Numerical Heat Transfer
(数值传热学)
Chapter 8 Numerical Simulation for Turbulent Flow and Heat Transfer（Chapter 9 in Textbook）

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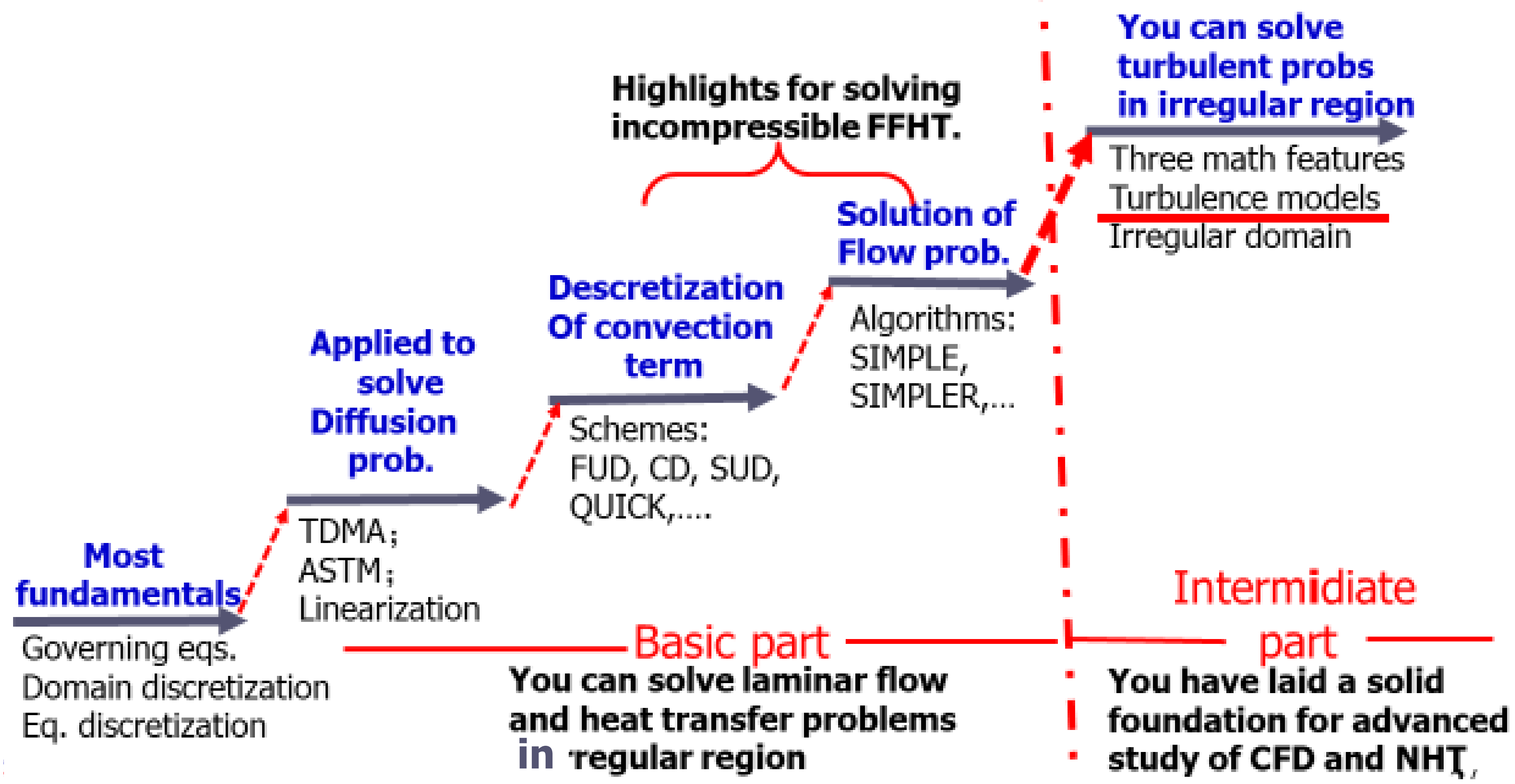
Road map of 40 hrs teaching for basic theory



8.1 Introduction to turbulence

8.1.1 Present understanding of turbulence

8.1.2 Classifications of turbulence simulation methods

8.1.3 Reynolds time-averages and their characteristics

8.1 Introduction to turbulence

8.1.1 Present understanding of turbulence

1. Turbulence is a highly complicated unsteady flow, within which all kinds of physical quantities are randomly varying with both time and space；

2. Transient Naiver-Stokes are valid for turbulent flows；

3. Turbulent flow field can be regarded as a collection of eddies（涡漩） with different geometric scales .

Eddy vs. vortex (漩涡)： Eddy is characterized by turbulent flow with randomness, and it covers a wide range of geometric scales;

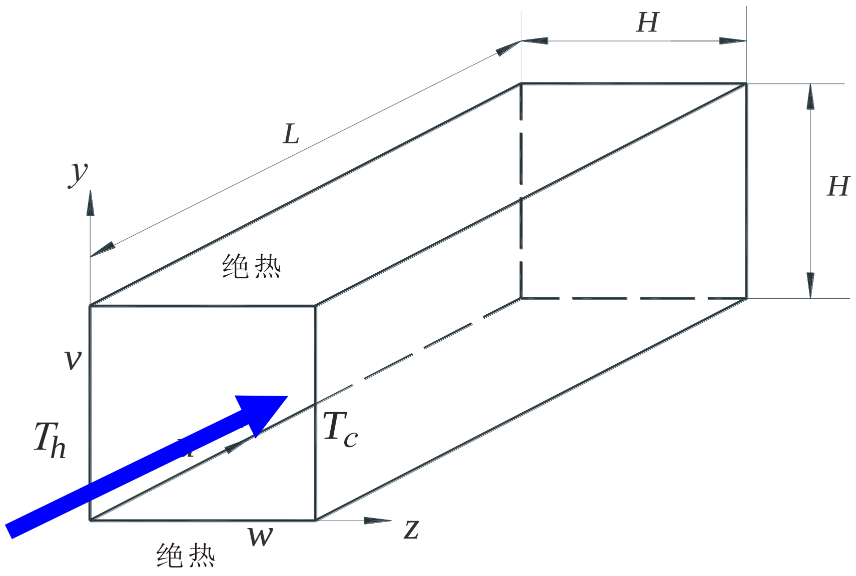
Vortex is kind of flow pattern caused by a specific solid outline characterized by a recirculation. Such vortex flow can be laminar or turbulent. The Chinese translations were proposed by Prof. G Z Liu（刘光宗）.

8.1.2 Classifications of turbulence simulation methods

Numerical turbulence model

- **N-S equation**
  - DNS Direct numerical simulation
  - LES Large eddy simulation
  - RANS — **R**eynolds **A**veraged **N-S** eqs.
    - Second moment closure
    - Algebraic stress model
    - Turbulent viscosity model
      - Mixing length theory
      - One-equation model
      - Two-equation model
      - Non-linear model. etc.
- **Boltzmann equation**

1.DNS In DNS very small time step and space step are needed to reveal the evolutions (演化) of eddies with different scales. Required computer resource is very high. Often high-performance computers (HPC) are needed.



For a fully developed mixed convection in a square duct (L=6.4H), when Re=6400, Gr= $10^4 \sim 10^7$ DNS is conducted with 4.194×10^6 nodes(=256×128×128), and 8×10^5 time steps are needed for statistical average.

2. LES

Basic idea： Turbulent fluctuations are mainly generated by large scale eddies, which are non-isotropic(各向异性) and vary with flow situation； Small scale eddies dissipate(耗散) kinetic energy (from mechanic to thermal energy), and are almost isotropic (各向同性). The N-S eqs. are used to simulate the large scale eddies and the behavior of small scale eddies is simulated by simplified model.

LES requires less computer resource than that of DNS, even though still quite high, and has been used for some engineering problems

For the above problem when simulated by LES only 128×80×80=819200 grids are needed (compared with 4.194×10^{6}).

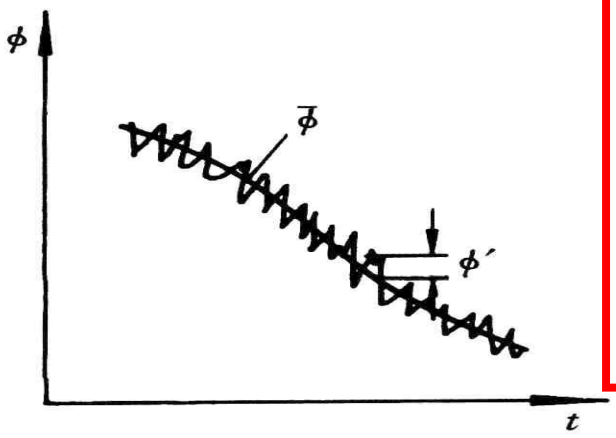
3. Reynolds time average N-S Eqs. methods

Expressing a transient term as the sum of average term and fluctuation(脉动) term. Time average is conducted for the transient N-S equations, and the time average terms of the fluctuations is expressed via some function of the average terms.

8.1.3 Reynolds time averages and their characteristics

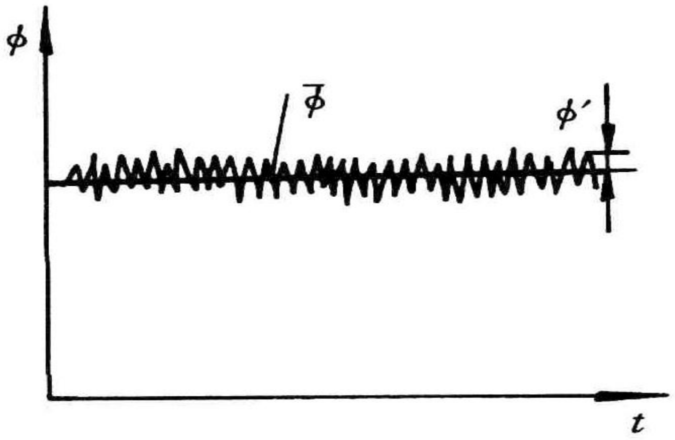
$$\phi = \overline{\phi} + \phi' \qquad \overline{\phi} = \frac{1}{\Delta t}\int_{t}^{t+\Delta t} \phi(t)dt$$

Δt is the time step, which should be large enough relative to the fluctuation but small enough with respect to the variation period of the time averaged quantity.



非稳态湍流

(a)
Unsteady



准稳态湍流

(b)
Quasi-steady

Characteristics of time average operations

1. $\overline{\phi'} \equiv 0;$ 2. $\overline{\overline{\phi}} = \overline{\phi};$ 3. $\overline{\overline{\phi}+\phi'} = \overline{\phi};$ 4. $\overline{\overline{\phi}\phi'} = \overline{\phi}\,\overline{\phi'} = 0$

5. $\overline{\phi f} = \overline{(\overline{\phi}+\phi')(\overline{f}+f')} = \overline{\phi}\,\overline{f} + \overline{\phi' f'}$

6. $\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \overline{\phi}}{\partial x};$

7. $\overline{\frac{\partial \phi'}{\partial x}} = \frac{\partial \overline{\phi'}}{\partial x} = 0$

8. $\overline{\frac{\partial (\phi f)}{\partial x}} = \frac{\partial (\overline{\phi}\,\overline{f})}{\partial x} + \frac{\partial (\overline{\phi' f'})}{\partial x}$

8.2 Time-averaged governing equation for incompressible convective heat transfer

8.2.1 Time average governing equation

8.2.2 Ways for determining additional terms

8.2.3 Governing equations with turbulent viscosity

8.2 Time-averaged governing equation for incompressible convective heat transfer

8.2.1 Time average governing equation

1. Continuity equation

$$\frac{\partial\overline{(\bar{u}+u')}}{\partial x}+\frac{\partial\overline{(\bar{v}+v')}}{\partial y}+\frac{\partial\overline{(\bar{w}+w')}}{\partial z}=\underbrace{\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}}_{=0}+\underbrace{\frac{\partial\overline{u'}}{\partial x}+\frac{\partial\overline{v'}}{\partial y}+\frac{\partial\overline{w'}}{\partial z}}_{=0}=0$$

Both time average velocity and time average fluctuation velocity satisfy continuity condition.

2. Momentum equation

Taking *x*-direction as an example：

$$\frac{\partial\overline{(\bar u+u')}}{\partial t}+\frac{\partial\overline{(\bar u+u')^2}}{\partial x}+\frac{\partial\overline{(\bar u+u')(\bar v+v')}}{\partial y}+\frac{\partial\overline{(\bar u+u')(\bar w+w')}}{\partial z}=-\frac{1}{\rho}\frac{\partial\overline{(\bar p+p')}}{\partial x}+$$

$$\nu\left[\frac{\partial^2\overline{(\bar u+u')}}{\partial x^2}+\frac{\partial^2\overline{(\bar u+u')}}{\partial y^2}+\frac{\partial^2\overline{(\bar u+u')}}{\partial z^2}\right]$$

According to the above characteristics, yielding

$$\frac{\partial\bar u}{\partial t}+\frac{\partial(\bar u^2)}{\partial x}+\frac{\partial\bar u\bar v}{\partial y}+\frac{\partial\bar u\bar w}{\partial z}+\frac{\partial\overline{(u')^2}}{\partial x}+\frac{\partial\overline{(u'v')}}{\partial y}+\frac{\partial\overline{(u'w')}}{\partial z}=$$

$$=-\frac{1}{\rho}\frac{\partial\bar p}{\partial x}+\nu\left(\frac{\partial^2\bar u}{\partial x^2}+\frac{\partial^2\bar u}{\partial y^2}+\frac{\partial^2\bar u}{\partial z^2}\right)$$

Moved to right hand side and combined with the corresponding viscous term

$$\frac{\partial\bar u}{\partial t}+\frac{\partial(\bar u^2)}{\partial x}+\frac{\partial\bar u\bar v}{\partial y}+\frac{\partial\bar u\bar w}{\partial z}=$$

$$-\frac{1}{\rho}\frac{\partial\bar p}{\partial x}+\frac{\partial}{\partial x}\left[\nu\frac{\partial\bar u}{\partial x}-\overline{(u')^2}\right]+\frac{\partial}{\partial y}\left[\nu\frac{\partial\bar u}{\partial y}-\overline{(u'v')}\right]+\frac{\partial}{\partial z}\left[\nu\frac{\partial\bar u}{\partial z}-\overline{(u'w')}\right]$$

Rewritten in a tensor （张量） form in Cartesian coordinate：

$$\frac{\partial(\rho\bar{u})}{\partial t}+\frac{\partial(\rho\bar{u}_i\bar{u}_j)}{\partial x_j}=-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}(\eta\frac{\partial\bar{u}_i}{\partial x_j}-\rho\overline{u_i^{'}u_j^{'}})\ (i=1,2,3)$$

Double subscript means summation over the subscript.

3. Other scalar (标量) variables

$$\frac{\partial(\rho\bar{\phi})}{\partial t}+\frac{\partial(\rho\bar{u}_j\bar{\phi})}{\partial x_j}=\frac{\partial}{\partial x_j}(\Gamma\frac{\partial\bar{\phi}}{\partial x_j}-\rho\overline{u_j^{'}\phi^{'}})+S$$

4. Discussion on the time averaged quantity

(1) Linear term remains unchanged during time average, while product term (乘积项) generates product of two fluctuations, representing the additional transport caused by fluctuation.

(2) Equations are not closed： for 3-D problem, there are only five equations, with 14 unknown variables：

Five time average variables — $\overline{u}, \overline{v}, \overline{w}, \overline{p}, \overline{\phi},$

Nine products of fluctuations

$\overline{u_i^{'} u_j^{'}}(i, j = 1,2,3)$（6 terms）; $\overline{u_i^{'} \phi^{'}}(i = 1,2,3)$（3 terms）

In order to close the above equations, additional relations must be added. **Such additional relations are called turbulence model, or closure model (封闭模型)。**

The concept of closure model was actually 1st proposed by Prof. P Y Chou (周培源) in 1945.

8.2.2 Ways of determining additional terms

1.Reynolds stress method

For the nine additional variables deriving their own governing equations.

However, in the derivation process new additional terms of higher order (product of three variables, four variables, etc…) are introduced.; If we still go along this direction then equations for much higher order products should be derived. ,,,,,. Thus we have to terminate such process at certain level. Historically some complicated models with more than 20 equations have been derived.

relation $\overline{w_i w'_k}$ were known. But unfortunately the equation of continuity (4.4) and the general dynamical equation of double correlation (7.6) are insufficient to yield a definite solution for $\overline{w_i w'_k}$, because of the presence of the triple correlation $\overline{w_i w_j w'_k}$ in (7.6).

Copied From the original paper of Prof. P. Y. Chou.

In the Reynolds stress models, the second moment model (model for the products of two fluctuation quantities) is quite famous and has been applied in some engineering problems. In the second moment model, for the product terms with two fluctuations their equations are derived, while for the terms with three or more fluctuations models are used to relate such terms with time average variables.

Prof. L X Zhou (周力行) in Tsinghua university contributed a lot in this regard.

2. Turbulent viscosity method

The product of fluctuations of two velocities is expressed via turbulent viscosity.

(1) Definition of turbulent viscosity

In 1877 Boussinesq introduced following equation, by mimicking(比拟) the **constitution equation** (本构方程) of laminar fluid flow:

$$(\tau_{i,j})_t = -\rho\overline{u_i' u_j'} = (-p_t\delta_{i,j}) + \eta_t\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) - \frac{2}{3}\eta_t\delta_{i,j}div\vec{U} \rightarrow 0$$

$$p_t = \frac{1}{3}\rho[\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2}] = \frac{2}{3}\rho k \qquad k = \frac{1}{2}[\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2}]$$

(2) Definition of turbulent diffusivity of other scalar variables

$$-\rho\overline{u_i'\phi'} = \Gamma_t\frac{\partial \overline{\phi}}{\partial x_i} \qquad \Gamma_t = \frac{\eta_t}{Pr_t}$$

Pr_t ---turbulent Prandtl number, usually treated as a constant.

Format Improvement of the General Governing Equation

The G.E. we just learned in the previous chapters is:

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\vec{U}) = div(\Gamma_{\phi}^{*} grad\phi) + S_{\phi}^{*} \quad (1)$$

Equation	ρ	ϕ	$\Gamma_\phi{}^*$	$S_\phi{}^*$
Continuity equation	ρ	1	0	0
Momentum eqn. (x direction)	ρ	u	μ	$\rho f_x - \frac{\partial p}{\partial x}$
Momentum eqn. (y direction)	ρ	v	μ	$\rho f_y - \frac{\partial p}{\partial y}$
Energy equation	ρ	T	λ / c_p	S_T / c_p

This type of general governing equation leads to some in convenience for practical applications. New equation will be used in our teaching code:

$$\frac{\partial(\rho^{*}\phi)}{\partial t} + div(\rho^{*}\phi\vec{U}) = div(\Gamma_{\phi} grad\phi) + S_{\phi}$$

Equation	ρ^*	ϕ	Γ_ϕ	$S_\phi{}^*$
Continuity equation	ρ	1	0	0
Momentum eqn. (x direction)	ρ	u	μ	$\rho f_x - \frac{\partial p}{\partial x}$
Momentum eqn. (y direction)	ρ	v	μ	$\rho f_y - \frac{\partial p}{\partial y}$
Energy equation	ρc_p	T	λ	S_T

That is, here we regard ρc_p in the energy equation as a general density:

$$\frac{\partial(\rho c_p T)}{\partial t} + \frac{\partial(\rho c_p u T)}{\partial x} + \frac{\partial(\rho c_p v T)}{\partial y} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + S_T.$$

Such a treatment is much better than taking Γ / c_p as a nominal diffusion coefficient and S_T / c_p as a nominal source term in energy eq.

Li W, Yu B, Wang Y， et al. Communications in Computational Physics, 2012, 12(5): 1482-1494

Please note:

Here we take

$\Gamma = \lambda,$

Rather than

$\Gamma = \lambda / c_P$

For laminar heat transfer we have

$$\Gamma_l = \lambda = \frac{\lambda}{c_p}\frac{\eta_l}{\eta_l}c_p = (\frac{\lambda}{c_p\eta_l})\eta_l c_p = \frac{\eta_l c_p}{(\frac{c_p\eta_l}{\lambda})} = \frac{\eta_l c_p}{\Pr_l}$$

Similarly: $\Gamma_t = \lambda_t = \eta_t c_p / \Pr_t$ for new governing equation

Therefore for turbulent viscosity model its major task is to find $\eta_t, \Pr_t$.

The name of engineering turbulence models comes from the number of PDEqs. included in the model to determine turbulence viscosity.

8.2.3 Governing equations of viscocity models

1. Governing equations

For simplicity of presentation, the symbol of time average "bar" is omitted hereafter:

$$\begin{cases} \dfrac{\partial u_k}{\partial x_k} = 0 \\ \dfrac{\partial u_i}{\partial t} + \dfrac{\partial(\rho u_k u_i)}{\partial x_k} = -\dfrac{\partial p_{eff}}{\partial x_i} + \dfrac{\partial}{\partial x_k}[\underbrace{(\eta_l + \eta_t)}_{\eta_{eff}}\dfrac{\partial u_i}{\partial x_k}] + S_i \quad ; p_{eff} = p + p_t \\ \dfrac{\partial(\rho^* \phi)}{\partial t} + \dfrac{\partial(\rho^* u_k \phi)}{\partial x_k} = \dfrac{\partial}{\partial x_k}[\underbrace{(\Gamma_l + \Gamma_t)}_{\Gamma_{eff}}\dfrac{\partial \phi}{\partial x_k}] + S_\phi \end{cases}$$

2. Differences from laminar governing equations:

(1) u_i, p, ϕ -Time average; (2) Replacing Γ by $\Gamma_{eff} = \Gamma + \Gamma_t$

(3) Replacing p by p_{eff} (4) In the source term S_i of u_i the additional terms caused by time averaging are included.

$$\Gamma_t = \lambda_t = \eta_t c_p / \Pr_t \qquad \rho^* \text{---} \rho c_p$$

In the Cartesian coordinates, the source terms of the three components are:

$$u:\ S=\frac{\partial}{\partial x}\left(\eta_{\mathrm{eff}}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\eta_{\mathrm{eff}}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\eta_{\mathrm{eff}}\frac{\partial w}{\partial x}\right)$$

$$v:\ S=\frac{\partial}{\partial x}\left(\eta_{\mathrm{eff}}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\eta_{\mathrm{eff}}\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\eta_{\mathrm{eff}}\frac{\partial w}{\partial y}\right)$$

$$w:\ S=\frac{\partial}{\partial x}\left(\eta_{\mathrm{eff}}\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\eta_{\mathrm{eff}}\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial z}\left(\eta_{\mathrm{eff}}\frac{\partial w}{\partial z}\right)$$

In laminar flow of constant properties, all source terms are zero, but for turbulent flow they are not zero .

η_{eff} is determined by the turbulence model.

3. Turbulent Prandtl number 20241105

Its value varies within a certain range， usually is taken as a constant, and $\Gamma_t = c_p\eta_t / \Pr_t$

8.3 Zero equation model and one equation model

8.3.1 Zero equation model

1. Turbulent additional stress of zero equation model
2. Equations for mixing length
3. Application range of zero eq. model

8.3.2 One equation model

1. Turbulent fluctuation kinetic energy as dependent variable
2. Prandtl-Kolmogorov equation
3. Governing equation of turbulent fluctuation kinetic energy
4. Boundary condition

8.3 Zero Equation Model and One Equation Model

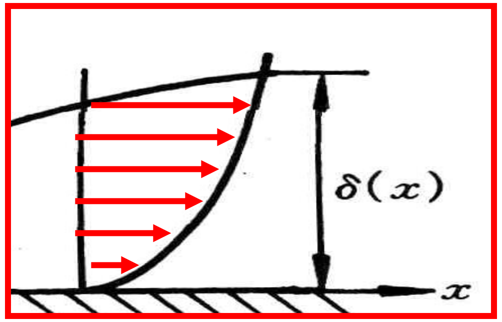
8.3.1 Zero equation model

1. Zero equation model for turbulent additional stress

In zero eq. model no PDE is involved to determine turbulent viscosity. The turbulent stress is expressed as:

Turbulent kinetic viscosity（运动粘性）

$$\tau_t = -\rho \overline{u_i' u_j'} = \rho \overline{u'v'} = \rho \nu_t \left(\frac{du}{dy}\right) = \rho\, l_m^2 \left|\frac{du}{dy}\right| \left(\frac{du}{dy}\right)$$



From dimensionality consideration

Cause of momentum exchange

From Newton shear stress eq.

where l_m is called mixing length, whose determination is the key of zero-eq. model.

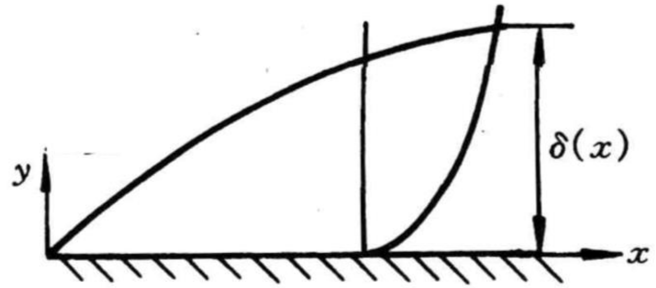
2.Equations for mixing length

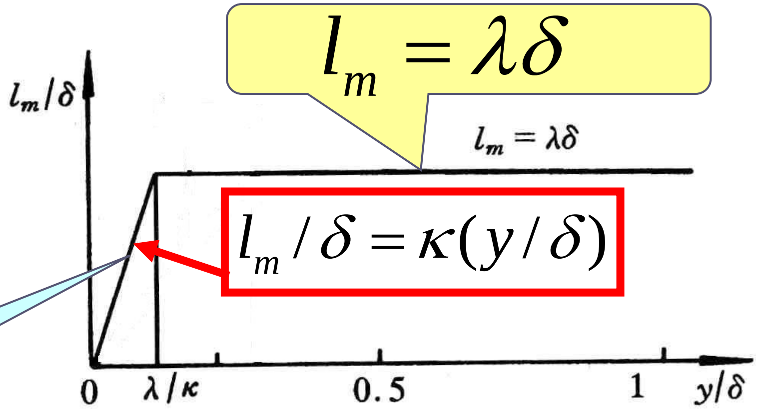
(1) Flow and HT over a plate l_m / δ vs. y/δ is a slope function (斜坡函数)：

At $y/\delta < \lambda/\kappa,\ l_m = \kappa y$

At $y/\delta \geq \lambda/\kappa,\ l_m = \lambda\delta$

Authors	κ	λ
Cebeci	0.41	0.08
P-S	0.435	0.09





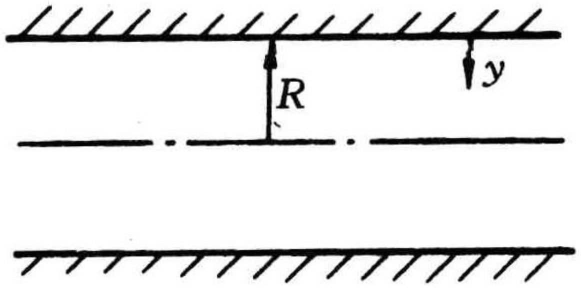
$l_m = \kappa y$

δ ---thickness of B.L.

(2) Turbulent HT in a circular tube--- Nikurads eq.

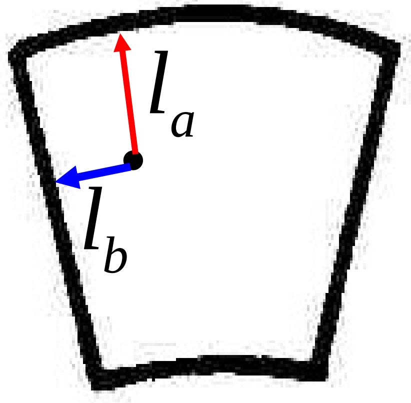
$$l_m / R = 0.14 - 0.08(1 - y/R)^2 - 0.06(1 - y/R)^4$$

Application range: $\mathrm{Re} = 1.1\times10^5 \sim 3.2\times10^6$



(3) Fluid in a duct corner

$$\frac{1}{l_m} = \frac{1}{l_a} + \frac{1}{l_b};\ l_a, l_b$$ from above eqs.



(4) Modification caused by molecular viscosity—van Driest eq.

$$l_m = \kappa y[1 - \exp(-\frac{y(\tau_m/\rho)^{1/2}}{A\nu})] = \kappa y[1 - \exp(-\frac{y^+}{A})], \quad A = 26$$

Correction caused by molecular viscosity

For $\frac{y^+}{A} = 6$, correction value＝0.997

3. Application range of zero eq. model

(1) Boundary layer flow & HT（Flow over a wing before separation)

(2) FF & HT in straight ducts;

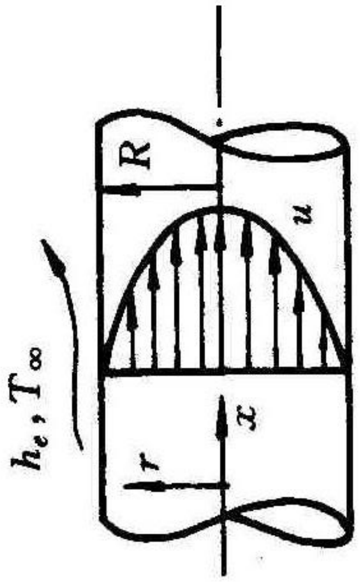
(3) Boundary layer type flow with weak recirculation.

Drawbacks of zero eq. model：

(1) At duct center line velocity gradient equals zero and according to this model turbulent viscosity is zero, but actually turbulent viscosity still exists.

(2) Effects of oncoming flow turbulence is not considered.

(3) Effects of turbulent flow itself is not considered.

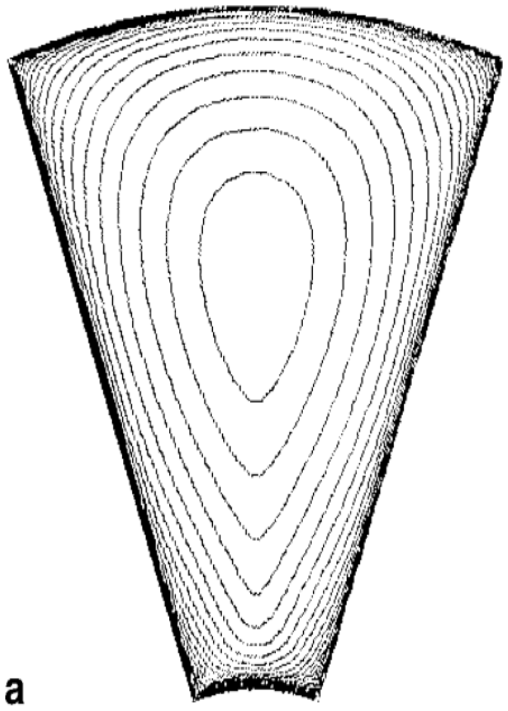


Li ZY, Hung TC, Tao WQ. Numerical simulation of fully developed turbulent flow and heat transfer in annular-sector ducts. Heat Mass Transfer,2002, 38 (4-5): 369-377 written in polar coordinates as [11]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma_{\text{eff}}\frac{\partial \phi}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\Gamma_{\text{eff}}\frac{\partial \phi}{r\partial \theta}\right)+S_\phi=0$$

$$\mathrm{Nu}=0.0281\mathrm{Re}^{0.75}$$

$$\frac{1}{L}=\frac{1}{L_{\text{p}}(r)}+\frac{1}{L_{\text{c}}(\theta)}$$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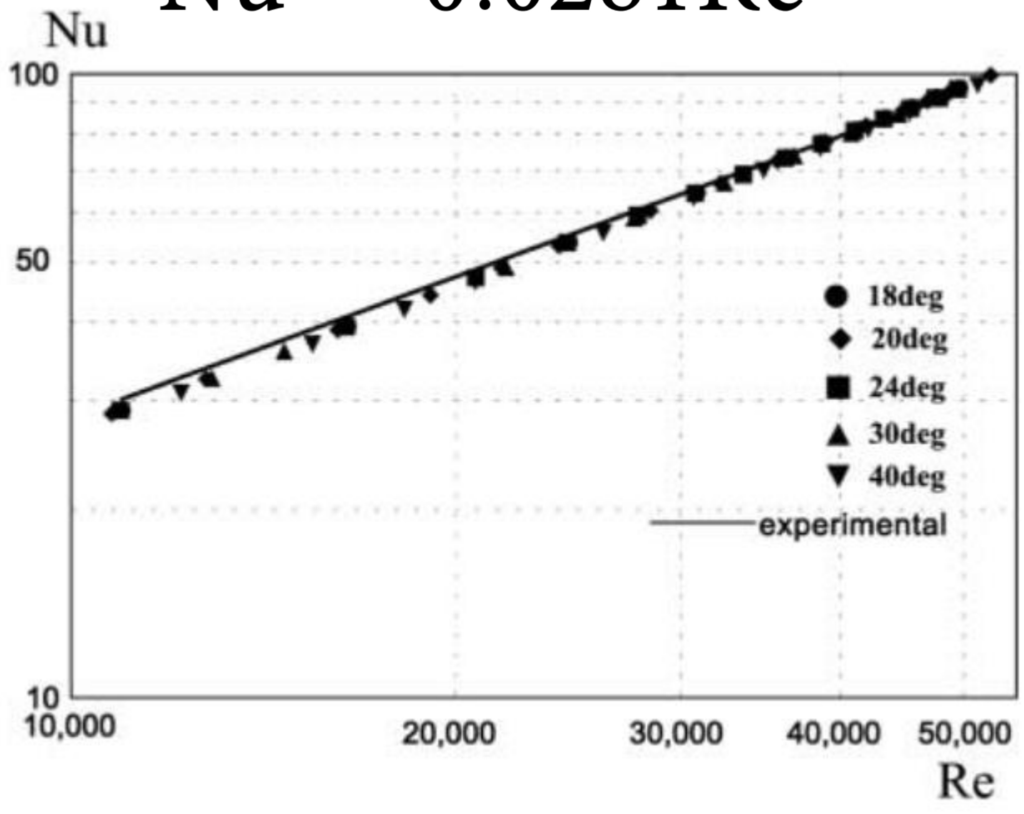


Implemented by adopting Teaching Codes.

8.3.2 One-equation model

1.Turbulent fluctuation kinetic energy is taken as a dependent variable to be solved by a PDE.

The most important feature of turbulence is fluctuation. Fluctuation kinetic energy *k* is an appropriate quantity to indicate fluctuation intensity(脉动强度). It is taken as a dependent variable for reflecting the effects of turbulence itself.

2.Prandtl-Kolmogorov equation

Mimicking (模仿) the molecular viscosity caused by the random motion of molecules, which is:

Molecular viscosity $\eta_l \propto \rho \bar{u} \bar{\lambda}$

Then the viscosity caused by turbulent fluctuation (turbulent viscosity) can be expressed by

$$\eta_t \propto \rho k^{1/2} l \longrightarrow \eta_t = C_\mu^{'} \rho k^{1/2} l$$

where l is the fluctuation scale, usually different from mixing length;

—Prandtl-Kolmogorov equation

Coefficient $C_\mu^{'}$ is within the range from 0.2 to 1.0;

In order to get the distribution of *k* a related PDE is required.

3.Governing equation of turbulent kinetic energy *k*

Starting from the definition of $k = 0.5(\overline{u_i^{'} u_i^{'}})$,conducting time-average operation for N-S equations, and introducing some assumptions, following governing equation for *k* can be obtained:

$$\underbrace{\rho\frac{\partial k}{\partial t}}_{\text{transient}} + \underbrace{\rho u_j\frac{\partial k}{\partial x_j}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial x_j}[(\eta_l + \frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}]}_{\text{diffusion}} \overbrace{+ \underbrace{\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})}_{\text{production}} - \underbrace{\rho(C_D\frac{k^{3/2}}{l})}_{\text{dissipation}}}^{\text{source}}$$

where σ_k is called turbulent Prandtl number of *k*, and its introduction can increase the application range of the model.

4. Boundary condition treatment: wall function method

8.4 Two-Equation Model

8.4.1 Second variables related to l

8.4.2 $k-\varepsilon$ governing equations

8.4.3 General governing equation for $k-\varepsilon$ model

8.4.4 Remarks

8.4 Two-Equation model

8.4.1 Second variables related to l

1. Five physical variables related to l

Z-variables	$k^{1/2}/l$	$k^{3/2}/l$	kl	k/l^2	ε/k
Proposed by	Kolmogorov [32]	Chou (周培源)[19]	Rodi, Spalding [38]	Spalding[39]	Wilox(1988)
Symbol	f	ε	kl	W	ω
Physical meaning	Eddy frequency	Energy dissipation	Product of energy and length scale	Mean square root of vorticity fluctuation	Energy dissipation per unit k energy

2. Two definitions of dissipation rate

(1) Strict definition given by Chou P.Y.

$$\varepsilon = \nu_l \overline{\left(\frac{\partial u_i'}{\partial x_k}\right)\left(\frac{\partial u_i'}{\partial x_k}\right)}$$

It represents dissipation rate per unit mass of isotropic small eddies, and is used in the derivation of its governing equation.

(2) Modeling definition The dissipation term is defined by

$$\varepsilon = C_D k^{3/2} / l$$

This is called modeling definition (模拟定义). It is used in the derivation process of ε-equation to simplify some complicated terms.; C_D is a dimensionless constant.

Understanding of its meaning： energy transit rate from large eddies to small eddies for unit volume is proportional to ρk , and 1/t , where the transit (**输运**) time t is proportional to $l/k^{1/2}$, thus

$$\rho\varepsilon \sim \rho k/(\frac{l}{k^{1/2}}) \sim \rho\frac{k^{3/2}}{l} = C_D\rho\frac{k^{3/2}}{l}$$

8.4.2 $k-\varepsilon$ governing equations

(1) ε equation

Starting from strict definition, $\varepsilon = \nu_l \overline{(\frac{\partial u_i'}{\partial x_k})(\frac{\partial u_i'}{\partial x_k})}$ conducting time average operation for N-S equation, and adopting some assumptions (including modeling definition), yielding

$$\frac{\partial(\rho\varepsilon)}{\partial t}+\frac{\partial(\rho u_j\varepsilon)}{\partial x_j}=\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_\varepsilon})\frac{\partial\varepsilon}{\partial x_j}]+C_1\frac{\varepsilon}{k}\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})-C_2\rho\frac{\varepsilon^2}{k}$$

transient | convection | diffusion | source

σ_ε Prandtl number of ε; C_1, C_2 are empirical (经验的) coefficients

(2) k equation After introducing ε

k equation can be re-written as

$$-\rho(C_D\frac{k^{3/2}}{l})$$

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}]+\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})-\rho\varepsilon$$

Source term

Introducing: $G=\frac{\eta_t}{\rho}\frac{\partial u_i}{\partial x_j}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$ called as unit mass production function

The source term of k eq. $\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})-\rho\varepsilon$ ⇒ $\rho G-\rho\varepsilon$

(3)Determination of turbulent viscosity of $k-\varepsilon$ model

$$\eta_t = C'_\mu \rho k^{1/2} l = C'_\mu C_D \rho k^{1/2+3/2} \frac{l}{C_D k^{3/2}} = C_\mu \rho k^2 / \varepsilon$$

$$C'_\mu C_D \to C_\mu \qquad \varepsilon = C_D \frac{k^{3/2}}{l}$$

8.4.3 General gov. eq. of $k-\varepsilon$ model

$$\frac{\partial(\rho^* \phi)}{\partial t} + div(\rho^* \boldsymbol{u} \phi) = div(\Gamma_\phi grad \phi) + S_\phi$$

ϕ represents: $u, v, w, T, k, \varepsilon$ (**u** velocity vector)

Most widely accepted values of model constants

C_1	C_2	C_μ	σ_k	σ_ε	σ_T
1.44	1.92	0.09	1.0	1.3	0.9-1.0

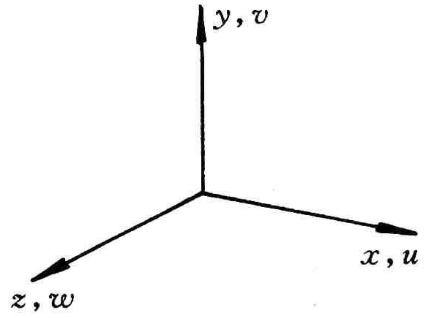
Γ_ϕ, S_ϕ depend on variable and coordinate：$u, v, w, T, k, \varepsilon$

For Cartesian Coordinate:

Text book, Page 350

In our new G.Eqs. for temp.:

$$\Gamma_t = \lambda_t = \eta_t c_p / \mathrm{Pr}_t$$

控制方程	$\frac{\partial(\rho u\phi)}{\partial x}+\frac{\partial(\rho v\phi)}{\partial y}+\frac{\partial(\rho w\phi)}{\partial z}=\frac{\partial}{\partial x}(\Gamma\frac{\partial\phi}{\partial x})+\frac{\partial}{\partial y}(\Gamma\frac{\partial\phi}{\partial y})+\frac{\partial}{\partial z}(\Gamma\frac{\partial\phi}{\partial z})+S$ 对 u，v，w，k，ε，T 广义扩散系数 Γ 为： u，v，w：$\Gamma=\eta_{\mathrm{eff}}=\eta+\eta_t$ k：$\Gamma=\eta+\frac{\eta_t}{\sigma_k}$ ε：$\Gamma=\eta+\frac{\eta_t}{\sigma_\varepsilon}$ T：$\Gamma=\frac{\eta}{Pr}+\frac{\eta_t}{\sigma_T}$ 
源项	u：$S=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}(\eta_{\mathrm{eff}}\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(\eta_{\mathrm{eff}}\frac{\partial v}{\partial x})+\frac{\partial}{\partial z}(\eta_{\mathrm{eff}}\frac{\partial w}{\partial x})$ v：$S=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}(\eta_{\mathrm{eff}}\frac{\partial u}{\partial y})+\frac{\partial}{\partial y}(\eta_{\mathrm{eff}}\frac{\partial v}{\partial y})+\frac{\partial}{\partial z}(\eta_{\mathrm{eff}}\frac{\partial w}{\partial y})$ w：$S=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}(\eta_{\mathrm{eff}}\frac{\partial u}{\partial z})+\frac{\partial}{\partial y}(\eta_{\mathrm{eff}}\frac{\partial v}{\partial z})+\frac{\partial}{\partial z}(\eta_{\mathrm{eff}}\frac{\partial w}{\partial z})$ k：$S=\rho G_k-\rho\varepsilon$ ε：$S=\frac{\varepsilon}{k}(c_1\rho G_k-c_2\rho\varepsilon)$ $G_k=\frac{\eta_t}{\rho}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right\}$ T：S 按实际问题而定

Diffusion coefficients

For the old general governing eq.

Source term

8.4.4 Remarks

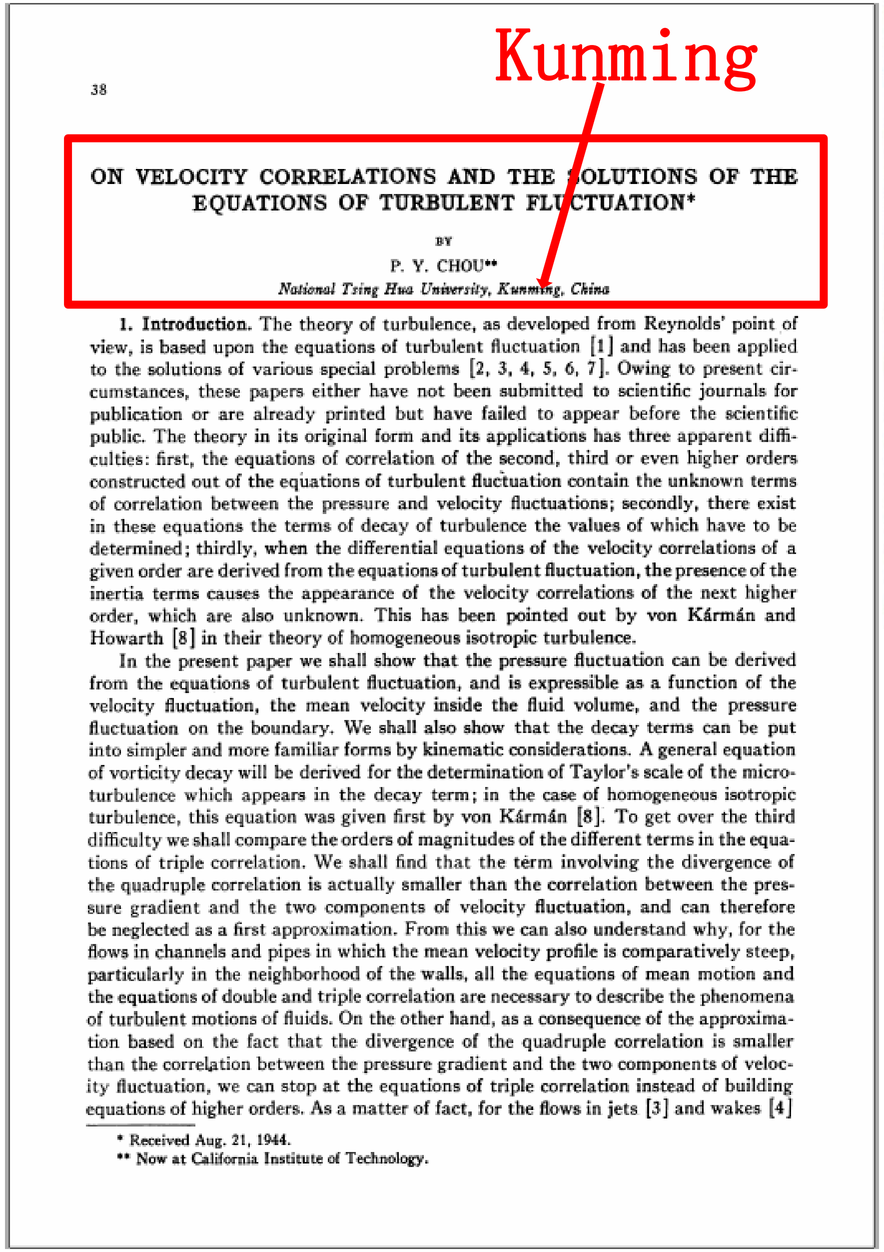
(1) Expansion of G term for 2D case

$$G=\frac{\eta_t}{\rho}\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=\frac{\eta_t}{\rho}\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}+\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right)=$$

$$\frac{\eta_t}{\rho}\left[\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)+\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)\right]$$

$$G=\frac{\eta_t}{\rho}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\}$$ There are 18 terms for 3D case.

(2) The above model is called standard $k-\varepsilon$ model. It can be applied to vigorously developed (旺盛发展） turbulent flow， also called as high-Re $k-\varepsilon$ model. Here Re is not the conventional Reynolds number defined by average velocity , will be discussed later.



Kunming

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ON VELOCITY CORRELATIONS AND THE SOLUTIONS OF THE EQUATIONS OF TURBULENT FLUCTUATION*

BY

P. Y. CHOU**

National Tsing Hua University, Kunming, China

1. **Introduction.** The theory of turbulence, as developed from Reynolds' point of view, is based upon the equations of turbulent fluctuation [1] and has been applied to the solutions of various special problems [2, 3, 4, 5, 6, 7]. Owing to present circumstances, these papers either have not been submitted to scientific journals for publication or are already printed but have failed to appear before the scientific public. The theory in its original form and its applications has three apparent difficulties: first, the equations of correlation of the second, third or even higher orders constructed out of the equations of turbulent fluctuation contain the unknown terms of correlation between the pressure and velocity fluctuations; secondly, there exist in these equations the terms of decay of turbulence the values of which have to be determined; thirdly, when the differential equations of the velocity correlations of a given order are derived from the equations of turbulent fluctuation, the presence of the inertia terms causes the appearance of the velocity correlations of the next higher order, which are also unknown. This has been pointed out by von Kármán and Howarth [8] in their theory of homogeneous isotropic turbulence.

In the present paper we shall show that the pressure fluctuation can be derived from the equations of turbulent fluctuation, and is expressible as a function of the velocity fluctuation, the mean velocity inside the fluid volume, and the pressure fluctuation on the boundary. We shall also show that the decay terms can be put into simpler and more familiar forms by kinematic considerations. A general equation of vorticity decay will be derived for the determination of Taylor's scale of the microturbulence which appears in the decay term; in the case of homogeneous isotropic turbulence, this equation was given first by von Kármán [8]. To get over the third difficulty we shall compare the orders of magnitudes of the different terms in the equations of triple correlation. We shall find that the term involving the divergence of the quadruple correlation is actually smaller than the correlation between the pressure gradient and the two components of velocity fluctuation, and can therefore be neglected as a first approximation. From this we can also understand why, for the flows in channels and pipes in which the mean velocity profile is comparatively steep, particularly in the neighborhood of the walls, all the equations of mean motion and the equations of double and triple correlation are necessary to describe the phenomena of turbulent motions of fluids. On the other hand, as a consequence of the approximation based on the fact that the divergence of the quadruple correlation is smaller than the correlation between the pressure gradient and the two components of velocity fluctuation, we can stop at the equations of triple correlation instead of building equations of higher orders. As a matter of fact, for the flows in jets [3] and wakes [4]

* Received Aug. 21, 1944.
** Now at California Institute of Technology.

（1902-1993）

被国际学术界认同的20世纪流体力学四位巨人(jiant)是： 美国的冯卡门 (von Karman)， 前苏联的柯尔莫哥洛夫(Kolmogorov)，英国的泰勒(G.I.Taylor)， 和中国的周培源。

门生也是巨人：包括胡宁、彭桓武、 何泽慧、 钱三强、 张宗燧、 钱伟长、 王竹溪、林家翘、 于光远等著名学者。

在1945年的这篇文章中，他给出了二阶耗散项的下列表达式：

$$2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}} = -\frac{2\nu}{3\lambda^2}(k-5)q^2 g_{ik} + \frac{2\nu k}{\lambda^2}\overline{u_i' u_k'}$$

8.5 Wall Function Method

8.5.1 Two ways for grid settlement near wall in turbulence simulation

8.5.2 Fundamentals of wall function method

8.5.3 Boundary conditions for standard k, ε model

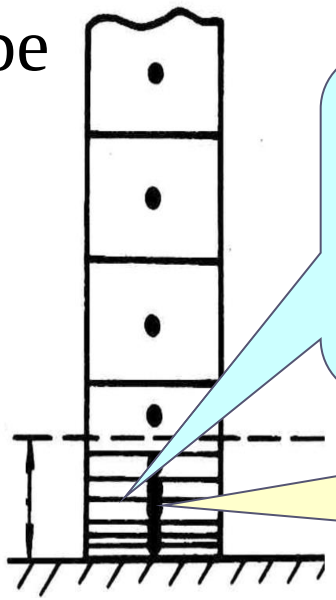
8.5.4 Cautions in implementing wall function method

8.5 Wall Function Method (壁面函数法)

8.5.1 Two ways for grid settlement (节点设置) near a wall in turbulence simulation

1.Setting enough number of grids in the viscous sublayer（>10 grids）

For this treatment *k* equation can be used from vigorous turbulent flow to the wall, and k_w=0 for its boundary condition. This treatment is used in low Reynolds number $k-\varepsilon$ model. Here Reynolds number is defined by $\mathrm{Re}_t = \rho k^2/\eta\varepsilon$.



Enough number of nodes should be set in viscous sub-layer.

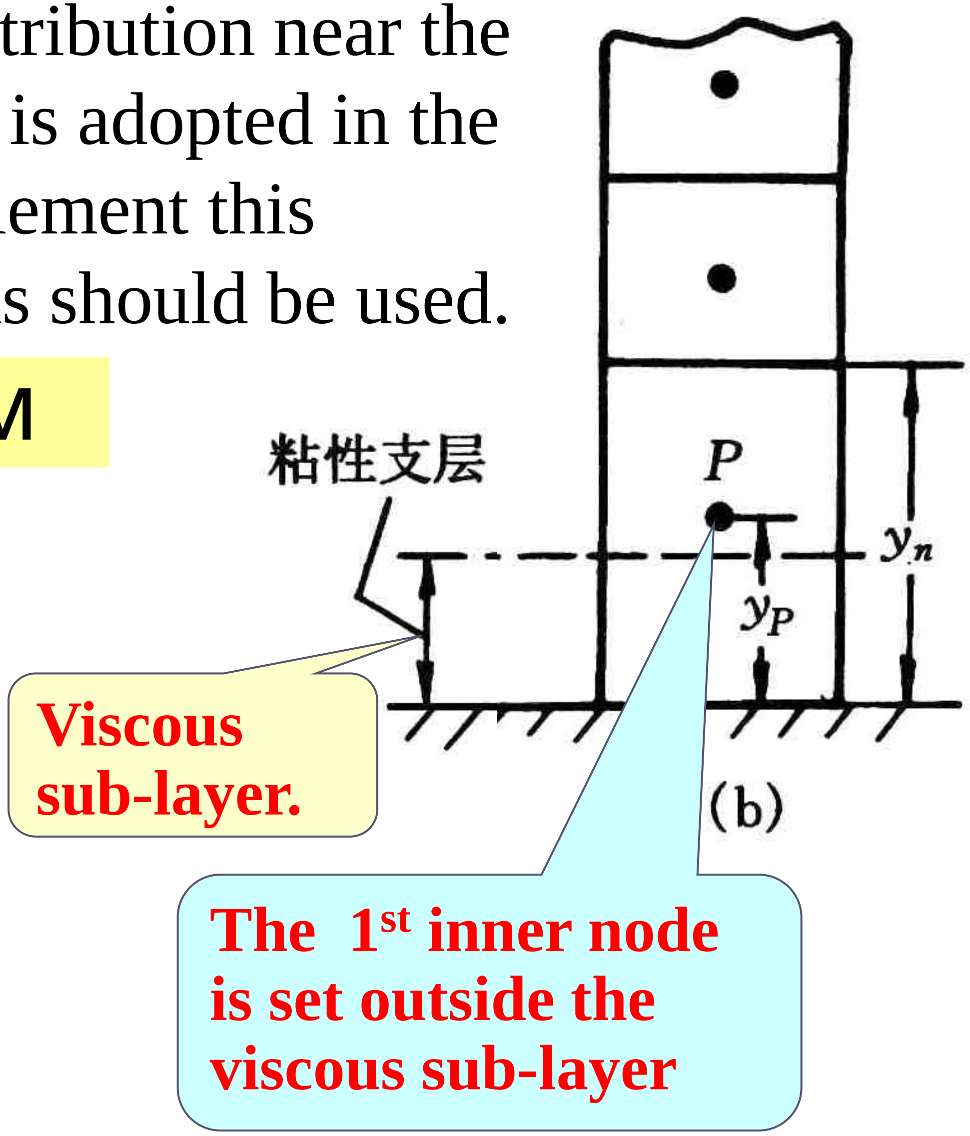
Viscous sublayer（粘性支层）

2. Set the 1st inner node outside the viscous sublayer

In this treatment velocity distribution near the wall should be assumed, and it is adopted in the high Re $k-\varepsilon$ model. To implement this practice Wall Function Methods should be used.

8.5.2 Fundamentals of WFM

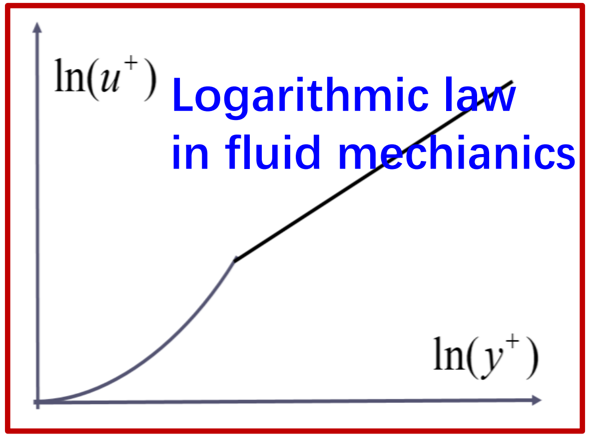
1) Assuming that the dimensionless velocity and temp. distributions outside the viscous sub-layer are of logarithmic law(对数律) type.



(1) Logarithmic law of velocity distribution in boundary layer

$$u^+ = \frac{u}{v^*} = \frac{1}{\kappa}\ln\left(\frac{yv^*}{\nu}\right) + B = \frac{1}{\kappa}\ln(y^+) + B = \frac{1}{\kappa}\ln(Ey^+)$$

$$v^* = \sqrt{\tau_w/\rho}, \kappa = 0.4 \sim 0.42, \quad B = 5.0 \sim 5.5$$



(2) New definitions of y^+, u^+

In order that the logarithmic law can reflect some characteristics of turbulence the law is reformed in two aspects as follows:

Replacing v^* by $C_\mu^{1/4}k^{1/2}$ to define y^+:

$$\frac{yv^*}{\nu} \Rightarrow y^+ = \frac{y(C_\mu^{1/4}k^{1/2})}{\nu}$$

Introducing $C_\mu^{1/4}k^{1/2}$ into u^+ definition

$$u^+ = \frac{u}{v^*} = \frac{u}{v^*}\frac{C_\mu^{1/4}k^{1/2}}{v^*} = \frac{u(C_\mu^{1/4}k^{1/2})}{\tau_w/\rho}$$

When dissipation and production of fluctuation kinetic energy are balanced, the above definitions are identical to conventional definition in fluid mechanics.

(3) Definition of dimensionless temperature: mimicking (模仿) the definition of u^+:

$$u^+ = \frac{u(C_\mu^{1/4} k^{1/2})}{\tau_w / \rho}$$

$$T^+ = \frac{(T - T_w)(C_\mu^{1/4} k^{1/2})}{(q_w / \rho c_p)}$$

Mimicking velocity

Mimicking stress

Required by dimension consistency

(4) Logarithmic laws of *u* & *T* in turbulence model

For $y^+ > 11.0$ following distributions are adopted:

$$u^+ = \frac{1}{\kappa} \ln(E y^+), \qquad \frac{1}{\kappa} \ln(E) = 5.0 \sim 5.5$$

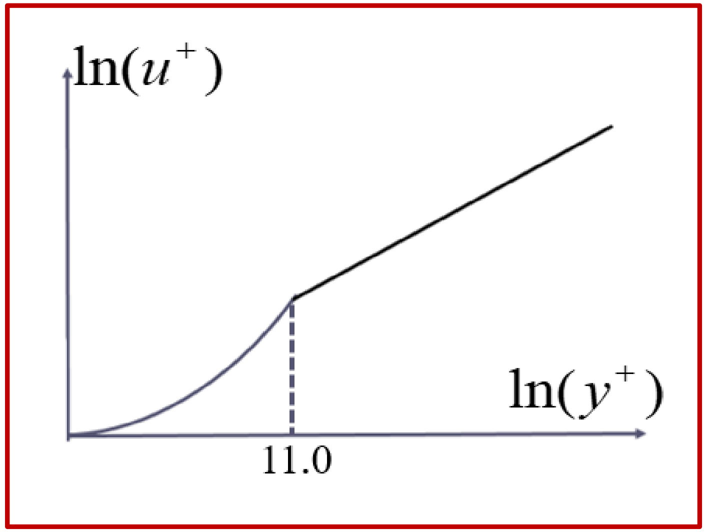
$$T^+ = \frac{\sigma_t}{\kappa}\ln(Ey^+) + P\sigma_t \qquad P = 8.96\left(\frac{\sigma_l}{\sigma_t} - 1\right)\left(\frac{\sigma_l}{\sigma_t}\right)^{-1/4}$$

$$\sigma_l = \Pr_l; \sigma_t = \Pr_t$$

If $\sigma_l = \sigma_t$ then $T^+ = u^+$

Then this is Reynolds analogy（雷诺比拟）.

For $y^+ < 11.0$, it is regarded as laminar sublayer.



2) Placing the 1st inner node P outside the viscous sub-layer, where logarithmic law valid ($y_P^+ > 11.0$)

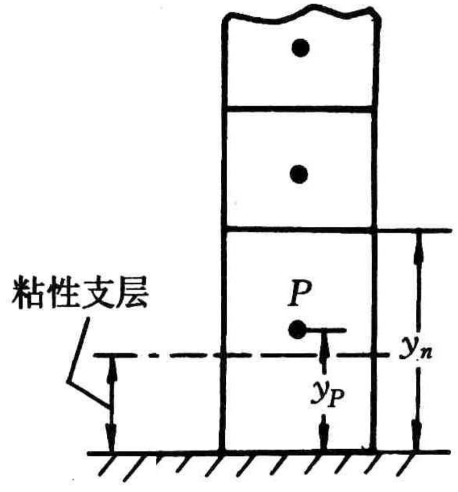
3) The effective turbulent viscosity and thermal conductivity between the 1st inner node and the wall should satisfy：

$$\tau_w = \eta_B \frac{u_P - u_W}{y_P}, \quad q_w = \lambda_B \frac{T_P - T_W}{y_P}$$

The equations of effective viscosity and thermal conductivity between the 1st inner node and the wall can be derived as follows:

(1) Equation for η_B: At point P, u^+ satisfy:

$$\frac{u_P(C_\mu^{1/4}k_P^{1/2})}{\tau_w/\rho}=\frac{1}{\kappa}\ln[Ey_P(\frac{C_\mu^{1/4}k_P^{1/2}}{\nu})]$$



(b)

This equation can be re-written as follows:

$$\tau_w=\frac{\rho u_P(C_\mu^{1/4}k_P^{1/2})}{\frac{1}{\kappa}\ln[Ey_P\frac{C_\mu^{1/4}k_P^{1/2}}{\nu}]}=\eta_B\frac{u_P-u_W}{y_P}$$

According to Point 3 (u_W = 0)

η_B equation can be derived from this equation.

$$\frac{\rho u_P (C_\mu^{1/4} k_P^{1/2})}{\frac{1}{\kappa}\ln[E y_P \frac{C_\mu^{1/4} k_P^{1/2}}{\nu}]} = \eta_B \frac{u_P}{y_P}$$

$$\eta_B = [\frac{y_P (C_\mu^{1/4} k_P^{1/2})}{\nu}](\rho \nu) \frac{1}{\frac{1}{\kappa}\ln(E y_P^+)} = (\frac{y_P^+}{u_P^+})\eta_l$$

y_P^+ η_l u_P^+

In the turbulent vigorous region $y_P^+ >> u_P^+$ above equation shows:

turbulent viscosity is y_P^+ / u_P^+ times of laminar viscosity.

For example $y_P^+ = 100, u_P^+ = \frac{1}{\kappa}\ln(100) + B = \frac{1}{0.4} 4.605 + 5.0 = 16.5$

Then: $\eta_B = (100/16.5)\eta_l = 6.06\eta_l$

(2) Equation for λ_B: At point P, T^+ satisfy:

$$\frac{(T_P - T_W)(C_\mu^{1/4} k_P^{1/2})}{q_w / \rho c_p} = \frac{\sigma_t}{\kappa}\ln(E y_P^+) + \sigma_t P$$

From which:

$$q_w = \frac{\rho c_p (T_P - T_W)(C_\mu^{1/4} k_P^{1/2})}{\frac{\sigma_t}{\kappa}\ln(E y_P^+) + \sigma_t P} = \lambda_B \frac{(T_P - T_W)}{y_P}$$

According to Point 3

$$\lambda_B = \frac{(C_\mu^{1/4} k_P^{1/2}) y_P}{\nu} \frac{\rho c_p \nu}{\lambda_l} \frac{\lambda_l}{\frac{\sigma_t}{\kappa}\ln(E y_P^+) + \sigma_t P} = \left(\frac{y_P^+}{T_P^+}\right) \Pr_l \lambda_l$$

y_P^+

$\frac{\eta c_p}{\lambda} = \Pr_l$

T_P^+

This is equivalent to magnify the molecular conductivity by $\left(\frac{y_P^+}{T_P^+}\right)\Pr_l$ times.

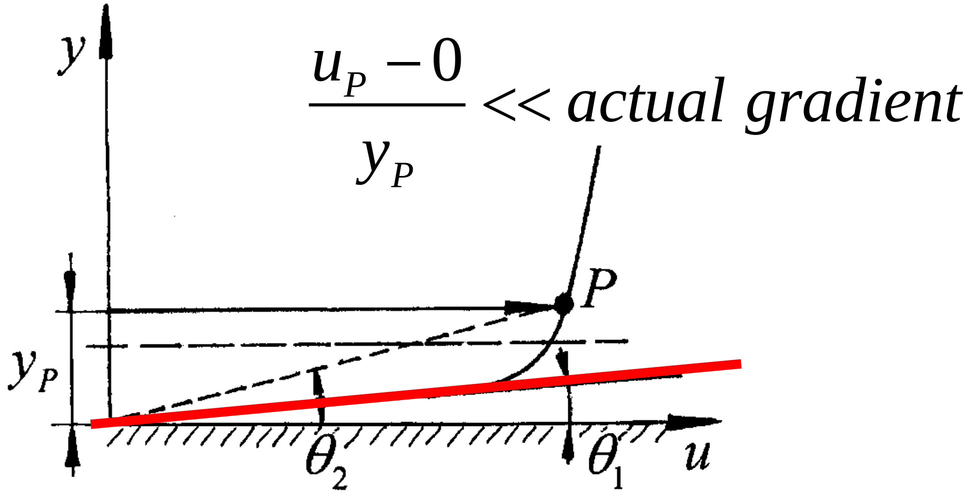
For $\Pr_l = 5.0, \Pr_t = 1.0, y_P^+ = 100,$

yielding $T_P^+ = 40.5, \frac{y_P^+}{T_P^+}\Pr_l = \frac{100}{40.5}\times 5.0 = 12.3$

The molecular conductivity is magnified by 12.3 times !

Why wall viscosity and conductivity η_B, λ_B should be magnified? This is because the 1st inner node is far from wall, leading to reduced wall gradient determined by FD method.

In WFM the magnified transport properties compensate (弥补) the reduced gradients near the wall so that their products will be approximately equal to the true values.



Wall functions refer to the expressions of η_B, λ_B

Fundamentals of Wall Function Method

1) Assuming that the dimensionless velocity and temp. distributions outside the viscous sub-layer are of logarithmic law(对数律) type.

For $y^+ > 11.0$

$$\begin{cases} u^+ = \dfrac{1}{\kappa}\ln(Ey^+), & \dfrac{1}{\kappa}\ln(E) = 5.0 \sim 5.5 \\ T^+ = \dfrac{\sigma_t}{\kappa}\ln(Ey^+) + P\sigma_t; & P = 8.96\left(\dfrac{\sigma_l}{\sigma_t} - 1\right)\left(\dfrac{\sigma_l}{\sigma_t}\right)^{-1/4} \end{cases}$$

2) Placing the 1st inner node P outside the viscous sub-layer, where logarithmic law valid ($y_P^+ > 11.0$)

3) The effective turbulent viscosity and thermal conductivity between the 1st inner node and the wall can be determined by following equations:

$$\eta_B = \left(\frac{y_P^+}{u_P^+}\right)\eta_l; \quad \lambda_B = \left(\frac{y_P^+}{T_P^+}\right)\Pr_l \lambda_l; \quad y_P^+ = \frac{y_P (C_\mu^{1/4} k_P^{1/2})}{\nu}$$

4) The boundary condition of k equation: $\left.\frac{\partial k}{\partial n}\right)_w = 0$

Because outside the sublayer the production of fluctuation kinetic energy is much larger than diffusion towards wall, hence diffusion to the wall is approximately taken zero.

5) The dissipation of fluctuation kinetic energy at 1st inner node is determined by the model equation:

$$\varepsilon_P = \frac{C_D k^{3/2}}{l} = \frac{C_\mu^{3/4} k_P^{\;3/2}}{\kappa y_P}$$

（see page 355 of text book）

For the 1st inner node dissipation rate is specified by above equation, and computation is limited within the region surrounded by the 1st inner nodes.

8.5.3 Boundary conditions of k,ε for standard $k-\varepsilon$ model

1. Inlet boundary

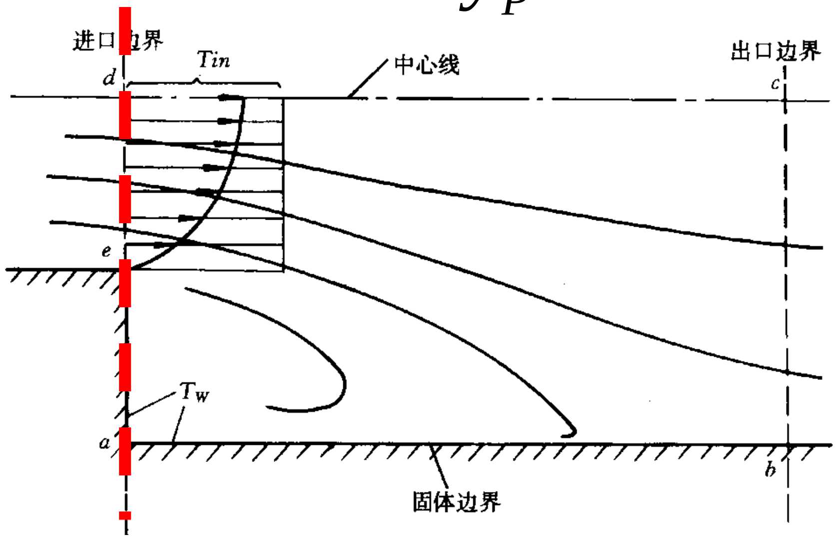
1) k: (1)Adopting test data； (2)Taking a percentage of kinetic energy of oncoming flow .For fully developed flow in ducts:0.5~1.5%;

2)ε： (1) Using model equation： $\varepsilon = \dfrac{C_\mu^{3/4} k^{3/2}}{\kappa y_P}$

(2)Using $\eta_t = C_\mu \rho k^2 / \varepsilon$

assuming $\dfrac{\rho u L}{\eta_t} = 100 \sim 1000$

yielding η_t with inlet u and L.



2. At central line: $\frac{\partial k}{\partial n}=\frac{\partial \varepsilon}{\partial n}=0$

3. Outlet: Adopting local one way coordinate assumption

3. Solid wall: Adopting wall function method

(1)Velocity— Velocity normal to wall $\partial \phi / \partial n)_w = 0$

Velocity parallel to wall $\phi_w = 0,$

And wall viscosity determined by WFM.

Remarks: here velocity is the dependent variable to be solved, not the one in the nonlinear part of convection term, for which wall velocities always equal zero: $u=v=0$.

(2) k— Adopting $\frac{\partial k}{\partial n}=0$ implemented via setting $\Gamma_B = 0$

(3) ε — Specifying the 1st inner node by

Then cutting connection with boundary.

$$\varepsilon_P = \frac{C_\mu^{3/4} k^{3/2}}{\kappa y_P}$$

8.5.4 Cautions in implementing wall function method

1) Approximate range of y_P^+, x_P^+

$$11.5 \sim 30 \le (y_P^+, x_P^+) \le 200 \sim 400$$

Logarithmic law is valid in this range

2) Underrelaxation

In the iteration process η_t, k, ε must be under-relaxed.

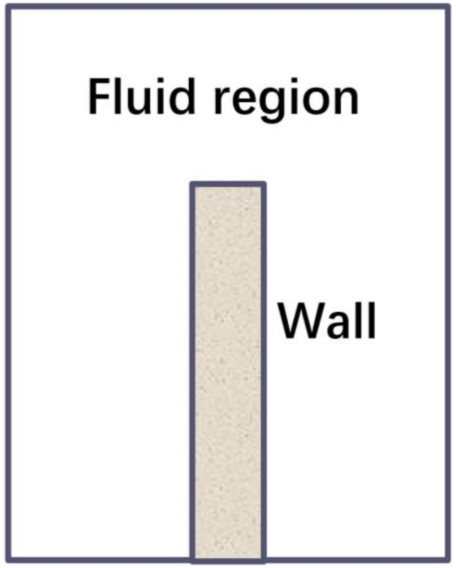
And it is organized within the solution process.

3) ε_P should be specified by large coefficient method

4) Source term treatment of k, ε

$$S_k = \rho G - \rho\varepsilon = \underbrace{\rho G}_{S_C} - \underbrace{(\rho\varepsilon / k^*)}_{S_P} k$$

$$S_\varepsilon = \frac{C_1 \rho \varepsilon G}{k} - \frac{C_2 \rho \varepsilon^2}{k} = \underbrace{\frac{C_1 \rho \varepsilon G}{k}}_{S_C} - \underbrace{\frac{C_2 \rho \varepsilon^*}{k}}_{S_P} \varepsilon$$

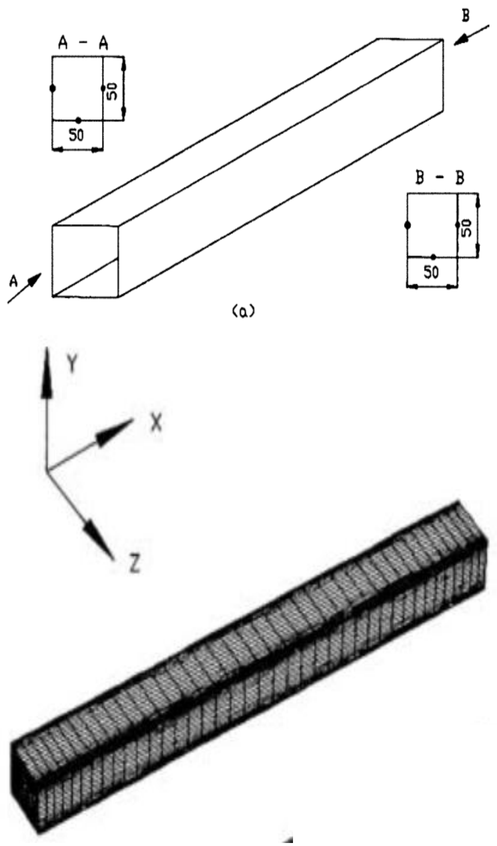


See pages 358－359 of textbook.

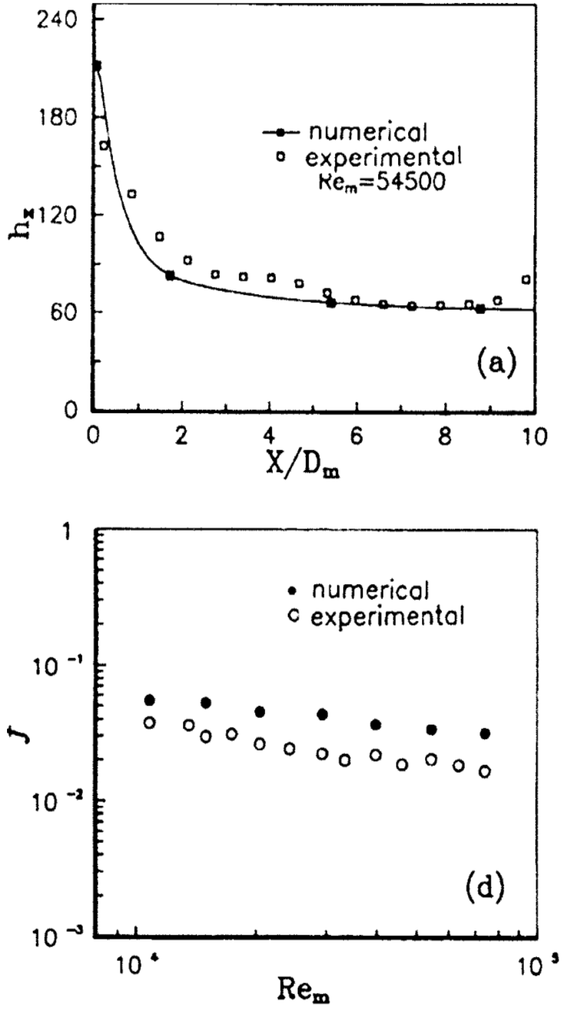
5) Treatment of solid located within fluid region

Application example of the *k*-Epsilon turbulence model

L.-B. Wang, Q.-W. Wang, Y.-L. He, W.-Q. Tao. Experimental and numerical study of developing turbulent flow and heat transfer in convergent/divergent square ducts. Heat & Mass Transfer, 2002, 38:399-408



The standard k-ε turbulent model is adopted. The outlet boundary condition of the computational domain is treated by local one-way method. The velocity and kinetic energy of turbulence are assumed to be uniform at the duct inlet. At the walls, the no-slip condition is used in conjunction with the wall functions method, which is implemented by the method provided in [25].



In the calculation, it was found that when the inlet k is in the range of 0.5–1.5% kinetic energy of flow, there is insignificantly effect on numerical results. The good agreement between numerical and experimental results was found when the y^+, z^+ are in the range of 11.5–40. Grid-independence studies were performed for case of $\mathrm{Re_m} = 100,000$ to all three ducts, and results for grid sizes of $41 \times 16 \times 16$, $61 \times 16 \times 16$, $41 \times 21 \times 21$ were obtained. The difference between the results is less than 1%, then the solutions presented in this paper were obtained using the $41 \times 21 \times 21$ grid.

Simulation is implemented by the extended version of our Teaching Codes

8.6 Low Reynolds Number *k*-epsilon Model

8.6.1 Application range of standard $k-\varepsilon$ model

8.6.2 Jones－Launder's considerations for low Re $k-\varepsilon$ model

8.6.3 Jones－Launder low Re $k-\varepsilon$ model

8.6.4 Other low Re $k-\varepsilon$ models

8.6 Low Reynolds Number k-epsilon Model

8.6.1 Application range of standard $k-\varepsilon$ model

1. Near wall velocity distribution obeys logarithmic law
2. Shear stress is distributed uniformly from wall to 1st inner node;
3. Production and dissipation are nearly balanced for fluctuation kinetic energy.

Above assumptions are valid only when $\mathrm{Re}_t = \dfrac{\rho k^2}{\eta \varepsilon} > 150$

If this Re_t less than 150， the standard $k-\varepsilon$ model can not be used. When approaching wall this Reynolds number becomes smaller and smaller.

In order that simulation can be conducted from vigorous part to the wall, model should be modified.

8.6.2 Jones－Launder's consideration of low Re $k-\varepsilon$ model

(1) Both molecular and turbulent diffusions should be considered;

(2) Effects of $Re_t = \dfrac{\rho k^2}{\eta \varepsilon}$ on coefficients in k-ε should be considered;

(3) Near a wall the dissipation of fluctuation kinetic energy is not isotropic, and this should be taken into account in k eq.

8.6.3 Jones－Launder low Re $k-\varepsilon$ model

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}]+\rho G-\rho\varepsilon-2\eta(\frac{\partial k^{1/2}}{\partial y})^2$$

D

$$\frac{\partial(\rho\varepsilon)}{\partial t}+\frac{\partial(\rho u_j\varepsilon)}{\partial x_j}=\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_\varepsilon})\frac{\partial\varepsilon}{\partial x_j}]+f_1C_1\frac{\rho G\varepsilon}{k}-f_2C_2\rho\frac{\varepsilon^2}{k}+2\frac{\eta_l\eta_t}{\rho}(\frac{\partial^2 u}{\partial y^2})^2$$

E

$$\eta_t=C_\mu f_\mu\rho k^2/\varepsilon$$

where $f_1=1.0$ $\quad f_2=1.0-0.3\exp(-\mathrm{Re}_t^2)$

$$f_\mu=\exp(-2.5/(1+Re_t/50)) \qquad Re_t=\frac{\rho k^2}{\eta\varepsilon}$$

Explanation: The vertical lines in Eqs.（9-47),(9-48) (page.363 of textbook) just show that the term is newly added, not the symbols of absolute value.

3. Explanations for additional terms

(1) $D=-2\eta(\frac{\partial k^{1/2}}{\partial y})^2$ (y is normal to wall)，for considering that near a wall the fluctuation kinetic energy is not isotropic, and with this term the condition of $\varepsilon_w=0$ can be used;

(2) The term E is for a better agreement with test data.

4. Boundary condition of J-L low Re model

$$k_w=\varepsilon_w=0$$

8.6.3 Other low Re $k-\varepsilon$ models

Since the proposal of J-L low Re model in 1972, more than 20 variants (变体) have been proposed. The major differences between them are in four aspects:

(1) Different values of the three modified coefficients:

$$f_1, f_2, f_\mu$$

(2) Different expressions of additional terms D and E；

(3) Different wall boundary condition for ε

$$\varepsilon = 0;$$

$$\frac{\partial \varepsilon}{\partial n} = 0$$

(4) Different values of coefficients C_1, C_2, C_μ and constants $\sigma_k, \sigma_\varepsilon$

Table 9-8 of Textbook

No	模型	简称	ε_w 条件	c_μ	c_1	c_2	σ_k	σ_ε	f_μ	f_1	f_2	D	E
1	高 Re 数	HR	壁面函数法	0.09	1.44	1.92	1.0	1.3	1.0	1.0	1.0	0	0
2	Janes/Launder	JL	0	0.09	1.44	1.92	1.0	1.3	$\exp[-2.5/(1+Re_t/50)]$	1.0	$1-0.3\exp(-Re_t^2)$	$2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$2\frac{\mu\mu_t}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)^2$
3	Launder/Sharma[78]	LS	0(附注 1)	0.09	1.44	1.92	1.0	1.3	$\exp[-3.4/(1+Re_t/50)^2]$	1.0	$1-0.3\exp(-Re_t^2)$	$2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$2\frac{\mu\mu_t}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)^2$
4	Hassid/Porch[79]	HP	0	0.09	1.45	2.0	1.0	1.3	$1-\exp(-0.0015Re_t)$	1.0	$1-0.3\exp(-Re_t^2)$	$2\eta\frac{k}{y^2}$	$-2\eta\left(\frac{\partial\varepsilon^{1/2}}{\partial y}\right)^2$
5	Hoffman[80]	HO	0	0.09	1.81	2.0	2.0	3.0	$\exp[-1.75/(1+Re_t/50)]$	1.0	$1-0.3\exp(-Re_t^2)$	$\frac{\eta}{y}\frac{\partial k}{\partial y}$	0
6	Dutoya/Michard[81]	DM	0	0.09	1.35	2.0	0.9	0.95	$1-0.86\exp[-(Re_t/600)^2]$	$1-0.04\exp\left[-\left(\frac{Re_t}{50}\right)^2\right]$	$1-0.3\exp\left[-\left(\frac{Re_t}{50}\right)^2\right]$	$2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$-c_2f_2(\varepsilon D/k)^2$
7	Chien[82]	CH	0	0.09	1.35	1.8	1.0	1.3	$1-\exp(-0.0115y^+)$	1.0	$1-0.22\exp\left[-\left(\frac{Re_t}{6}\right)^2\right]$	$2\eta\frac{k}{y^2}$	$-2\eta(\varepsilon/y^2)\exp(-0.5y^+)$
8	Reynolds[83]	RE	$\nu\frac{\partial^2 k}{\partial y^2}$	0.084	1.0	1.83	1.69	1.3	$1-\exp(-0.0198Re_y)$(附注 2)	1.0	$\left\{1-0.3\exp\left[-\left(\frac{Re_t}{6}\right)^2\right]\right\}f_\mu$	0	0
9	Lam/Bremhost[84](Dirichlet)	LB	$\nu\frac{\partial^2 k}{\partial y^2}$	0.09	1.44	1.92	1.0	1.3	$[1-\exp(-0.0165Re_y)]^2\times\left(1+\frac{20.5}{Re_t}\right)$	$1+(0.05/f_\mu)^3$	$1-\exp(-Re_t^2)$	0	0
10	Lam/Bremhost[84](Neumann)	LB1	$\frac{\partial\varepsilon}{\partial y}=0$	0.09	1.44	1.92	1.0	1.3	同 LB	同 LB	同 LB	0	0

Table 9-8 in Textbook（Continued）

续表 9－8

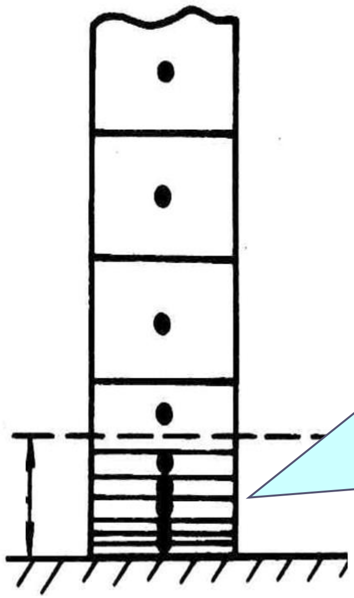
No	模型	简称	ε_w 条件	c_μ	c_1	c_2	σ_k	σ_ε	f_μ	f_1	f_2	D	E
11	Nagano/Hishida[86]	NH	0	0.09	1.45	1.90	1.0	1.3	$[1-\exp(-y^+/26.5)]^2$	1.0	$1-0.3\exp(-Re_t^2)$	$2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$\eta\eta_t(1-f_\mu)\left(\frac{\partial^2 u}{\partial y^2}\right)^2$
12	Myong/Kosagi[86]	MK	$\nu\frac{\partial^2 k}{\partial y^2}$ (附注 3)	0.09	1.40	1.80	1.4	1.3	$(1+3.45Re_t^{1/2})\times[1-\exp(-y^+/70)]$	1.0	$\left[1-\frac{2}{9}\exp\left(\frac{Re_t}{6}\right)^2\right]\times[1-\exp(-y^+/5)]^2$	0	0
13	Abid[87]	AB	$\nu\frac{\partial^2 k}{\partial y^2}$	0.09	1.45	1.83	1.0	1.4	$\tanh(0.008Re_y)\left(1+\frac{4}{Re_t^{3/4}}\right)$	1.0	$1-\frac{2}{9}\exp\left(1-\frac{Re_t^2}{36}\right)\cdot\left[1-\exp\left(\frac{-Re_y}{12}\right)\right]$	0	0
14	Abe \ Kondoh Nagano[88]	AKN	$2\nu\frac{R_p}{y_p^2}$	0.09	1.5	1.9	1.4	1.4	$\left\{1+5/Re_\tau^{3/4}\exp\left[1-\left(\frac{Re_\tau}{200}\right)^2\right]\right\}[1-\exp(-y^*/14)]^2$(附注 4)	1.0	$\left\{1-0.3\exp\left[-\left(\frac{Re_t}{6.5}\right)^2\right]\right\}\cdot[1-\exp(-y^*/3.1)]^2$	0	0
15	Fan \ Barnett Lakshmin-arayana[89]	FBL	$\frac{\partial\varepsilon}{\partial y}=0$	0.09	1.4	1.8	1.0	1.3	$0.4f_w/\sqrt{Re_t}+(1-0.4f_w/\sqrt{Re_t})\cdot[1-\exp(Re_y/42.63)]^3$ (附注 5)	1.0	$f_w^2\left\{1-0.22\exp\left[-\left(\frac{Re_t}{6}\right)^2\right]\right\}$	0	0
16	Cho/Goldstein[90]	CG	$\frac{\partial\varepsilon}{\partial y}=0$	0.09	1.44	1.92	1.0	1.3	$1-0.95\exp(-5\times10^{-5}Re_t)$	1.0	$1-0.222\exp\left(\frac{-Re_t^2}{36}\right)$	0	(附注 6)

Further remarks for the low Re number $k-\varepsilon$ model

1. The Reynolds number refers to the turbulent Reynolds number, defined by

$$Re_t = \frac{\rho k^2}{\eta \varepsilon}$$

2. The grid number within the laminar sublayer should be larger than 10;



Enough number of nodes should be set in viscous sub-layer.

3. For J-L low Reynolds number model, the wall boundary condition of k and ε are:

$$k_w = \varepsilon_w = 0$$

8.7 Brief Introduction to Recent Developments

8.7.1 Developments in $k-\varepsilon$ two-equation model

8.7.2 Brief introduction to second moment model

8.7.3 Near wall region treatment of different models

8.7.4 Chen model for indoor air movement

8.7.5 $\overline{V^2}$ - f turbulence model for highly inhomogeneous turbulent flow

8.7 Brief Introduction to Recent Developments

8.7.1 Developments of $k-\varepsilon$ two-eq. model

1. Non-linear $k-\varepsilon$ model

Boussinesq's constitution eq.

$$(\tau_{i,j})_t = -\rho\overline{u_i^{'}u_j^{'}} = (-p_t\delta_{i,j}) + \eta_t\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)$$

In Boussinesq's constitution eq. every term is of 1st power (一次方)---linear leading to $\tau_{xx} = \tau_{yy}$ for fully developed turbulent flow in parallel plate duct, which does not agree with test results.

Speziale et al. proposed a non-linear model in 1987, see Reference [95] of Chapter 9 of the textbook.

2. Multi-scale $k-\varepsilon$ model

In the standard $k-\varepsilon$ model only one geometric scale is used. Actually turbulent flow fluctuations cover a wide range of time scales and geometric scales. A simple improvement is adopting two geometric scales: big eddies for carrying kinetic energy(载能涡) and small eddies for dissipating energy(耗能涡). See Reference [108] in the textbook.

3. Renormalized group (重整化群) model

Starting from transient N-S eq. Yakhot-Orzag adopted spectral analysis (谱分析)method and derived *k*-epsilon equations with different coefficients and constants.

See Ref.[113] in the textbook.

3. Realizable $k-\varepsilon$ model (可实现)

In the standard *k*-epsilon model when fluid strain is very large the normal stress will be negative, which is not **rea**lizable; In order to establish all-cases realizable model the coefficient C_μ should be related with strain. (应变) See Ref. [115] in the textbook。

8.7.2 Brief introduction to second moment model (二阶矩模型）

For the products with two fluctuations, $-\rho\overline{u_i^{'}u_j^{'}}$, their governing eqs. are derived; for products with more than two fluctuations, say $u_i^{'}u_j^{'}u_k^{'}$, models are introduced to close the model.

1. Original form of Reynolds stress equation

$$\frac{\partial \overline{u_i^{'} u_j^{'}}}{\partial t} + u_k \frac{\partial \overline{u_i^{'} u_j^{'}}}{\partial x_k} = P_{i,j} + \pi_{i,j} + D_{i,j} - \varepsilon_{i,j}$$

where

$$P_{i,j} = -(\overline{u_i^{'} u_k^{'}} \frac{\partial u_j}{\partial x_k} + \overline{u_i^{'} u_k^{'}} \frac{\partial u_i}{\partial x_k})$$

Production term

$$\pi_{i,j} = \overline{\frac{p^{'}}{\rho}(\frac{\partial u_j^{'}}{\partial x_i} + \frac{\partial u_i^{'}}{\partial x_j})}$$

Redistribution term

$$D_{i,j} = -\frac{\partial}{\partial x_k}(\overline{u_i^{'} u_j^{'} u_k^{'}} - \nu \frac{\partial \overline{(u_i^{'} u_j^{'})}}{\partial x_k} + \delta_{i,k} \frac{\overline{u_j^{'} p^{'}}}{\rho})$$

Diffusion term

The above three terms $P_{i,j}, D_{i,j}, \pi_{i,j}$ have to be simplified or modeled . Different treatments lead to different second moment models.

3. Eqs. and constants in 2nd moment closure for convective heat transfer

(1) 3-D time average governing eqs.---16:

5 time average eqs. for five variables: u, v, w, p, T

6 time average fluctuation stress eqs.

$\overline{u_i^{'}u_j^{'}}(i, j = 1, 2, 3)$ (6 terms);

3 eqs. for additional heat flux

$\overline{u_i^{'}\phi^{'}}(i = 1, 2, 3)$ (3 terms)

1 eq. for k， and

1 eq. for ε

(2) Nine empirical constants.

8.7.3 Near wall region treatment of different models

All the above improvements are only for the vigorous part of turbulent flow; for near wall region the molecular viscosity must be taken into account. At present following methods are used：

1. Adopting WFM；

2. Adopting two-layer model： several choices

(1) With Re_t=150 as a deviding line(分界线): adopting one of the above model when it is larger than 150 ; if Re_t is less than 150 low *Re* k-epsilon model is used.

(2) In near wall region *k* equation model is used, and in the vigorous part above model is adopted.

Emphasis should be paid for the near wall region.

8.7.4 Chen model for indoor air movement

Q Y Chen proposed following simple model for indoor air turbulent flow：

$$\eta_t = 0.03874 \rho v l$$

$\rho -$ Air density

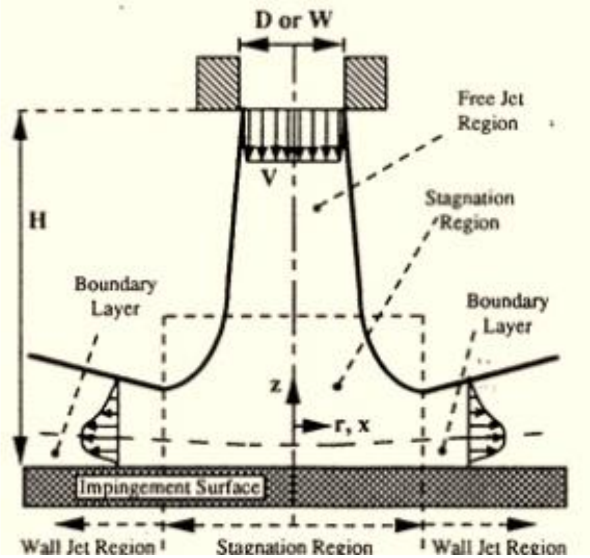
$v -$ Local time average velocity

$l -$ The shortest distance to the wall

Qingyang Chen, Weiran Xu. A zero-equation model for indoor airflow simulation. Energy and Building, 1998, 28, 137-144

8.7.5 $\overline{v^2}-f$ Turbulence model for highly inhomogeneous turbulent flow

For highly inhomogeneous flow and heat transfer, such as jet impingement flow, this $\overline{v^2}-f$ turbulence model may obtain reasonable simulation results.



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Home Work 8（2024-2025）

Please finish your homework independently !!!

Please hand in on Nov. 19， 2022

Problem 8-1

Estimate the value of the turbulent effective pressure based on the following data: Air is going through a wind tunnel with velocity of 55 ms^{-1} and pressure of one bar. The turbulent intensity $(\sqrt{\overline{(u')^2}}/u)$ is isotropic ($\overline{(u')^2}=\overline{(v')^2}=\overline{(w')^2}$) and equals 5%.

Problem 8-2

In a 2-D boundary layer flow ,if the generation of turbulence kinetic energy and the dissipation are balanced each other, try to show:

$$\sqrt{\tau_{\mathrm{w}} / \rho} = \mathrm{C}_{\mu}^{1/4} k^{1/2}$$

Problem 8-3

Analyze the dimension and unit for the coefficient and constant in the k-Epsilon turbulence model : $C_{\mu}, C_1, C_2, \sigma_k, \sigma_{\varepsilon}, \sigma_T$.

Problem 8-4

A very simple turbulence model proposed by Chen is as follows:

$$\eta_t = 0.03874 \rho v l$$

$\rho -$ Air density

$v -$ Local time average velocity

$l -$ The shortest distance to the wall

In a conventional working space, the air velocity usually varies from 0.5ms^{-1} to 2-3ms^{-1}. Calculate the turbulent viscosity by Chen's model for air average velocity of 1. m^{s-1} with *l*=0.15m ,and velocity of 2. m^{s-1} with *l*=0.25 m.

Problem 8-5

In a wind tunnel of square cross section (0.5m × 0.5m) the air average velocity is 55 ms^{-1}. The isotropic turbulence intensity is 6 %.

Calculate the turbulent Reynolds number by following equation,

$$\mathrm{Re}_t = \frac{\rho k^2}{\eta \varepsilon}$$

Assume that air pressure is one bar and the dissipation rate can be estimated from following equation with y_P of 0.35m,

$$\varepsilon = \frac{C_\mu^{3/4} k^{3/2}}{\kappa y_P}$$

Compare Re_t and the conventional Reynolds number defined by,

$$\mathrm{Re} = \frac{uD}{\nu}$$

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渡彼岸!

People in the same boat help each other to cross to the other bank, where....