p}{u_\tau} = \frac{y_p u_\tau}{\nu}$ gives $\tau_w \approx \mu \frac{U_p}{y_p}$, the resolved wall gradient.

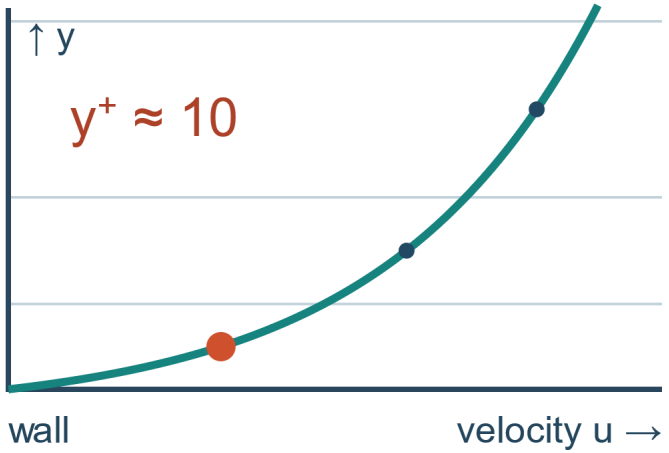
Definitions: $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$, $y^+ = \frac{y_p u_\tau}{\nu}$. Here τ_w denotes the wall-shear magnitude.

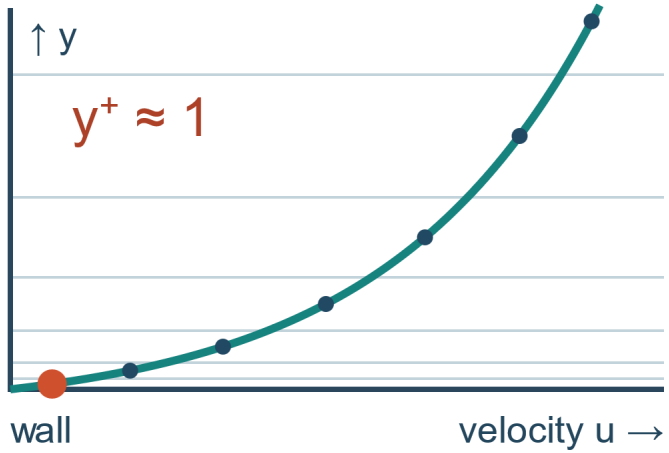
How y^+-Insensitive Treatment Works

Use a continuous relation $\frac{U_p}{u_\tau} = F\left(\frac{y_p u_\tau}{\nu}\right)$. F connects the two limits below.

Large y^+ → Intermediate y^+ → Small y^+







Recover the log law: $F(y^+) \approx \frac{\ln(y^+)}{\kappa} + B$. Model the unresolved wall shear.

Blend the viscous and log limits smoothly, with consistent turbulence boundary conditions.

Recover $F(y^+) \approx y^+$ and $\tau_w \approx \mu \frac{U_p}{y_p}$.

Resolve the viscous layer on the fine mesh.

The continuous relation reduces y^+ sensitivity to grid resolution.

8.6 Low Reynolds Number k-epsilon Model

8.6.1 Application range of standard $k-\varepsilon$ model

8.6.2 Jones-Launder's considerations for low Re $k-\varepsilon$ model

8.6.3 Jones-Launder low Re k-ε model

8.6 Low Reynolds Number k-epsilon Model

8.6.1 Application range of standard k-ε model

1. Near wall velocity distribution obeys logarithmic law
2. Shear stress is distributed uniformly from wall to 1st inner node;
3. Production and dissipation are nearly balanced for fluctuation kinetic energy.

Above assumptions are valid only when $\mathrm{Re}_t = \dfrac{\rho k^2}{\eta \varepsilon} > 150$

If this Re_t less than 150, the standard $k-\varepsilon$ model can not be used. When approaching wall, this Reynolds number becomes smaller and smaller.

In order that simulation can be conducted from vigorous part to the wall, model should be modified.

8.6.2 Jones-Launder's consideration of low *Re* *k*-ε model

(1) Both molecular and turbulent diffusions should be considered;

(2) Effects of $\mathrm{Re}_t = \dfrac{\rho k^2}{\eta\varepsilon}$ on coefficients in *k*-ε should be considered;

(3) Near a wall, dissipation of fluctuation kinetic energy is not isotropic, and this should be taken into account in *k* eq.

8.6.3 Jones – Launder low Re *k*-ε model

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}]+\rho G-\rho\varepsilon-\underbrace{2\eta(\frac{\partial k^{1/2}}{\partial y})^2}_{D}$$

$$\frac{\partial(\rho\varepsilon)}{\partial t}+\frac{\partial(\rho u_j\varepsilon)}{\partial x_j}=\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_\varepsilon})\frac{\partial\varepsilon}{\partial x_j}]+f_1C_1\frac{\rho G\varepsilon}{k}-f_2C_2\rho\frac{\varepsilon^2}{k}+2\frac{\eta_l\eta_t}{\rho}(\frac{\partial^2u}{\partial y^2})^2$$

E

$$\eta_t=C_\mu f_\mu\rho k^2/\varepsilon \qquad \mathrm{Re}_t=\frac{\rho k^2}{\eta\varepsilon}$$

where $f_1=1.0$ $f_2=1.0-0.3\exp(-\mathrm{Re}_t^2)$ $f_\mu=\exp(-2.5/(1+\mathrm{Re}_t/50))$

3. Explanations for additional terms

(1) $D=-2\eta(\frac{\partial k^{1/2}}{\partial y})^2$ (y is normal to wall), for considering that near a wall fluctuation kinetic energy is not isotropic, and with this term the condition of $\varepsilon_w=0$ can be used;

(2) The term E is for a better agreement with test data.

4. Boundary condition of J-L low *Re* model

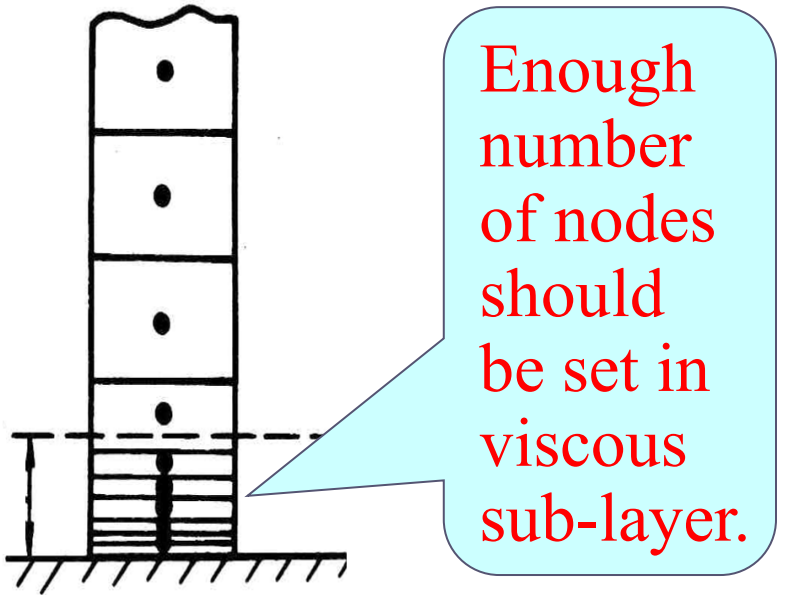
$$k_w=\varepsilon_w=0$$

Remarks For the low *Re* number *k*-ε model

1. The Reynolds number refers to the turbulent Reynolds number, defined by

$$\mathrm{Re}_t = \frac{\rho k^2}{\eta \varepsilon}$$

2. The grid number within the laminar sublayer should be larger than 10;



3. The four coefficients, $C_1, C_2, \sigma_k, \sigma_\varepsilon$, have rather wide ranges of variation, as long as they can meet a bunch of test data.

8.7 Introduction of k-omega Model

8.7.1 Characteristic of standard k-ε model

1. The standard k-ε model is a **high-Reynolds-number** turbulence model.
2. It is mainly developed for fully turbulent regions and free shear flows.
3. It cannot directly resolve the near-wall region. Additional wall treatments or low-Reynolds-number corrections are required.
4. The standard k-ε model often shows poor performance **under adverse pressure gradients**.
5. To overcome these limitations, another two-equation turbulence model (**k-ω model**) was developed.

8.7.1 Standard k–ω Model (Wilcox, 1988)

1. Second variable: specific dissipation rate (比耗散率)

The k-ω model is a two-equation turbulence model.

The second variable, $\omega \sim \varepsilon/k$, is called the specific dissipation rate.

ω can also be interpreted as a characteristic frequency scale of turbulence

$$\omega \sim \frac{1}{k/\varepsilon}$$

k/ε: characteristic time scale of turbulent fluctuations

2. Transport equation for k

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta^*\rho\omega k+\frac{\partial}{\partial x_j}\left[(\eta+\frac{\eta_t}{\sigma_{k1}})\frac{\partial k}{\partial x_j}\right]$$

3. Transport equation for ω

$$\frac{\partial(\rho\omega)}{\partial t}+\frac{\partial(\rho u_j \omega)}{\partial x_j}=\alpha_1\frac{\omega}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta_1\rho\omega^2+\frac{\partial}{\partial x_j}\left[(\eta+\frac{\eta_t}{\sigma_{\omega 1}})\frac{\partial \omega}{\partial x_j}\right]$$

$\tau_{i,j}$ denotes the turbulent stress; β^*=0.09; α_1= 5/9、β_1= 3/40、σ_{k1} = 2、$\sigma_{\omega 1}$ = 2;

$\eta_t = \rho k/\omega$.

8.7.1 Standard *k*-ω Model: Comparison with *k*-ε

1. Common structure and variable transformation

With the Boussinesq hypothesis, the standard k-ω and k-ε models share the same mathematical form for the k equation.

The ω equation can be transformed into a form similar to the ε equation using

$$\omega = \varepsilon / (\beta^* k)$$

2. Main differences

A cross-diffusion term involving $\nabla k \cdot \nabla \omega$ appears.

The turbulence-model constants are also different.

3. Near-wall behavior

Near the wall: k decreases ↓ ε increases ↑

Therefore, ω becomes more sensitive to near-wall effects.

8.7.2 Near-wall asymptotic behavior: *k-ω* .vs. standard *k-ε*

A. *k-ω*: near-wall solution exists

1. Leading-order equation

$$\nu\frac{d^2\omega}{dy^2}-\beta\omega^2\approx 0$$

diffusion destruction

2. Assume a power-law solution

$$\omega = A\nu y^{-p}$$

3. Match the powers of *y*

diffusion ~ $y^{-(p+2)}$; destruction ~ y^{-2p}

$p=2;\ 6A=\beta A^2 \rightarrow A=6/\beta$

$$\omega \sim 6\nu/(\beta y^2)$$

B. High-Re *k-ε*: wrong wall behavior

1. Leading-order requirement

$$\nu\frac{d^2\varepsilon}{dy^2}-C_{\varepsilon 2}\frac{\varepsilon^2}{k}\approx 0$$

diffusion dissipation

2. A smooth, finite wall value requires

$$\varepsilon=\varepsilon_w+by+cy^2+\cdots$$

3. Compare the orders $k\sim ay^2$

diffusion $\nu\frac{d^2\varepsilon}{dy^2}\sim O(1)$ dissipation $\frac{\varepsilon^2}{k}\sim y^{-2}$

The two terms cannot balance.

The *k-ω* model can resolve the near-wall region directly and provides better near-wall performance

Kaul, U. K. (2010). Effect of Inflow Boundary Conditions on the Solution of Transport Equations for Internal Flows. AIAA Paper 2010-4743.

Standard k–ω: near-wall behavior

Near-wall prediction

In principle, k–ω predicts near-wall flow better than k–ε.

Direct integration to the wall

The k–ω model can be integrated directly to the wall without additional damping functions or wall functions. However, this requires fine near-wall meshes ($y^+ \sim 1$). In practical CFD simulations, automatic near-wall treatments are often employed to accommodate different mesh resolutions.

Its main limitation

Free-shear-flow predictions are sensitive to freestream boundary conditions.

Wilcox (2006): modified k–ω equations

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta^*\rho\omega k+\frac{\partial}{\partial x_j}\left[(\eta+\frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial(\rho\omega)}{\partial t}+\frac{\partial(\rho u_j \omega)}{\partial x_j}=\alpha_1\frac{\omega}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta_1\rho\omega^2+\frac{\partial}{\partial x_j}\left[(\eta+\frac{\eta_t}{\sigma_\omega})\frac{\partial \omega}{\partial x_j}\right]+\frac{\rho\sigma_d}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$$

Cross-diffusion coefficient σ_d

$$\sigma_d=\begin{cases}0, & \frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}\le 0\\ \frac{1}{8}, & \frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}>0\end{cases}$$

Minimize dependence on free-stream (ω)

Improve turbulent shear-layer predictions

Preserve near-wall behavior

BSL k–ω: combine two formulations

Near-wall region: k–ω

Use the standard k–ω formulation for near-wall accuracy.

Outer region: k–ε

Use the standard k–ε formulation for free-shear-flow behavior.

Rewrite the k–ε model as k and ω transport equations before blending.

BSL k–ω: k–ε rewritten in k–ω form

Outer-region transport equations

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta^*\rho\omega k+\frac{\partial}{\partial x_j}\left[(\eta+\frac{\eta_t}{\sigma_{k2}})\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial(\rho\omega)}{\partial t}+\frac{\partial(\rho u_j \omega)}{\partial x_j}=\alpha_2\frac{\omega}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta_2\rho\omega^2+\frac{\partial}{\partial x_j}\left[(\eta+\frac{\eta_t}{\sigma_{\omega 2}})\frac{\partial \omega}{\partial x_j}\right]$$

$$+\rho\sigma_{\omega 2}\frac{2}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$$

BSL k–ω: blending with F_1

Blend the near-wall and outer-layer formulations through F_1.

$$F_1 \times (k - \omega)$$

Near-wall formulation

$$(1 - F_1) \times (k - \varepsilon \rightarrow k - \omega)$$

Outer-layer formulation

BSL k–ω: resulting transport equations

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta^*\rho\omega k+\frac{\partial}{\partial x_j}\left[(\eta+\frac{\eta_t}{\widetilde{\sigma}_k})\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial(\rho\omega)}{\partial t}+\frac{\partial(\rho u_j \omega)}{\partial x_j}=\widetilde{\alpha}\frac{\omega}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j}-\widetilde{\beta}\rho\omega^2+\frac{\partial}{\partial x_j}\left[(\eta+\frac{\eta_t}{\widetilde{\sigma}_\omega})\frac{\partial \omega}{\partial x_j}\right]$$

$$+2(1-F_1)\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$$

The equations retain the *k*–*ω* form; model coefficients are determined by F_1.

SST k–ω: limit turbulent viscosity

BSL k-ω model combines near-wall k-ω with outer-region k-ε.

Under adverse pressure gradients, BSL may overpredict shear stress and turbulent viscosity.

SST k-ω model addresses this by limiting turbulent viscosity:

$$\eta_t = \frac{\rho a_1 k}{\max(a_1 \omega,\ \sqrt{2} S F_2)}$$

F_2 is an auxiliary function.

Home Work 8（2026-2027）

Please finish your homework independently !!!

Please hand in on Nov. 26，2026

Problem 8-1

Estimate the value of the turbulent effective pressure based on the following data: Air is going through a wind tunnel with velocity of 35 ms^{-1} and pressure of one bar. The turbulent intensity $\left(\sqrt{\overline{(u')^2}}/u\right)$ is isotropic ($\overline{(u')^2}=\overline{(v')^2}=\overline{(w')^2}$) and equals 5%.

Problem 8-2

In a 2-D boundary layer flow, if the generation of turbulence kinetic energy and the dissipation are balanced each other, try to show:

$$\sqrt{\tau_w / \rho} = C_\mu^{1/4} k^{1/2}$$

Problem 8-3

Analyze the dimension and unit for the coefficient and constant in the k-Epsilon turbulence model : $C_\mu, C_1, C_2, \sigma_k, \sigma_\varepsilon, \sigma_T$.

Problem 8-4

A very simple turbulence model proposed by Chen is as follows:

$$\eta_t = 0.03874\rho v l$$

ρ – Air density

v – Local time average velocity

l – The shortest distance to the wall

In a conventional working space, the air velocity usually varies from 0.5ms^{-1} to 2-3ms^{-1}. Calculate the turbulent viscosity for air average velocity of 1.5 ms^{-1} with l=0.1m ,and velocity of 2.5 ms^{-1} with l=0.2m.

Problem 8-5

In a wind tunnel of square cross section (0.5m × 0.5m) the air average velocity is 50 ms^{-1}. The isotropic turbulence intensity is 5%.

Calculate the turbulent Reynolds number by following equation,

$$\mathrm{Re}_t = \frac{\rho k^2}{\eta \varepsilon}$$

Assume that air pressure is one bar and the dissipation rate can be estimated from following equation with y_P of 0.5m,

$$\varepsilon = \frac{C_\mu^{3/4} k^{3/2}}{\kappa y_P}$$

Compare Re_t and the conventional Reynolds number defined by,

$$Re = \frac{uD}{\nu}$$

本组网页地址：http://nht.xjtu.edu.cn 欢迎访问！

Teaching PPT will be loaded on ou website



同舟共济 渡彼岸！

People in the same boat help each other to cross to the other bank, where....