Numerical Heat Transfer

(数值传热学)

Chapter 7 Mathematical and Physical Characteristics of Discretized Equations (Chapter 3 of Textbook)

Instructor Tao, Wen-Quan

Key Laboratory of Thermo-Fluid Science & Engineering
Int. Joint Research Laboratory of Thermal Science & Engineering
Xi'an Jiaotong University
Innovative Harbor of West China, Xian

2024-Oct.-29

Contents

7.1 Consistence, Convergence and Stability of Discretized Equations

7.1.1 Truncation error and consistence(相容性)

7.1.2 Discretization error（离散误差） and convergence(收敛性)

7.1.3 Round off error （舍入误差）and stability（稳定性） of initial problems（初值问题）

7.1.4 Examples

7.1 Consistence, Convergence and Stability of Discretized Equations

7.1.1 Truncation error and consistence（相容性）

1. Accurate (analytical) solution of the discretized equations (离散方程的精确解)

It refers to the numerical solution without any round-off (舍入) error introduced in the solution procedure, denoted by ϕ_i^n .

It is assumed that Taylor expansion can be applied to the accurate numerical solutions ϕ_i^n;

2. Differential vs. difference operators (微分与差分算子)

(1) Differential operator (微分算子):

Implementing(执行) some differential (微分) and/or arithmetic(算术) operations on function $\phi(i,n)$ at a point (i,n):

$$L(\phi)_{i,n} = (\rho\frac{\partial \phi}{\partial t} + \rho u\frac{\partial \phi}{\partial x} - \Gamma\frac{\partial^2 \phi}{\partial x^2} - S)_{i,n}$$

Then $L(\phi)_{i,n} = 0$ --1-D transient model diffu-convec. equation.

(2) Difference operator(差分算子) :

Implementing some difference (差分) and/or arithmetic operations on function ϕ_i^n at point (i,n)

$$L_{\Delta x,\Delta t}(\phi_i^n) = \rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} + \rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x} - \Gamma\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{\Delta x^2} - S_i^n$$

Then $L_{\Delta x,\Delta t}(\phi_i^n)=0$ ---discretized form of 1-D transient model equation by **f**orward **t**ime and **c**entral **s**pace scheme —FTCS

3. Truncation error (T.E.截断误差) of the discretized equation

T.E. is the difference between differential and difference operators (微分算子与差分算子的差).

(1) Definition — $\text{T.E.} = L_{\Delta x,\Delta t}(\phi_i^n) - L(\phi)_i^n$

(2) Analysis—Expanding $\phi_i^{n+1}, \phi_{i\pm1}^n$ at point (i,n) by Taylor series (with respect to both space and time), substituting the series into the discretized equation and rearranging into the form of two operators;

For1-D model equation discretized by FTCS we have following results:

$$\underbrace{\rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}+\rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}-\Gamma\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{\Delta x^2}-S_i^n}_{L_{\Delta x,\Delta t}(\phi_i^n)}-\{\underbrace{\rho\frac{\partial\phi}{\partial t}+\rho u\frac{\partial\phi}{\partial x}-\Gamma\frac{\partial^2\phi}{\partial x^2}-S}_{L(\phi)_i^n}\}_{i,n}=\underbrace{O(\Delta t,\Delta x^2)}_{T.E.}$$

How to get this result? First discussing the transient term

Transient term of FD form

$$\rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=\rho\frac{\cancel{\phi_i^n}+\left(\frac{\partial\phi}{\partial t}\right)_{i,n}\cancel{\Delta t}+\left(\frac{\partial^2\phi}{\partial t^2}\right)_{i,n}\frac{\cancel{\Delta t}^2}{2!}+\ldots-\cancel{\phi_i^n}}{\cancel{\Delta t}}$$

i.e.

$$\rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}-\rho\left(\frac{\partial\phi}{\partial t}\right)_{i,n}=\frac{1}{2}\frac{\partial^2\phi}{\partial t^2}\Delta t+\ldots=O(\Delta t)$$

Second, for the convection term of FD form:

$$\rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}=\rho u\left[\frac{\cancel{\phi_i^n}+\Delta x\frac{\partial\phi}{\partial x}+\frac{1}{2}\Delta x^2\cancel{\frac{\partial^2\phi}{\partial x^2}}+O(\Delta x^3)}{2\Delta x}-\frac{(\cancel{\phi_i^n}-\Delta x\frac{\partial\phi}{\partial x}+\frac{1}{2}\Delta x^2\cancel{\frac{\partial^2\phi}{\partial x^2}}+O(\Delta x^3))}{2\Delta x}\right]$$

$$=\frac{\cancel{2}\rho u\frac{\partial\phi}{\partial x}\cancel{\Delta x}+O(\cancel{\Delta x}^3)}{\cancel{2\Delta x}}=\rho u\frac{\partial\phi}{\partial x}+O(\Delta x^2)$$

Thus: $$\rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}-\rho u\left(\frac{\partial\phi}{\partial x}\right)_{i,n}=O(\Delta x^2)$$

Then for diffusion term : $$\Gamma\frac{\phi_{i+1}^n-\phi_i^n+\phi_{i-1}^n}{\Delta x^2}-\Gamma\frac{d^2\phi}{dx^2}=O(\Delta x^2)$$

Assuming that the source term does not introduce any truncation error, then:

The T.E. of FTCS scheme for 1-D model equation:

$$\boxed{O(\Delta t,\Delta x^2)}$$

Its mathematical meaning is:

Existing two positive constants， K_1， K_2， when $\Delta t\to 0,\Delta x\to 0$ the difference between the two operators will be less than $(K_1\Delta t+K_2\Delta x^2)$.

4. Consistence (相容性) of discretized equations

If the T.E. of a discretized equation approaches zero when $\Delta t \to 0, \Delta x \to 0$ then:

the discretized equation is said to be in consistence with the partial differential equation (PDE).

When T.E. is in the form of $O(\Delta t^n, \Delta x^m)(n, m > 0)$ the discretized equations possess(**具有**) consistence;

However when T.E. contains $\Delta t / \Delta x$ only when the time step approaches zero much faster than space step , the consistence can be guaranteed **(保证)**.

7.1.2 Discretization error and convergence

1. Discretization error(离散误差) ρ_i^n

$$\rho_i^n = \phi(i,n) - \phi_i^n$$

Analytical solution of **PDE**

Analytical solution of **FDE**

2. Factors affecting discretization error

(1) T.E.: The higher the order, the smaller the value of ρ_i^n for the same grid system;

(2) Grid step: For the same order of accuracy, a finer grid system leads to less numerical error.

For conventional engineering simulation, usually:

2nd order for diffusion term and 2nd or 3rd order for convection term are used. For direct simulation of turbulent flow much higher scheme schemes are needed!

3. Convergence (收敛性) of the discretized equations

When $\Delta t \to 0, \Delta x \to 0$ if $\rho_i^n \to 0$ then it is said: the discretized equations possess convergence.

Proving convergence mathematically for a specific scheme is not easy.

It should be noted :that above descriptions of consistence and convergence are only qualitatively (定性地）, not in the strict mathematical sense. But enough for engineering students.

（Quantitatively----定量地）

7.1.3 Round-off error(舍入误差) and stability of initial problems

1. Round-off error ε_i^n

$$\varepsilon_i^n = \phi_i^n - \phi_i^n$$

ϕ_i^n -- actual solution from computer we can obtain

2. Factor affecting round-off error

Length of computer word; Numerical solution method

3. Total error of numerical solutions

$$\phi(i,n)-\phi_i^n=\underline{\phi(i,n)-\phi_i^n}+\underline{\phi_i^n-\phi_i^n}=\rho_i^n+\varepsilon_i^n$$

For most engineering problems, generally ρ_i^n is predominant **(占优).**

4. Stability of initial problems

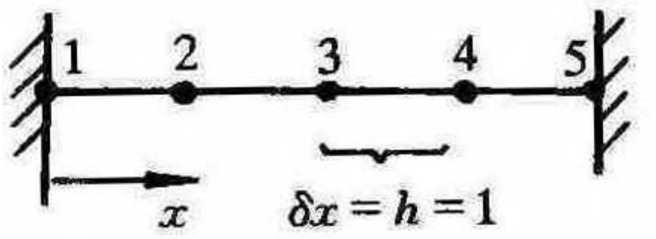
The solution procedure of an initial problem is of marching **(步进)** type; if errors introduced at any time level are enlarged **(放大)** in the subsequent **（随后的）** simulation such that the solutions become **i**nfinite（无限）, this scheme is called **unstable**（**不稳定**）；Otherwise the scheme for the initial problem is stable.

Stability is an **inherent (固有的)** character of a scheme, no matter what kind of error is introduced.

7.1.4 Example

[Example 3-1 of Textbook] Effect of T.E. and grid number

$$\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}-2\phi=0\,,\ \phi(0)=0;\ \ \phi(4)=1$$



Find: Values of nodes 2, 3 and 4.

Solution: By FDM: replacing $\dfrac{d^2\phi}{dx^2}$, $\dfrac{d\phi}{dx}$ by FD Exp.

First way: for all three points 2nd order scheme is adopted, then the FD Eqs can be established ;

Second way: for Node 3 fourth order scheme is adopted; Nodes 2 and 4----second order scheme is used.

The analytical solution

$$\phi = \frac{e^{x} - e^{2x}}{e^{4} - e^{-8}}$$

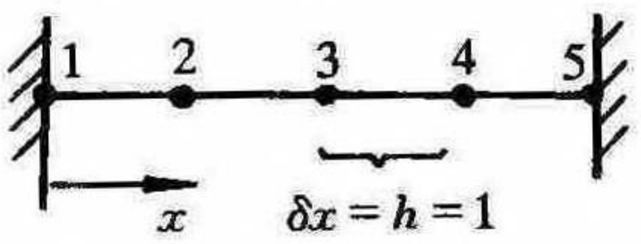


Table 3-1 in the textbook

格　式	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
精确解	0	0.047 3	0.135 0	0.367 9	1
$(i=2,3,4)$ 二阶格式	0	0.058 2	0.155 2	0.394 4	1
$(i=3)$ 四阶格式	0	0.050 5	0.134 8	0.391 8	1

The fourth order scheme is only adopted at Node 3, while solution accuracy is greatly improved

Table3-2 of Textbook Effect of grid fineness

区间数	4	8	16	32	64	精确解
$\phi_{x=1}$	0.058 2	0.050 2	0.048 0	0.047 5	0.047 3	0.047 3
$\phi_{x=2}$	0.155 2	0.140 4	0.136 4	0.135 3	0.135 0	0.135 0
$\phi_{x=3}$	0.394 4	0.375 2	0.369 7	0.368 3	0.367 9	0.367 9

Solution of 32 intervals (**区间**) may be regarded as grid-independent!

[Ex. 3-3 of Textbook] Instability of explicit scheme

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2},\quad 0<x<1$$

$$t\le 0,\ T=2x,\ 0\le x\le 0.5; T=2(1-x), 0.5\le x\le 1$$

Boundary condition: $t>0,\ T(0,t)=T(1,t)=0$

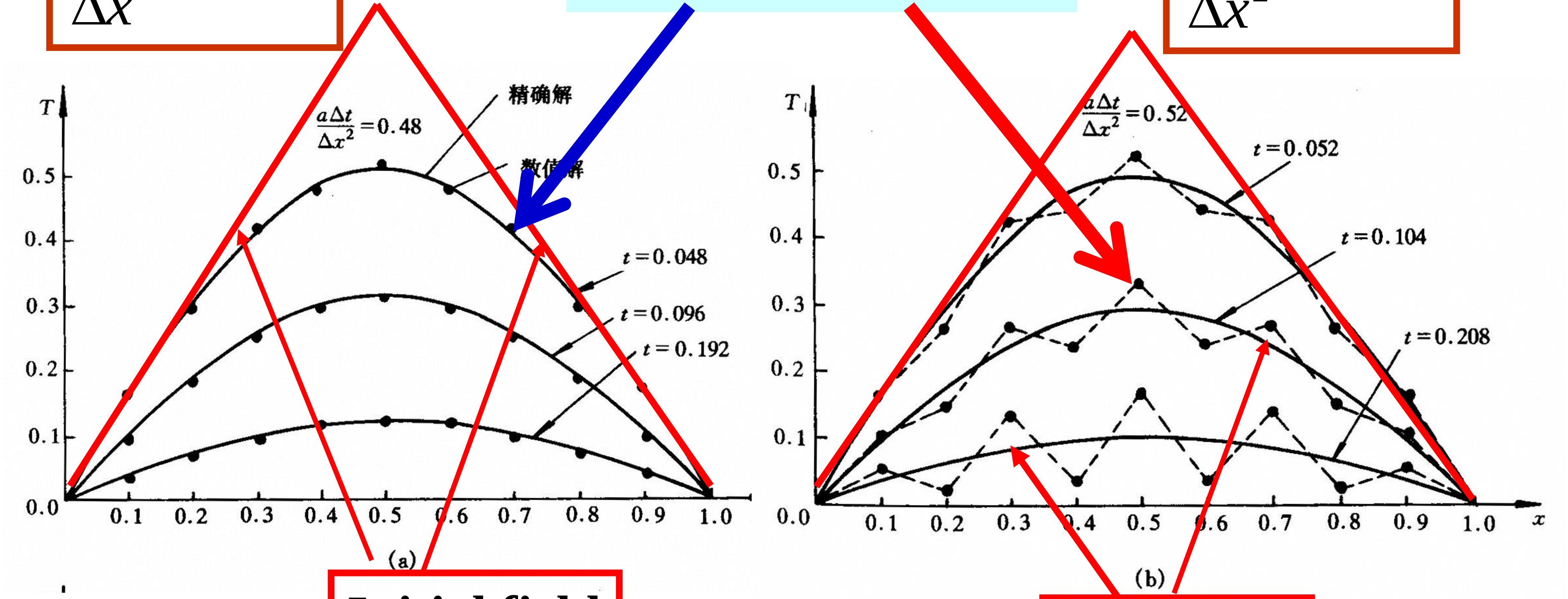
Solution: Numerical solutions are conducted for

$$\frac{a\Delta t}{\Delta x^2}=0.48,\ \text{and}\ 0.52$$

$$\frac{a\Delta t}{\Delta x^2}=0.48$$

Numerical solution

$$\frac{a\Delta t}{\Delta x^2}=0.52$$



Initial field

Analytical

Numerical solutions are converged ,but oscillating .This is because of the physically meaningless coefficients of the algebraic equations .

7.2 von Neumann Method for Analyzing Stability of Initial Problems

7.2.1 Propagation of error vector with time

7.2.2 Discrete Fourier expansion

7.2.3 Basic idea of von Neumann analysis

7.2.4 Examples of von Neumann analysis

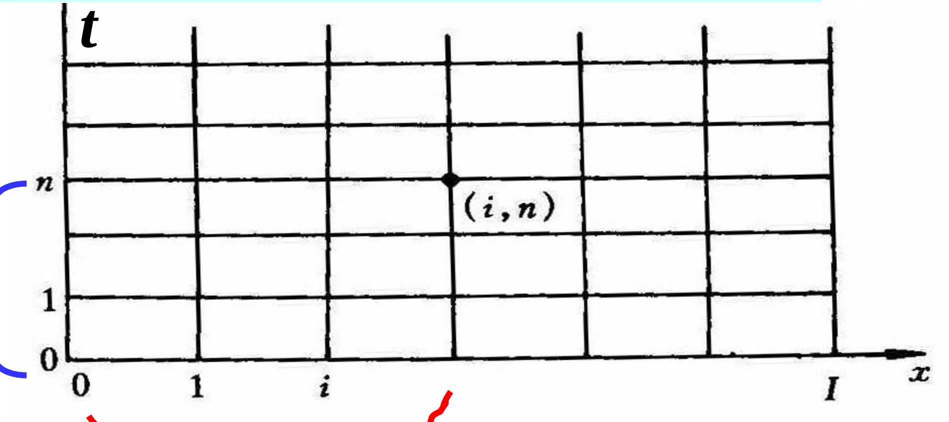
7.2.5 Discussion on von Neumann analysis

7.2 von Neumann Method for Analyzing Stability of Initial Problems

7.2.1 Propagation (传递) of error vector with time

1 Matrix expression of discretized equations

$$\begin{cases} \dfrac{\partial T}{\partial t} = a\dfrac{\partial^2 T}{\partial x^2},\ 0 < x < L \\ T(x,0) = F(x) \\ T(0,t) = f_1(t),\ T(L,t) = f_2(t) \end{cases}$$



$n\Delta t$

$i\Delta x$

$$\left.\begin{cases} \dfrac{T_i^{n+1} - T_i^n}{\Delta t} = a\dfrac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2},\ i = 1,2,,3,....(I-1) \\ T_i^0 = F(x_i),\ i = 0,1,2,3.....I \quad \text{Initial condition} \\ T_0^n = f_1(n\Delta t),\ T_I^n = f_2(n\Delta t),\ n = 1,2,..... \end{cases}\right\} \quad \text{(a)}$$

Boundary condition

Set $\frac{a\Delta t}{\Delta x^2}=r$, the difference eqs. can be expressed as

$$\boxed{T_i^{n+1}=T_i^n(1-2r)+r(T_{i+1}^n+T_{i-1}^n),\ i=1,2,....(I-1)}$$

For a fixed time level n, the above eqs. can be re-written for each inner point as follows:

$$\left\{\begin{array}{ll}
i=1, & T_1^{n+1}=T_1^n(1-2r)+r(T_0^n+T_2^n)\\
i=2, & T_2^{n+1}=T_2^n(1-2r)+r(T_1^n+T_3^n)\\
i=3, & T_3^{n+1}=T_3^n(1-2r)+r(T_2^n+T_4^n)\\
\cdots\cdots\cdots\cdots\cdots & \\
i=I-2, & T_{I-2}^{n+1}=T_{I-2}^n(1-2r)+r(T_{I-1}^n+T_{I-3}^n)\\
i=I-1, & T_{I-1}^{n+1}=T_{I-1}^n(1-2r)+r(T_I^n+T_{I-2}^n)
\end{array}\right.$$

Expressed in matrix (矩阵)form:

$$\underbrace{\begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \\ T_3^{n+1} \\ \cdot \\ \cdot \\ \cdot \\ T_{I-2}^{n+1} \\ T_{I-1}^{n+1} \end{bmatrix}}_{\vec{T}^{n+1}} = \underbrace{\begin{bmatrix} (1-2r) & r & & & & \\ r & (1-2r) & r & & \mathbf{0} & \\ & r & (1-2r) & r & & \\ \cdot & & & & & \\ \cdot & & \mathbf{0} & & & \\ \cdot & & & & & \\ & & & r & (1-2r) & r \\ & & & & r & (1-2r) \end{bmatrix}}_{\vec{A}} \underbrace{\begin{bmatrix} T_1^{n} \\ T_2^{n} \\ T_3^{n} \\ \cdot \\ \cdot \\ \cdot \\ T_{I-2}^{n} \\ T_{I-1}^{n} \end{bmatrix}}_{\vec{T}^{n}} + \underbrace{\begin{bmatrix} rT_0^{n} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ rT_I^{n} \end{bmatrix}}_{\vec{g}}$$

Column vector(列矢量) **Matrix(矩阵)**

Simply in vector form:

$$\left.\begin{aligned} \vec{T}^{n+1} &= \vec{A}\vec{T}^{n} + \vec{g} \\ \vec{T}^{0} &= \vec{F} \end{aligned}\right\} \qquad \textbf{(b)}$$

$\vec{A}$ represents a transformation(变换) from $\vec{T}^{n}$ to $\vec{T}^{n+1}$

2 Propagation (传递) of error vector with time

Assuming that no error is introduced at the boundary, while it is introduced at the initial condition . Then the error components at each node form(形成) an error vector, denoted by $\vec{\varepsilon}^{\,0}$:

For the exact solution:

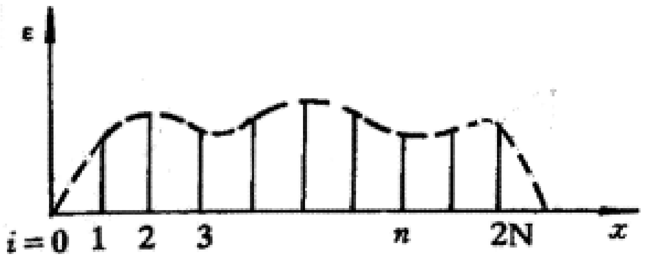
$$\left.\begin{aligned} \vec{T}^{n+1} &= \vec{A}\vec{T}^{n} + \vec{g} \\ \vec{T}^{0} &= \vec{F} \end{aligned}\right\} \text{(b)}$$

Denoting the solution with error by $\vec{\tilde{T}}$:

then
$$\left.\begin{aligned} \vec{\tilde{T}}^{n+1} &= \vec{A}\vec{\tilde{T}}^{n} + \vec{g} \\ \vec{\tilde{T}}^{0} &= \vec{F} + \vec{\varepsilon}^{\,0} \end{aligned}\right\} \text{(c)}$$

(c)—(b) $\rightarrow$
$$\left\{\begin{aligned} \vec{\tilde{T}}^{n+1} - \vec{T}^{n+1} &= \vec{A}\left(\vec{\tilde{T}}^{n} - \vec{T}^{n}\right) \\ \vec{\tilde{T}}^{0} - \vec{T}^{0} &= \vec{\varepsilon}^{\,0} \end{aligned}\right.$$

That is: $\vec{\varepsilon}^{\,n+1} = \vec{A}\vec{\varepsilon}^{\,n}$, with $\vec{\varepsilon}^{\,0}$ being specified (给定) **(d)**

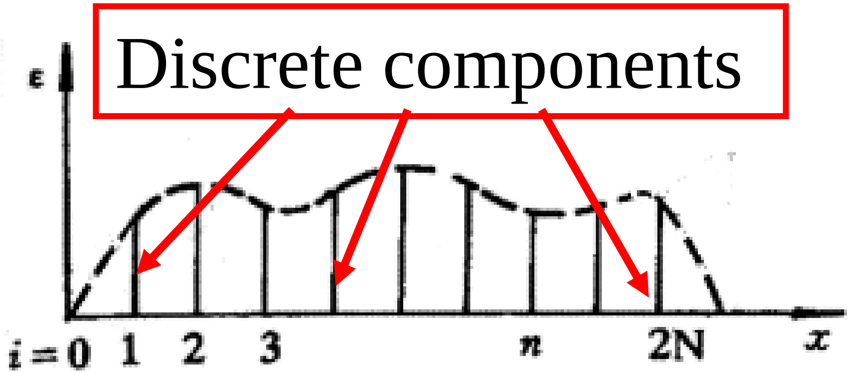


Thus the propagation of error vector can be described by the same matrix $\vec{A}$ under the condition that:

No error is introduced at the boundary!

3. Expression of error vector （误差矢量的表示方法)

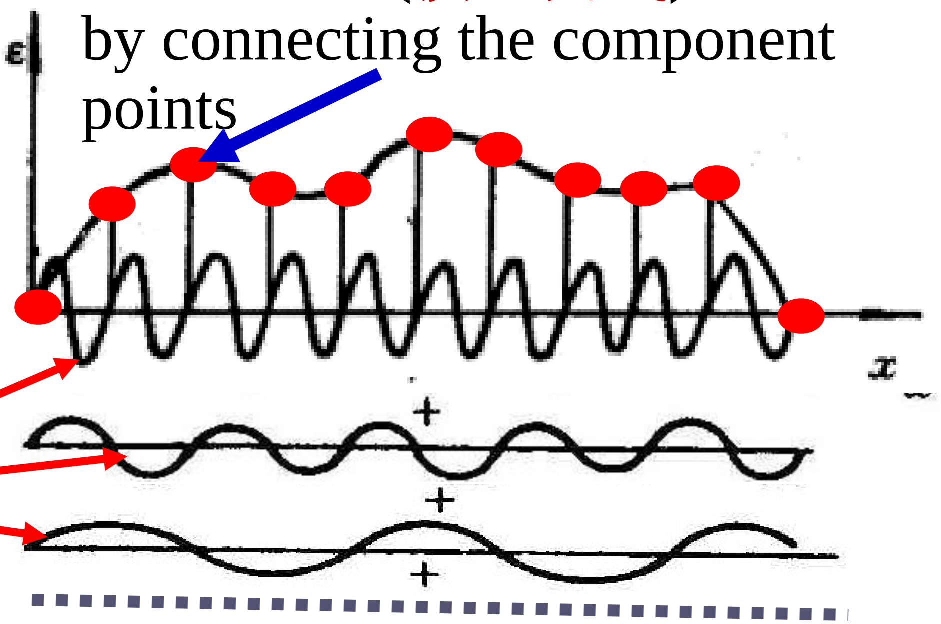
(1) Expressed by discrete components（**离散分量**)



(2) Expressed by harmonic components (谐波分量)

The error curve can be expressed by a summation of harmonic components

Error curve (误差曲线) obtained by connecting the component points



7.2.2 Discrete Fourier expansion(离散傅里叶展开)

1. Expansion equation: Similar to Fourier expansion for continuous function within the region $[-l, l]$, (2N+1) pair of numbers (数对) ,(x_i, y_i) , can be expressed by a summation of harmonic components（谐波分量）：

Continuous Fourier exp.(连续傅里叶展开)

Fourier exp. for finite pair of numbers (有限个数对的傅氏展开)

Continuous func.within $[-l, l\ ]$

$$f(x) = y = \sum_{n=-\infty}^{\infty} C_n e^{I(\frac{2n\pi}{2l})x}$$

$$I = \sqrt{-1}$$

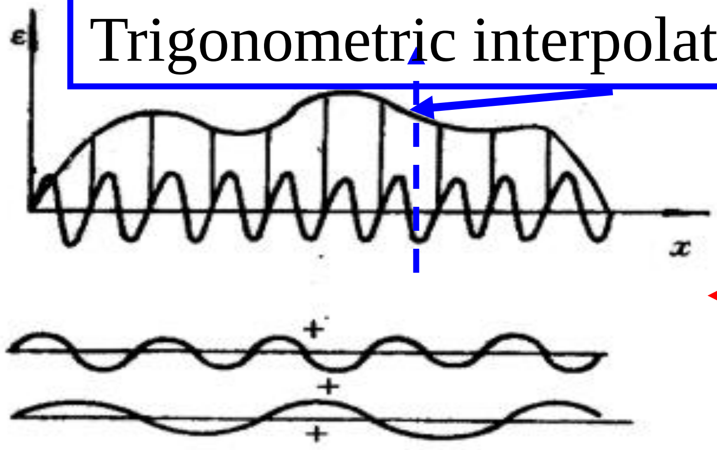
(2N+1) pair of numbers

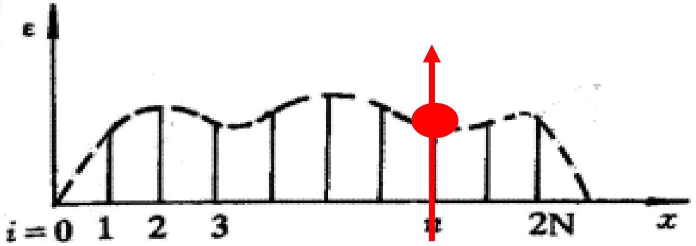
$$y_i = \sum_{k=-N}^{N} C_k e^{I(\frac{2k\pi}{2N+1})x_i}$$

subscripts $i = 0, 1, 2, \ldots\ldots 2N$

Sum of harmonic components

Trigonometric interpolation





$$x = x_i, y = y_i;$$

When x is between x_i, y_i is the interpolation by finite terms of trigonometric (三角) functions

2. Expression of harmonic component

Corresponding to the term $(\frac{2n\pi}{2l})x$ in Fourier expansion

$$(\frac{2k\pi}{2N+1})x_i = (\frac{2k\pi}{2N+1})i\Delta x = i(\frac{2k\pi}{2N+1})\Delta x = i\,k_x\,\Delta x = i\ \theta_k$$

Then $C_k e^{I(\frac{2k\pi}{2N+1})x_i} = C_k e^{I\,i\,\theta_k}$

k_x —wave number， $k_x\lambda = 2\pi$， θ —phase angle

$C_k e^{Ii\theta_k}$ harmonic component， C_k — amplitude(**振幅**)

In transient problem it is a function of time, denoted by $\psi(t)$

The general expression of harmonic component is then

$\psi(t)e^{I\,i\,\theta}$ --Getting this form of harmonic components is the purpose of discussion on discrete Fourier-expansion

7.2.3 Basic idea of von Neumann analysis

1. Basic idea

The numerical error is considered as a kind of disturbances(**扰动**)， which can be decomposed(**分解**) into a finite number (**有限个**)of harmonic components; If some discretized scheme can guarantee that the amplitude of any component will be attenuated (**衰减**) or at least be kept unchanged in the calculation procedure then the scheme is stable; Otherwise it is unstable.

2. Analysis method

How to implement (**实施**) this idea? Replacing the dependent variable by the expression of a harmonic component , finding the ratio of amplitude of two subsequent time levels, and demanding (要求) that

$$\left|\frac{\psi(t+\Delta t)}{\psi(t)}\right| \le 1$$

The condition of this inequality is the criterion of scheme stability.

The ratio is called magnified factor (放大因子)

7.2.4 Examples

1. Stability analysis for FTCS of 1-D conduction eq.

Replacing T in the discretized eq. by $\varepsilon(t) = \psi(t)e^{Ii\theta}$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = a\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

yields $\frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} e^{Ii\theta} = a\psi(t)\frac{e^{I(i+1)\theta} - 2e^{Ii\theta} + e^{I(i-1)\theta}}{\Delta x^2}$

Divided by $e^{Ii\theta}$ and from Euler Eq. $\boxed{e^{I\theta} = \cos\theta + I\sin\theta}$

Rearranging, $\frac{\psi(t+\Delta t)}{\psi(t)} = 1 - 2\left(\frac{a\Delta t}{\Delta x^2}\right)(1-\cos\theta)$

$$\boxed{1-\cos\theta = 2(\sin\frac{\theta}{2})^2} \rightarrow \boxed{\frac{\psi(t+\Delta t)}{\psi(t)} = 1-4(\frac{a\Delta t}{\Delta x^2})\sin^2(\frac{\theta}{2})}$$

Stability condition requires:

$$-1 \le \frac{\psi(t+\Delta t)}{\psi(t)} \le 1$$

i.e.,

$$-1 \le \underbrace{1-4(\frac{a\Delta t}{\Delta x^2})\sin^2(\frac{\theta}{2}) \le 1}_{\text{Automatically satisfied}}$$

Thus, it is required:

$$-1 \le 1-4(\frac{a\Delta t}{\Delta x^2})\sin^2(\frac{\theta}{2}) \rightarrow \boxed{4(\frac{a\Delta t}{\Delta x^2})\sin^2(\frac{\theta}{2}) \le 2}$$

This requirement should be satisfied for all possible values of θ , the most severe case is $\sin^2(\theta/2) = 1$

$$\boxed{4\left(\frac{a\Delta t}{\Delta x^2}\right)\sin^2\left(\frac{\theta}{2}\right)\le 2} \quad \text{if } \sin^2\left(\frac{\theta}{2}\right)=1 \longrightarrow \frac{a\Delta t}{\Delta x^2}\le\frac{1}{2}$$

The above analysis method is called von Neumann method: concept is clear , and its implementation is easy !

Discussion: The above derived stability criterion can be applied only for internal nodes, because it is assumed that at the boundary no error is introduced ; For the 2nd and 3rd kinds of B.C. the stability criterion may be obtained from the discretized equations obtained by balance method by requiring that the coefficient of neighbors must be positive !

2. Stability criterion of FTCS scheme of 1-D model eq.

Replacing ϕ by $\varepsilon(t)=\psi(t)e^{Ii\theta}$ in the discretized eq.

$$\rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}+\rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}=\Gamma\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{\Delta x^2}$$

$$\rho\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}e^{Ii\theta}+\rho u\psi(t)\frac{e^{I(i+1)\theta}-e^{I(i-1)\theta}}{2\Delta x}=$$

$$\Gamma\psi(t)\frac{e^{I(i+1)\theta}-2e^{Ii\theta}+e^{I(i-1)\theta}}{\Delta x^2}$$

Rearranging, yields:

$$\frac{\psi(t+\Delta t)}{\psi(t)}=\mu=1-\frac{1}{2}\left(\frac{u\Delta t}{\Delta x}\right)\underbrace{(e^{I\theta}-e^{-I\theta})}_{2I\sin\theta}+\left(\frac{a\Delta t}{\Delta x^2}\right)\underbrace{(e^{I\theta}-2+e^{-I\theta})}_{(2\cos\theta-2)}$$

Set $c=\dfrac{u\Delta t}{\Delta x}$ (Courant number) and $r=\dfrac{a\Delta t}{\Delta x^2}$

Courant was the supervisor of Professor G J Zhu **(朱公瑾)**

《柯士微积分》

$$\frac{\psi(t+t)}{\psi(t)} = \underline{1-2r+2r\cos\theta} - \underline{Ic\sin\theta}$$ Complex **variable**

Stability requires: $\left|1-2r+2r\cos\theta - Ic\sin\theta\right| \le 1$

How to get stability criterion? Analysis and graphics. The later has advantages of clear concept and easy to be implemented.

The locus **(轨迹)** of the complex represents an elliptic circle

$$\mu = \underline{1-2r} + \underline{2r}\cos\theta - I\underline{c}\sin\theta$$

Graphics: $\boxed{\text{Center location}}$ $\boxed{\text{Radius of long axis}}$ $\boxed{\text{Radius of short axis}}$

For $|\mu| \le 1$, the locus of the E.C. (elliptic circle, **椭圆**)

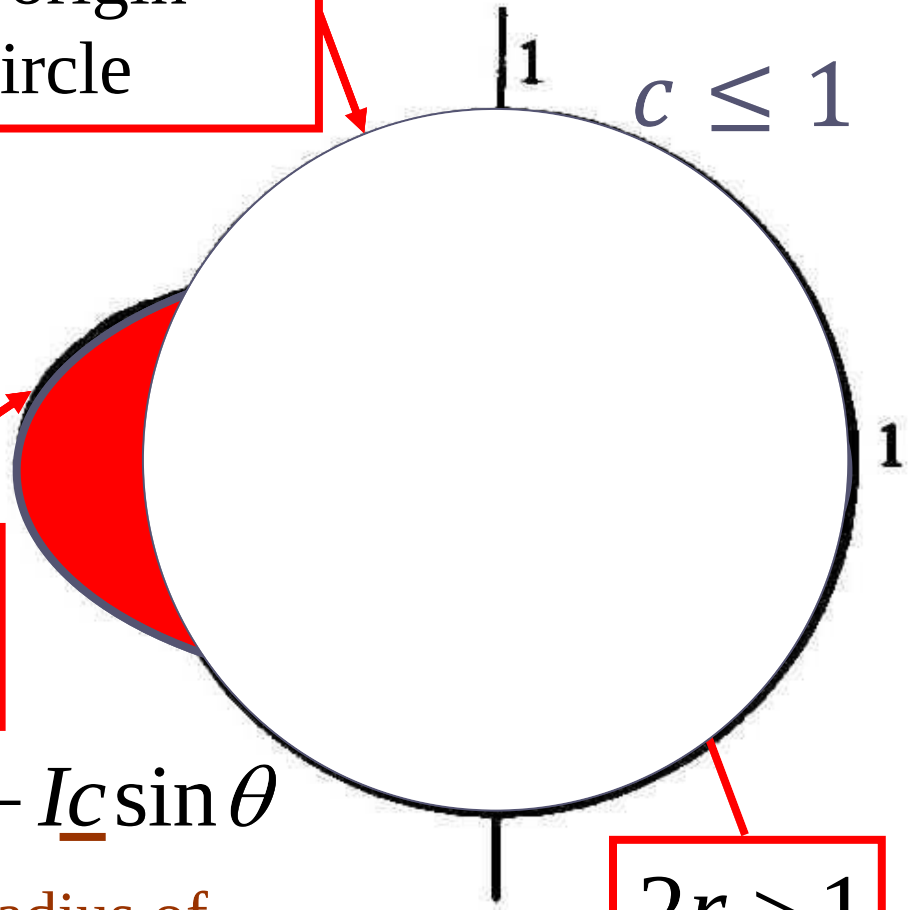
must be within the unit circle with its center at coordinate origin **(原点)**.

Possible Case 1

Circle with its center at origin and radius of 1---Unit circle

$\cancel{(a)\ 2r>1, c\le 1}$

Locus of the elliptic circle (椭圆轨迹)



$2r>1$

$$\mu = \underline{1-2r} + \underline{2r}\cos\theta - I\underline{c}\sin\theta$$

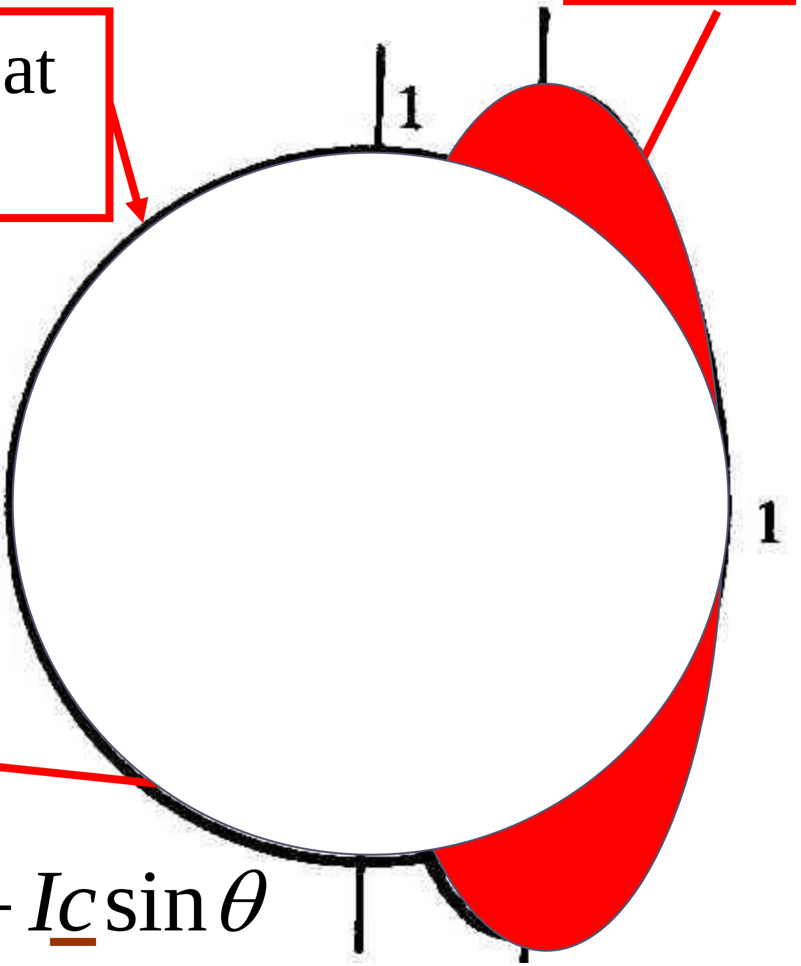
Center | Radius of long axis | Radius of short axis

Possible Case 2

$c > 1$

Circle with its center at origin and radius of 1

$\cancel{(b)\ 2r \le 1, c > 1}$



Locus of the elliptic circle

$$\mu = \underline{1 - 2r} + \underline{2r}\cos\theta - I\underline{c}\sin\theta$$

Center

Radius of long axis

Radius of short axis

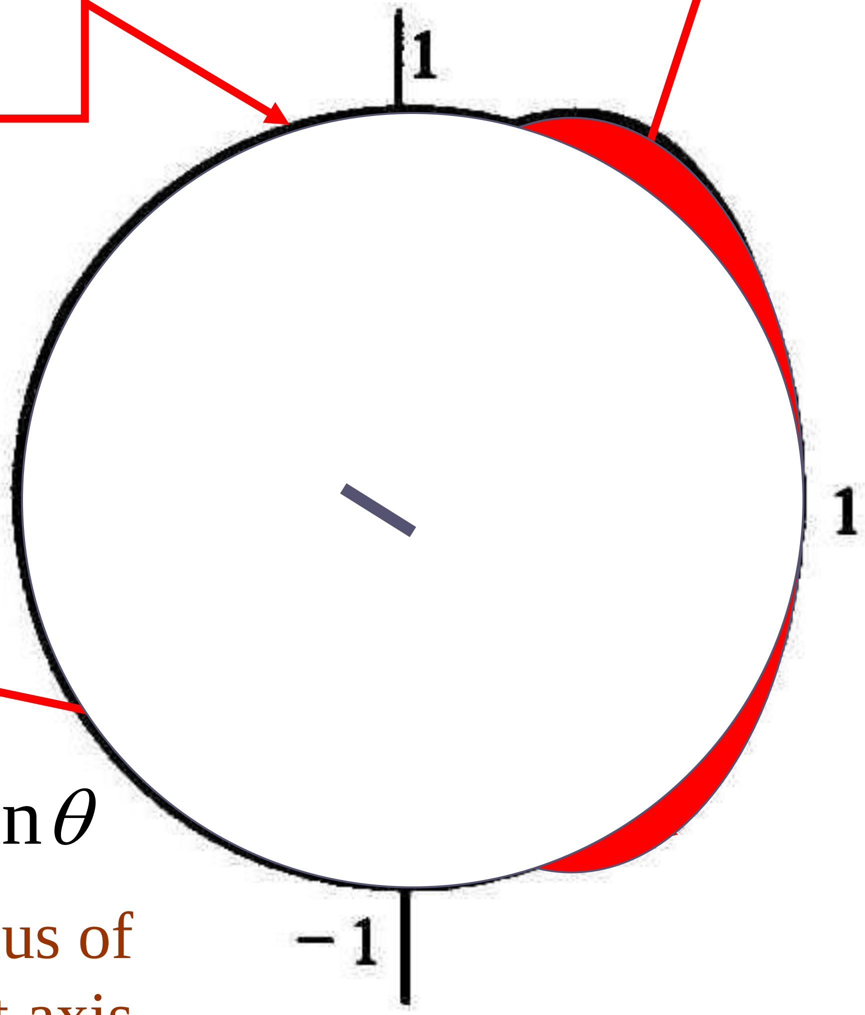
Possible Case 3

Circle with its center at origin and radius of 1

$c < 1$

$\cancel{(c)\ 2r \le 1, c \le 1}$

Locus of the elliptic circle



$$\mu = \underline{1 - 2r} + \underline{2r}\cos\theta - \underline{Ic}\sin\theta$$

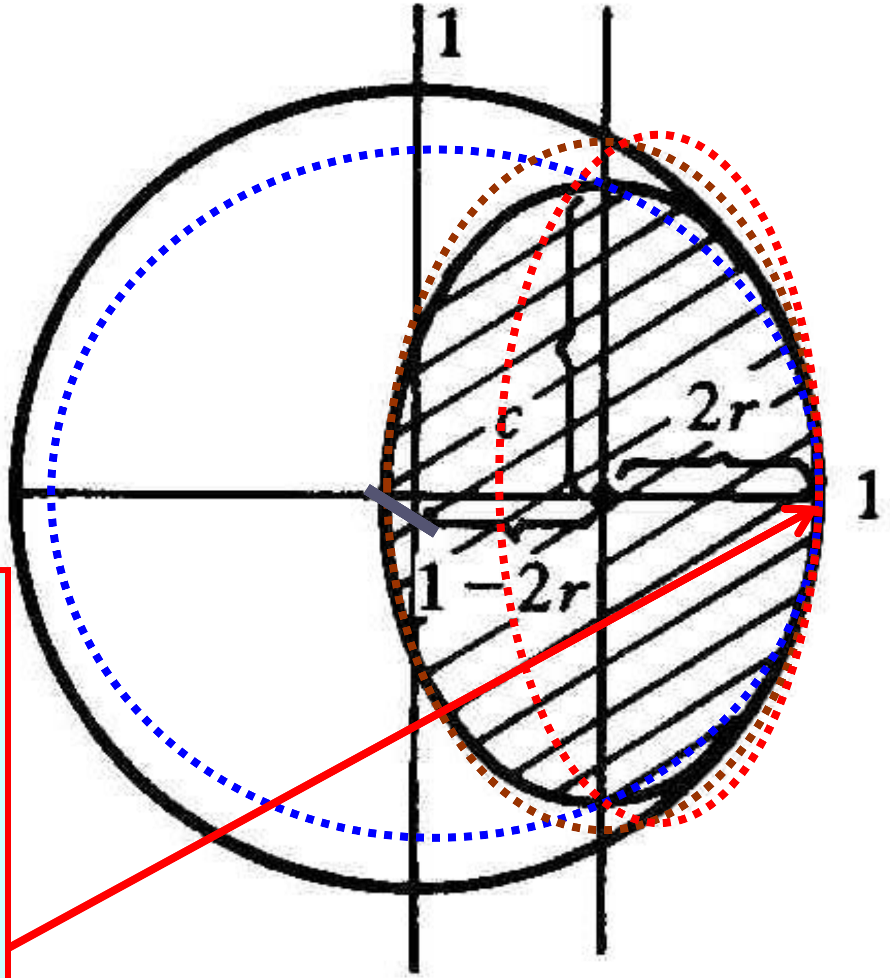
Center

Radius of long axis

Radius of short axis

Possible Case 4

$$(d)\ 2r \le 1, c^2 \le 2r$$



The curvature(**曲率**) radius at the right end of the elliptical circle should be less than and at most equal to 1.

Proof: Magnified factor is: $\mu = \underline{1-2r+2r\cos\theta} - \underline{Ic\sin\theta}$

The parameter equation (**参数方程**) of the elliptic circle is

$$x = 1-2r+2r\cos\theta,\ y = c\sin\theta$$

The curvature radius (**曲率半径**) is:

$$R = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\left|\dot{x}\ddot{y}-\ddot{x}\dot{y}\right|}$$

Dot (•) stands for derivatives

At the right end,

where $\theta = 0$, it is required that $R \le 1$, yields:

$$\boxed{R = \frac{c^3}{2rc} \le 1 \to c^2 \le 2r}$$

Thus the stability condition of FTCS for 1-D model eq.:

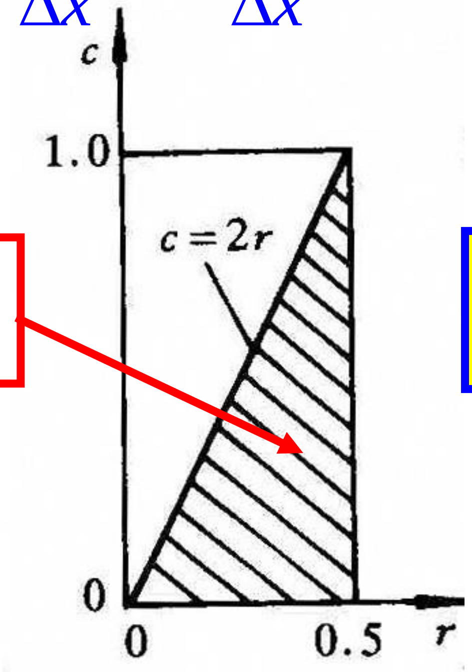
$2r \le 1;$ $c^2 \le 2r$

Discussion: Historically it was considered that:

From $2r \le 1;\ c \le 1$ → $c \le 2r$

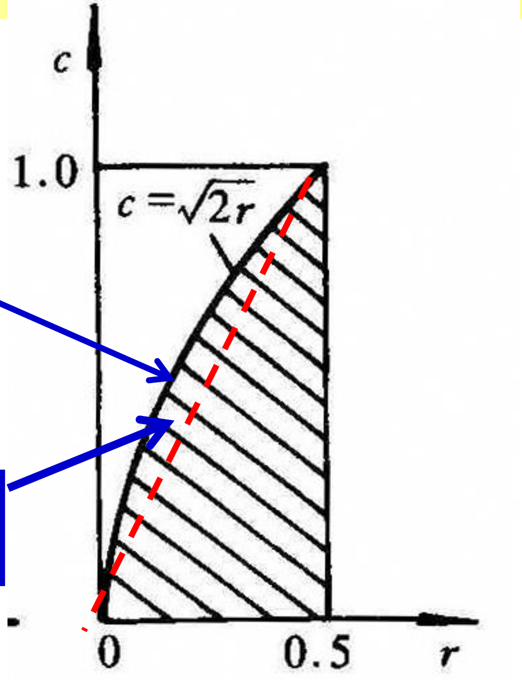
$$c \le 2r \rightarrow \frac{u\Delta t}{\Delta x} \le 2\frac{a\Delta t}{\Delta x^2} \rightarrow \frac{u\Delta x}{a} \le 2$$ **or** $Re_\Delta \le 2;\ Pe_\Delta \le 2$

$c \le 2r$



$c^2 \le 2r$

Extra area!



7.2.5 Application discussion

1.It is applicable to linear transient problem, leading to the maximum allowable (允许的) time step;

2.For non-linear transient problems （transient NS Eqs.） locally linearized (局部线性化) approximation may be adopted : Analyzing the problem as it was linear and making a reduction of the resulting time step, say taking 80%;

3. von-Neumann analysis method is a very useful analysis tool. It has be used to reveal the major concept of MG method **(多重网格).**

7.3 Conservation of Discretized Equations

7.3.1 Definition and analyzing model

7.3.2 Direct summation method

7.3.3 Conditions for guaranteeing conservation of discretized equations

7.3.4 Discussion—expected but not necessary (期待而非必须)

7.3 Conservation of Discretized Equations

7.3.1 Definition and analyzing model

1. Definition

If the summation of a certain number of discretized equations over a finite volume （有限大小体积）satisfies conservation requirement , these discretized equations are said to possess conservation (**离散方程具有守恒性**).

2. Analyzing model---advection equation

It is easy to show that CD of diffusion term possesses conservation. Discussion is only performed for the equation which only has transient term and convective term (advection equation, **平流方程**).

Advection equation

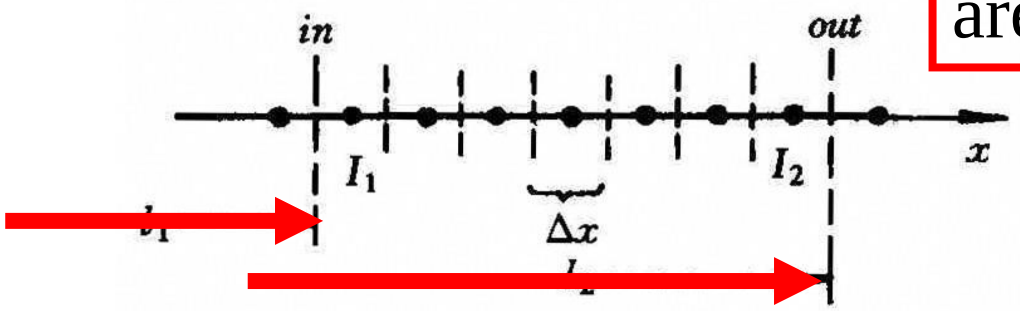
$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{\partial (u\phi)}{\partial x} = 0 & \text{(Conservative)} \\ \dfrac{\partial \phi}{\partial t} + u\dfrac{\partial \phi}{\partial x} = 0 & \text{(Non-conservative)} \end{cases}$$

7.3.2 Direct summation method (直接求和法)

Summing up FTCS scheme of advection eq. of conservative form over the region of $[I_1, I_2]$:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -\frac{u_{i+1}\phi_{i+1} - u_{i-1}\phi_{i-1}}{2\Delta x}$$

Time level of the spatial terms are not shown



$$\sum_{I_1}^{I_2}\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=-\sum_{I_1}^{I_2}\frac{u_{i+1}\phi_{i+1}-u_{i-1}\phi_{i-1}}{2\Delta x}=-\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2\Delta x}$$

$$\sum_{I_1}^{I_2}\underbrace{(\phi_i^{n+1}-\phi_i^n)\Delta x}=-\Delta t\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2}$$

Increment(**增值**) of ϕ within Δt and $[l_1, l_2]$

Is it equal to the net amount of ϕ entering the space region by convection within the same time period?

Analyzing should be made for the right hand terms of the equation to see whether this is true:

$$-\Delta t\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2}=\frac{\Delta t}{2}\sum_{I_1}^{I_2}[(u\phi)_{i-1}-(u\phi)_{i+1}]$$

For the term $\sum_{I_1}^{I_2}[(u\phi)_{i-1}-(u\phi)_{i+1}]$ directly summing up: for the left end, we have:

$$i=I_1 \qquad (u\phi)_{I_1-1} \qquad \cancel{-(u\phi)_{I_1+1}}$$

$$i=I_1+1 \qquad (u\phi)_{I_1} \qquad \cancel{-(u\phi)_{I_1+2}}$$

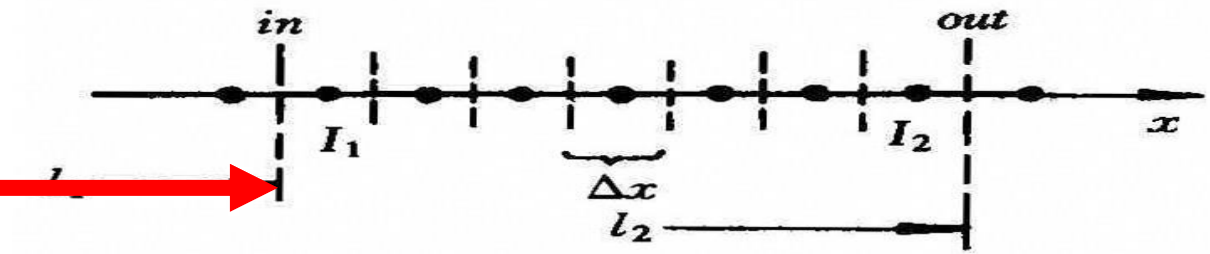
$$i=I_1+2 \qquad \cancel{(u\phi)_{I_1+1}} \qquad \cancel{-(u\phi)_{I_1+3}}$$

$$i=I_1+3 \qquad \cancel{(u\phi)_{I_1+2}} \qquad \cancel{-(u\phi)_{I_1+4}}$$

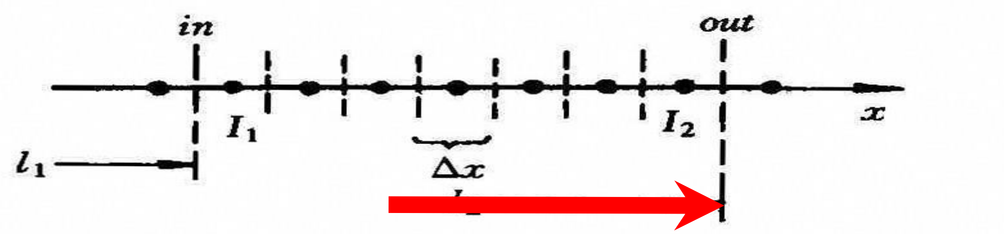
$$i=I_1+4 \qquad \cancel{(u\phi)_{I_1+3}}$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$(u\phi)_{I_1-1}+(u\phi)_{I_1}$$



For the right end:



$$i = I_2 - 3 \qquad \cancel{(u\phi)_{I_2-4}} \qquad \cancel{-(u\phi)_{I_2-2}}$$

$$i = I_2 - 2 \qquad \cancel{(u\phi)_{I_2-3}} \qquad \cancel{-(u\phi)_{I_2-1}}$$

$$i = I_2 - 1 \qquad \cancel{(u\phi)_{I_2-2}} \qquad -(u\phi)_{I_2}$$

$$i = I_2 \qquad \cancel{(u\phi)_{I_2-1}} \qquad -(u\phi)_{I_2+1}$$

$$-[(u\phi)_{I_2} + (u\phi)_{I_2+1}]$$

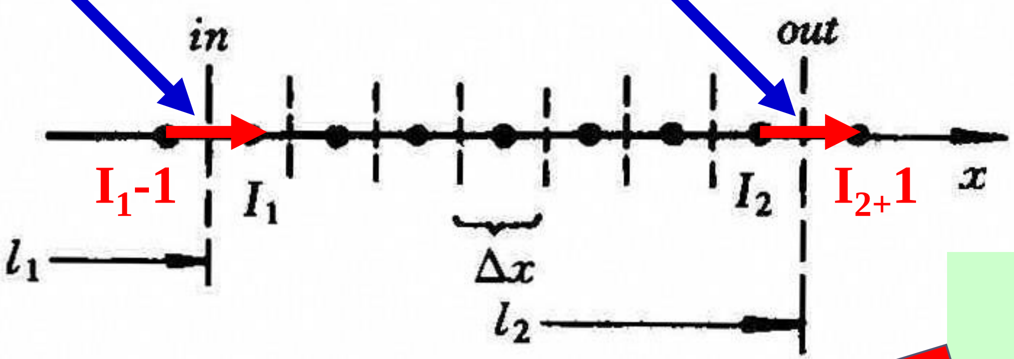
Then: $\dfrac{\Delta t}{2}\sum_{I_1}^{I_2}[(u\phi)_{i-1} - (u\phi)_{i+1}]$

$$= \frac{\Delta t}{2}\{\underbrace{[(u\phi)_{I_1-1} + (u\phi)_{I_1}]}_{\text{Left end of domain}} - \underbrace{[(u\phi)_{I_2} + (u\phi)_{I_2+1}]}_{\text{Right end of domain}}\}$$

Further: $$\frac{\Delta t}{2}\{[(u\phi)_{I_1-1}+(u\phi)_{I_1}]-[(u\phi)_{I2}+(u\phi)_{I_2+1}]\}=$$

$$\Delta t\{[\frac{(u\phi)_{I_1-1}+(u\phi)_{I_1}}{2}]-[\frac{(u\phi)_{I2}+(u\phi)_{I_2+1}}{2}]\}$$

CD-uniform grid



$$=\Delta t(\phi\, flowin-\phi\, flowout)$$

It should be noted that the interface flow rate should be determined by the same scheme as the convection term.

Thus the central difference discretization of the convective term possesses conservative feature.

7.3.3 Conditions for guaranteeing conservation

1.Governing equation should be conservative

For non-conservative form: $$\frac{\partial \phi}{\partial t}+u\frac{\partial \phi}{\partial x}=0$$

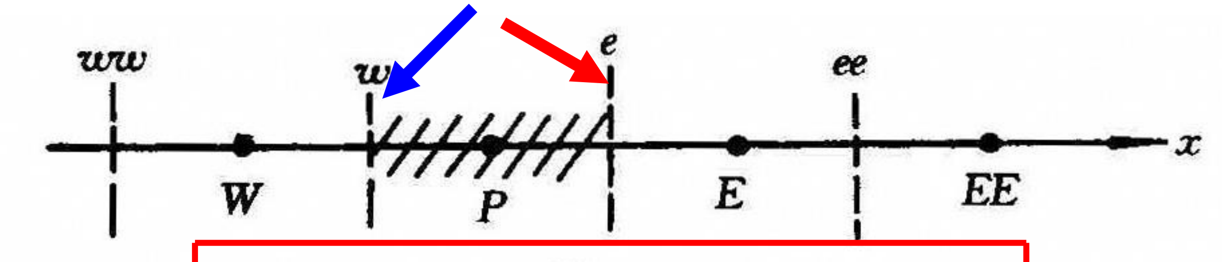
Its FTCS scheme is $$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=-u_i\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x}$$

By direct summation, the above results do not possess conservation because of no cancellation (**抵消**) can be made for the product terms. Only when u and ϕ have the same subscript , the cancellation of inner terms can be done.

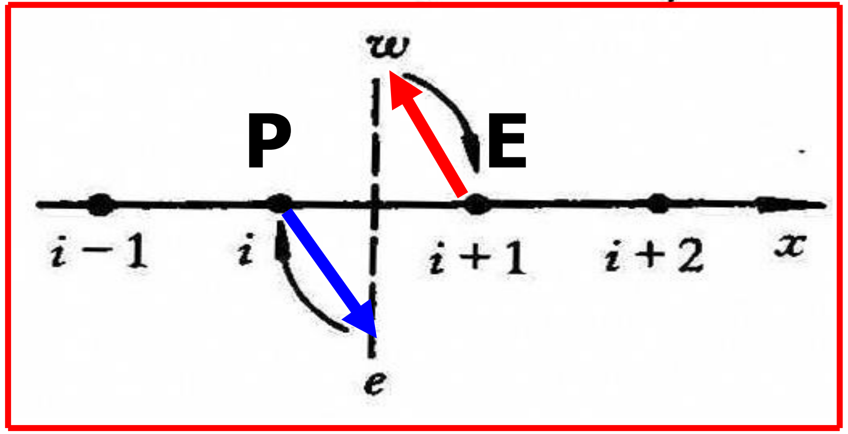
2. Dependent variable and its 1st derivative are continuous at interface

Meaning of “Continuous”

Different interfaces viewed from point P



The same interface viewed from two points P and E



By “Continuous” we mean:

$$(\phi_e)_P = (\phi_w)_E;$$

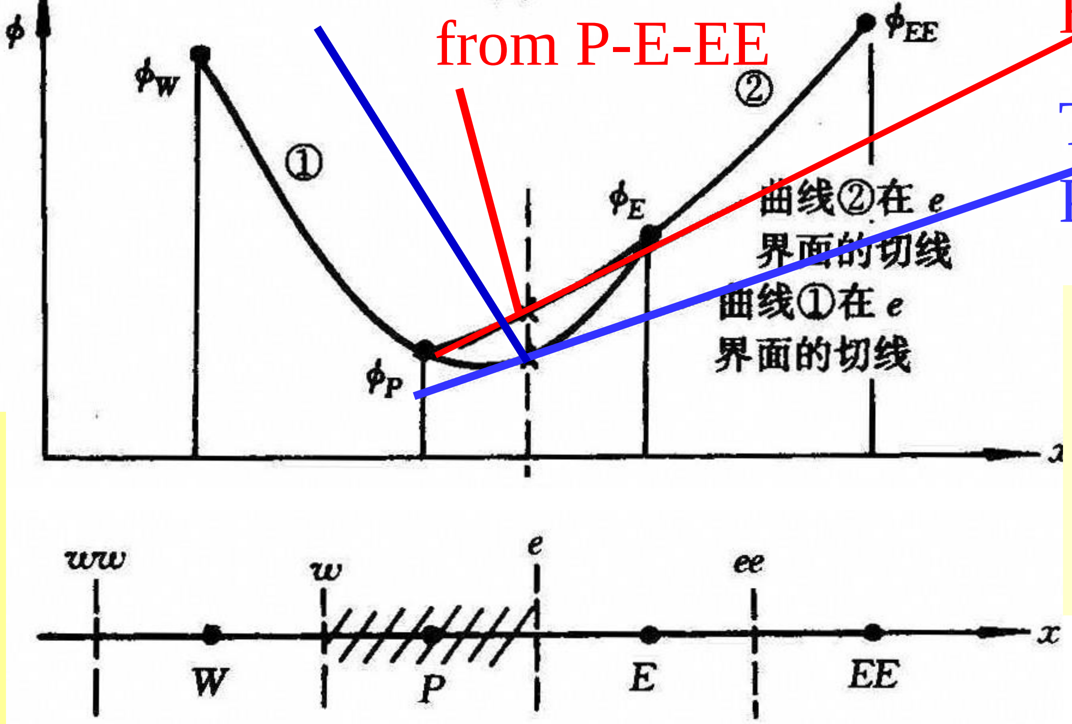
$$[(\frac{\delta\phi}{\delta x})_e]_P = [(\frac{\delta\phi}{\delta x})_w]_E$$

The piecewise linear profile can meet this condition.

Interface-biased quadratic (界面偏向的二次插值) can not satisfy such requirement

For west side of the interface, W, P and E are used for interpolation

For east side of the interface, P , E and EE are used for interpolation



Both first order derivatives and interface values are not equal!

7.3.4 Discussion－Conservation is expected but not necessary for all simulation.(希望而非必须)

Contents

7.4 Transportive (迁移)Character of Discretized Equations

7.4.1 Essential (基本的) difference between convection and diffusion

7.4.2 CD of diffusion term can propagate(传播) disturbance all around (四面八方) uniformly

7.4.3 Analysis of transport character of discretized scheme of convection term

7.4.4 Upwind scheme of convection term possesses transport character

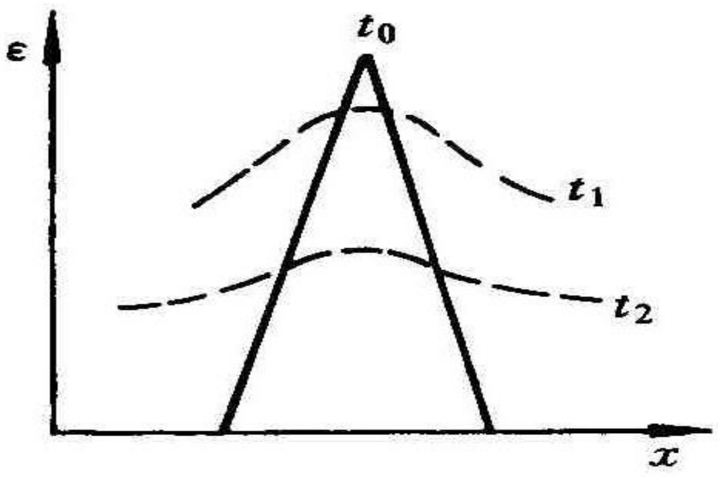
7.4.5 Discussion on transport character of discretized convection term

7.4 Transportive Property of Discretized Equations

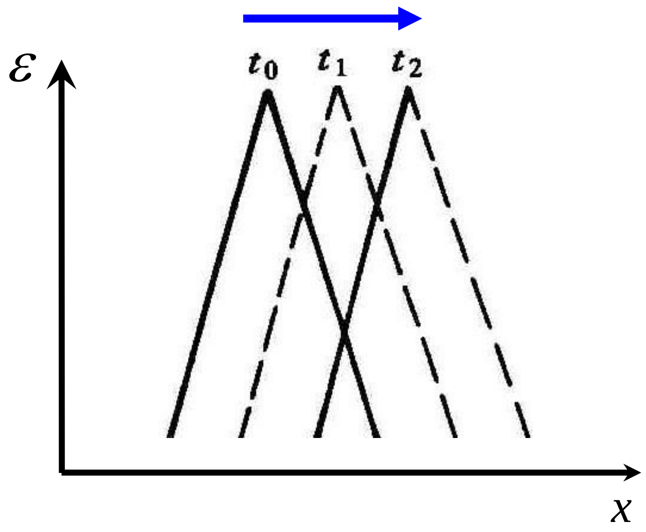
7.4.1 Essential difference between convection and diffusion

Diffusion—Random thermal motions of molecules, no bias(偏向) in direction;

Convection—Directional moving of fluid element, always from upstream to downstream(**从上游到下游**)



(a)



(b)

7.4.2 CD of diffusion term can propagate disturbances all around (四面八方) uniformly

1. FTCS scheme of diffusion eq.

$$\frac{\partial \phi}{\partial t}=\Gamma\frac{\partial^2\phi}{\partial x^2} \longrightarrow \frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=\Gamma\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{\Delta x^2}$$

$$\phi_i^{n+1}=\phi_i^n(1-2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2})+\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}(\phi_{i-1}^n+\phi_{n+1}^n)$$

2. Discrete disturbance analysis (离散扰动分析法)

（1） Assuming a uniform and zero initial field ;

（2） Assuming that a disturbance ε occurs at a point i, at some instant, n, while at all other points and all subsequent time levels no any disturbances;

（3）Analyzing the transfer of the disturbance by the studied scheme.

3. Implementation of discrete disturbance analysis

For point i at (n+1) instant：

Known: $\phi_i^n = \varepsilon,\ \phi_{i-1}^n = \phi_{i+1}^n = 0,$

$$\phi_i^{n+1} = \overset{=\varepsilon}{\phi_i^n}\left(1-2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right) + \frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\left(\overset{0}{\cancel{\phi_{i-1}^n}} + \overset{0}{\cancel{\phi_{i+1}^n}}\right)$$

$$\phi_i^{n+1} = \varepsilon\left(1-2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right) \xrightarrow[\text{Stability requires}]{\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2} \le 0.5} 0 < \phi_i^{n+1} < \varepsilon$$

Physically reasonable

For Point (i ＋1) at (n+1) instant：

$$\frac{\phi_{i+1}^{n+1}-\cancelto{0}{\phi_{i+1}^{n}}}{\Delta t}=\frac{\Gamma}{\rho}\frac{\cancelto{0}{\phi_{i+2}^{n}}-2\cancelto{0}{\phi_{i+1}^{n}}+\underbrace{\phi_{i}^{n}}_{=\varepsilon}}{\Delta x^{2}}$$

$$\phi_{i+1}^{n+1}=\varepsilon\left(\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^{2}}\right)$$

$\longleftarrow$ $\boxed{\textbf{Physically reasonable}}$

For Point (i －1) at (n+1) instant：

$$\frac{\phi_{i-1}^{n+1}-\cancelto{0}{\phi_{i-1}^{n}}}{\Delta t}=\frac{\Gamma}{\rho}\frac{\underbrace{\phi_{i}^{n}}_{=\varepsilon}-2\cancelto{0}{\phi_{i-1}^{n}}+\cancelto{0}{\phi_{i-2}^{n}}}{\Delta x^{2}}$$

$$\phi_{i-1}^{n+1}=\varepsilon\left(\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right)$$

Physically reasonable

$$\phi_{i+1}^{n+1}=\phi_{i-1}^{n+1}$$

Disturbance is transported onto two directions uniformly by diffusion term

7.4.3 Analysis of transport character (迁移特性) of discretized convective term

1. Definition－If a scheme can only transfer disturbance towards the downstream (下游) direction, then it possesses the transport character **(具有迁移特性)** ;
2. Analysis－Applying discrete disturbance analysis to an advection equation with the studied scheme;
3. CD does not possess transport character.

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} \rightarrow \frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -u\underbrace{\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}}_{\text{CD}}$$

$(u>0)$

For Point (i+1) at (n+1) instant:

$$\frac{\phi_{i+1}^{n+1}-\cancelto{0}{\phi_{i+1}^n}}{\Delta t} = -u\frac{\cancelto{0}{\phi_{i+2}^n}-\phi_i^n}{2\Delta x} \rightarrow \phi_{i+1}^{n+1} = (\frac{u\Delta t}{2\Delta x})\varepsilon \qquad (\phi_i^n=\varepsilon)$$

Disturbance is transferred downstream!

Physically reasonable!

For Point (i-1) at (n+1) instant:

$$\frac{\phi_{i-1}^{n+1}-\cancelto{0}{\phi_{i-1}^n}}{\Delta t} = -u\frac{\phi_i^n-\cancelto{0}{\phi_{i-2}^n}}{2\Delta x} \rightarrow \phi_{i-1}^{n+1} = -(\frac{u\Delta t}{2\Delta x})\varepsilon \;?\qquad (\phi_i^n=\varepsilon)$$

Disturbance is transferred upstream， and its sign is the opposite to the original one!

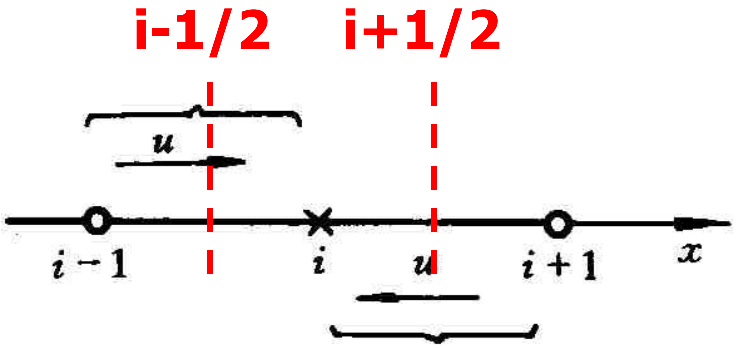
CD of convective term does not possess transport character!

7.4.4 Upwind scheme (迎风格式) of convective term possesses transport character

1. Definitions in FVM and FDM

$$\text{FDM:} \left.\frac{\partial \phi}{\partial x}\right)_i = \begin{cases} \dfrac{\phi_i - \phi_{i-1}}{\delta x}, & u > 0 \\ \dfrac{\phi_{i+1} - \phi_i}{\delta x}, & u < 0 \end{cases}$$

$$\text{FVM:} \phi_{i+1/2} = \begin{cases} \phi_i, & u > 0 \\ \phi_{i+1}, & u < 0 \end{cases}$$



2. FUD possesses transport character

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} \xrightarrow[\textbf{FDM}]{u>0} \frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -u\frac{\phi_i^n-\phi_{i-1}^n}{\Delta x}$$

For point (i+1) at (n+1) instant

$$\frac{\phi_{i+1}^{n+1}-\overset{0}{\phi_{i+1}^{n}}}{\Delta t} = -u\frac{\overset{0}{\phi_{i+1}^{n}}-\phi_i^n}{\Delta x} \quad = \varepsilon$$

Thus: $\phi_{i+1}^{n+1} = \varepsilon(\frac{u\Delta t}{\Delta x})$ ← **Physically reasonable**

For point (i-1) at (n+1) instant:

$$\frac{\phi_{i-1}^{n+1}-\overset{0}{\phi_{i-1}^{n}}}{\Delta t} = -u\frac{\overset{0}{\phi_{i-1}^{n}}-\overset{0}{\phi_{i-2}^{n}}}{\Delta x}$$

Thus $\phi_{i-1}^{n+1} = 0$ ← **Physically required**

Disturbance is not transferred upstream; FUD possesses transport character.

7.4.5 Discussion on transportive character of discretized convective term

1. Transportive character (T.C.) is an important property of discretized convective term; Those who possess T.C. are absolutely stable;

2. Within the stable range, CD is superior to (优于) FUD; Strong convection may lead solution by CD oscillating while solution by FUD is always physically plausible!

3. For those schemes who do not possess T.C. in order to get an absolutely stable solution the coefficients of the scheme should satisfy certain conditions. （替代教材73页4－5行的“凡是不具有迁移特性的对流项…因而只是条件地稳定）；

4. Numerical solution with FUD often has large false-diffusion error; FUD is not recommended for the final solution; while in the deb**u**gging (调试) stage it may be used for its absolutely stability. Upwind idea once was widely used to construct higher-order schemes.

Contents

7.5 Stability analysis of discretized diffusion-convection equation

7.5.1 Three kinds of instability in numerical simulation

1. Instability of explicit scheme for initial problem

Too large time step of explicit scheme will introduce oscillating results; Purpose of stability study is to find the allowed maximum time step; for 1-D diffusion problem:

$$\frac{a\Delta t}{\Delta x^2} \le 0.5$$

2. Instability of iterative solution procedure of ABEqs.

If iterative procedure can not converge, such procedure is called unstable! Unstable procedure can not get a solution!

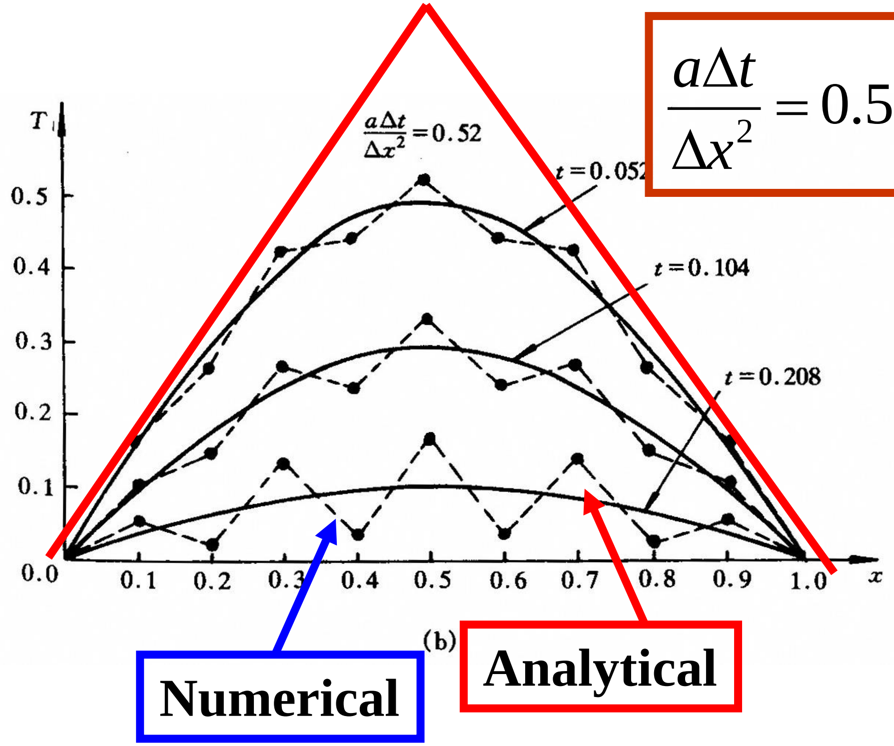
3. Instability caused by discretized convective term

For CD, QUICK,TUD large space step, high velocity may cause to oscillating (wiggling) (振荡的) results. It is called **convective instability**. The purpose of stability study for convection scheme is to find the related critical Peclet number. The consequence (后果）of the three instabilities:

1. Transient instability of explicit scheme: oscillating solutions , and these are the actual solutions of the ABEqs. solved.

2. Instability of solution procedure for ABEqs.: no solution at all.

$$\frac{a\Delta t}{\Delta x^2} = 0.52$$



von Neumann method can be adopted to analyze such instability, see

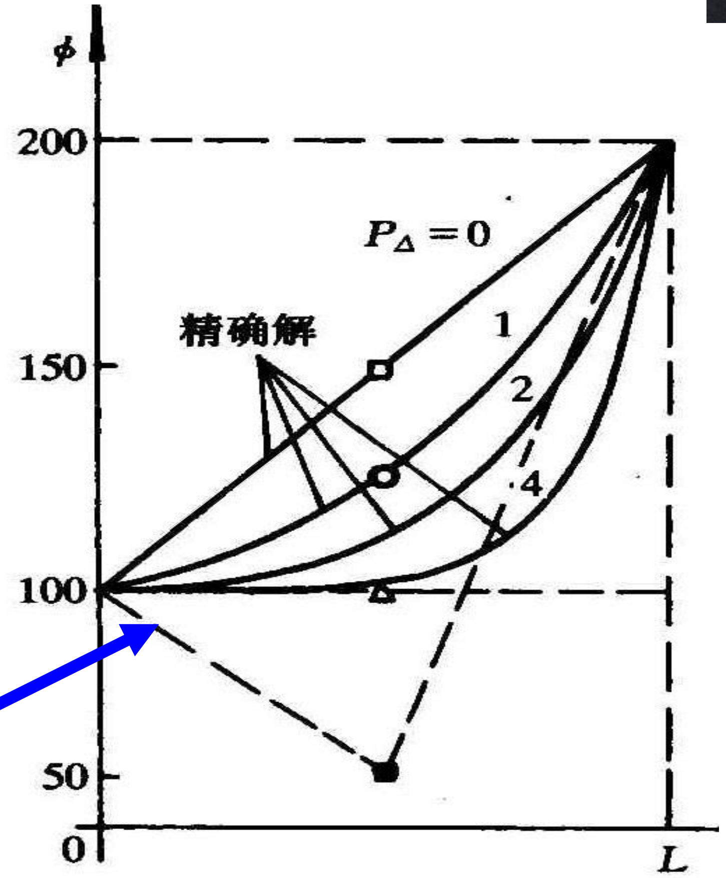
Ni MJ, Tao WQ, Wang SJ .Stability analysis for discretized steady convective-diffusion equation. Numerical Heat Transfer, Part B,1999, 35 (3): 369-388

倪明玖

3. Convective instability :

leading to oscillating solutions and they are the actual solution of the ABEqs.

The problem is caused by unphysical coefficients of the discretized equations.



Actual solution of the scheme

5.7.2 Sign preservation principle for analyzing convective instability

1. Basic idea:

An iterative solution procedure of the ABEqs. of diffusion-convection problem is a marching process (步进过程), from step to step, like the solution procedure of the explicit scheme of an initial problem;

If any disturbance (扰动) at a node is transported in such a way that its effect on the neighboring node is of the opposite sign（符号相反）then the final solution will be oscillating.

Tao W Q, Sparrow EM. The transportive property and convective numerical stability of the steady-state diffusion-convection finite difference equation. Numerical Heat Transfer, 1987, 11:491-497

Sparrow EM

Thus to avoid oscillating results we should require that any disturbance at a node should be transported in such a way that its effect on the neighboring nodes must have the same sign as the original disturbance, i.e., sign is preserved!

2. Analysis method:

(1) The iterative solution procedure of the discretized diffusion-convection equation is modeled by the marching process of the explicit scheme of an initial problem;

(2) Stability is an inherent (固有的) character, which can be tested by adding any disturbance ;

(3) The studied scheme is used to discretize the convection term of 1-D transient diffusion-convection equation with explicit scheme

and diffusion term is by CD; The transfer of a disturbance to the next time level is determined by the discrete disturbance analysis method.

(4) Stability of the convection scheme requires that the effect of any disturbance at any time level on the neighboring point at the next time level **must has the same sign**.

3. Implementation procedure

(1) Applying the studied scheme to the explicit scheme of 1-D transient diffusion-convection equation ;

(2) Adopting the discrete disturbance analysis method to determine the transportation of disturbance ε_i^n introduced any time level n and node i ;

(3) Stability of the studied scheme requires:

$$\frac{\phi_{i\pm1}^{n+1}}{\varepsilon_i^n} \geq 0$$ **(Sign preservation principle 符号不变原则，SPP)**

If above equation is unconditionally valid, the scheme is absolutely stable; Otherwise the condition that makes the above equation valid gives the critical Peclet number, beyond which the scheme will lead to osillating solution.

(4) We have shown that disturbance transportation by diffusion via CD is $\Gamma\Delta t / \rho\Delta x^2$ ， hence discrete disturbance analysis can be only conducted for the studied convection scheme, and then adding the two effect terms together.

4. Implementation example

Stability analysis for TUD scheme:

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} \quad \boxed{u \ge 0} \quad \frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -u\frac{2\phi_{i+1}^n+3\phi_i^n-6\phi_{i-1}^n+\phi_{i-2}^n}{6\Delta x}$$

Disturbance analysis for the convection term

For node(i+1) (downstream):

$$\frac{\phi_{i+1}^{n+1}-\underset{0}{\phi_{i+1}^n}}{\Delta t} = -u\frac{2\underset{0}{\phi_{i+2}^n}+3\underset{0}{\phi_{i+1}^n}-6\underset{\varepsilon_i^n}{\phi_i^n}+\underset{0}{\phi_{i-1}^n}}{6\Delta x}$$

Thus: $$\phi_{i+1}^{n+1} = (\frac{u\Delta t}{\Delta x})\varepsilon_i^n$$ ← **Physically reasonable**

Disturbance is transported by convection downstream!

For node (i－1) (upstream):

$$\frac{\phi_{i-1}^{n+1}-\cancel{\phi_{i-1}^{n}}^{\,0}}{\Delta t}=-u\frac{\cancel{2\phi_i^n}^{\,\varepsilon_i^n}+3\cancel{\phi_{i-1}^n}^{\,0}-6\cancel{\phi_{i-2}^n}^{\,0}+\cancel{\phi_{i-3}^n}^{\,0}}{6\Delta x}$$

$$\text{Thus:}\quad \phi_{i-1}^{n+1}=-\frac{1}{3}\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon_i^n$$

Physically wrong!

Disturbance is transported upstream with opposite sign!

For node(i+1): $\dfrac{\dfrac{\Gamma\Delta t}{\Delta x^2}+\left(\dfrac{u\Delta t}{\Delta x}\right)\varepsilon_i^n}{\varepsilon_i^n}>0$ **Automatically satisfied!**

For node (i－1):

$$\frac{\dfrac{\Gamma\Delta t}{\Delta x^2}-\dfrac{1}{3}\left(\dfrac{u\Delta t}{\Delta x}\right)\varepsilon_i^n}{\varepsilon_i^n}\ge 0$$

Valid only when $\dfrac{\rho u\Delta x}{\Gamma}\le 3$

$$\longrightarrow\ \frac{\rho u\Delta x}{\Gamma}=P_{\Delta cr}\equiv 3!$$

Leonard (1981) once analyzed the stability character of TUD and concluded that it is inherently stable(固有地稳定）. However numerical practice shows it is only conditionally stable.

5. Summary of analysis results

Stability of seven schemes (Table 5-3 of Textbook)

No	Scheme	Definition of scheme	Transferred by convection		Stability condition
			Up	Down	
1	**FUD**	$\left.\frac{\partial\phi}{\partial x}\right\|_i \cong \frac{\phi_i-\phi_{i-1}}{\Delta x}, u>0$ $\frac{\phi_{i+1}-\phi_i}{\Delta x}, u<0$	0	$\left(\frac{u\Delta t}{\Delta x}\right)\epsilon$	**Abs. stable**
2	**CD**	$\left.\frac{\partial\phi}{\partial x}\right\|_i \cong \frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x}$	$-\frac{1}{2}\left(\frac{u\Delta t}{\Delta x}\right)\epsilon$	$\frac{1}{2}\left(\frac{u\Delta t}{\Delta x}\right)\epsilon$	$P_\Delta \leqslant 2$

No	Scheme	Definition of scheme	Transferred by convection		Stability condition
			Up	Down	
3	**SUD**	$\left.\frac{\partial\phi}{\partial x}\right\|_i \cong \frac{\phi_i-\phi_{i-1}}{\Delta x}+\frac{\phi_i-2\phi_{i-1}+\phi_{i-2}}{2\Delta x}, u>0$ $\frac{\phi_{i+1}-\phi_i}{\Delta x}+\frac{\phi_i-2\phi_{i+1}+\phi_{i+2}}{2\Delta x}, u<0$	0	$2\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	**Abs. stable**
4	**TUD**	$\left.\frac{\partial\phi}{\partial x}\right\|_i \cong \frac{2\phi_{i+1}+3\phi_i-6\phi_{i-1}+\phi_{i-2}}{6\Delta x}, u>0$ $\frac{-\phi_{i+2}+6\phi_{i+1}-3\phi_i-2\phi_{i-1}}{6\Delta x}, u<0$	$-\frac{1}{3}\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	$\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	$P_\Delta \leqslant 3$
5	**Fromm**	$\phi_{i+1/2}=\frac{1}{4}(\phi_{i+1}+4\phi_i-\phi_{i-1})$	$-\frac{1}{4}\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	$\frac{1}{4}\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	$P_\Delta \leqslant 4$

No	Scheme	Definition of scheme	Transferred by convection: Up	Transferred by convection: Down	Stability condition
6	QUICK	$\phi_{i+1/2}=\dfrac{\phi_i+\phi_{i+1}}{2}-\dfrac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{8}, u>0$ $\dfrac{\phi_i+\phi_{i+1}}{2}-\dfrac{\phi_{i+2}-2\phi_{i+1}+\phi_i}{8}, u<0$	$\dfrac{-3}{8}$ $\left(\dfrac{u\Delta t}{\Delta x}\right)\epsilon$	$\dfrac{7}{8}$ $\left(\dfrac{u\Delta t}{\Delta x}\right)\epsilon$	$P_\Delta \leqslant \dfrac{8}{3}$
7	**Expon. scheme**	**Discretized form of 1-D diffusion-convection eq.** $a_P\phi_P = a_E\phi_E + a_W\phi_W$ $a_E=\dfrac{\rho u}{\exp(P_\Delta)-1}$, $a_W=\dfrac{\rho u\exp(P_\Delta)}{\exp(P_\Delta)-1}$ $a_P=a_E+a_W+a_P^0$, $b=a_P^0\phi_P^0$, $a_P^0=\dfrac{\rho\Delta x}{\Delta t}$	**Total effects of Dif-Con** $\dfrac{a_E}{a_P^0}\epsilon(\geqslant 0)$	$\dfrac{a_W}{a_P^0}\epsilon(\geqslant 0)$	**Abs. stable**

7.5.3 Discussion on the analysis results of schemes

1) For those schemes possessing transportive property the SPP is always satisfied, and the schemes are absolutely stable, such as **FUD**，**SUD**；

2) For those schemes containing downstream node they do not possess transportive property, and are often conditionally stable. Only when the coefficients in the interpolation satisfy certain conditions they can be absolutely stable: **CD**，**TUD**，**QUICK**，**FROMM**；

3) For conditionally stable schemes, the larger the coefficients of the downstream nodes the smaller the critical Peclet number.

CD: $\phi_e = \dfrac{\phi_E + \phi_P}{2}$; For situation of positive velocity,

Coefficient of downstream node is 1/2, $P_{\Delta cr} = 2$

QUICK: $\phi_{i+1/2} = \dfrac{1}{8}(3\phi_{i+1} + 6\phi_i - \phi_{i-1})$

Coefficient of downstream node is 3/8, $P_{\Delta cr} = 8/3$

TUD: $\dfrac{\partial \phi}{\partial x})_i = \dfrac{2\phi_{i+1} + 3\phi_i - 6\phi_{i-1} + \phi_{i-2}}{6\Delta x}$

Coefficient of downstream node is 2/6, $P_{\Delta cr} = 6/2$

$= 3$

FROMM: $\phi_e = \dfrac{1}{4}(\phi_{i+1} + 4\phi_i - \phi_{i-1})$

Coefficient of downstream node is 1/4, $P_{\Delta cr} = 4$

There is some inherent relationship!

4) All the above analyses for convective stability are based on the following six conditions:

(1) 1-D problem;

(2) Linear problem （u, Γ are known constants)；

(3) Two-point boundary problem；

(4) No non-constant source term；

(5) Uniform grid system；

(6) Diffusion term is discretized by CD.

The resulted critical Peclet is the smallest； Violation(违反) of any above condition will enhance stability.

7.5.4 Summary of discussion on convective scheme

1.For conventional fluid flow and heat transfer problems, in the debugging process (调试过程）:

FUD or PLS may be used; For the final computation QUICK or SGSD is recommended, and defer correction is used for solving the ABEqs.

2. For direct numerical simulation （DNS ） of turbulent flow, schemes of fourth order or more are often used;

3. When there exists a sharp variation of properties, higher order and bounded schemes (高阶有界格式） should be used.

Recent advances in scheme construction can be found in:

金巍巍

Jin W W, Tao W Q. Numerical Heat Transfer, Part B, 2007, 52(3): 131-254
Jin W W, Tao W Q. Numerical Heat Transfer, Part B, 2007, 52(3): 255-280

Home Work 7（2024-2025）

Please finish your homework independently !!!

Please hand in on Nov. 12

Problem 7-1

A general Crank-Nicolson scheme of the 1-D transient heat conduction is as follows,

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=a\left[\theta\frac{T_{i+1}^{n+1}-2T_i^{n+1}+T_{i-1}^{n+1}}{(\Delta x)^2}+(1-\theta)\frac{T_{i+1}^{n}-2T_i^{n}+T_{i-1}^{n}}{(\Delta x)^2}\right]$$

where θ is the weighting factor, $0\le\theta\le1$.

Using von-Neumann analysis method to find the initial stability conditions for the scheme.

Problem 7-2

In the 2-D diffusion-convection equation:

$$\rho\frac{\partial\phi}{\partial t}+\rho(u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y})=\Gamma(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2})$$ u,v,ρ,Γ all are known constants.

Its one discretized scheme is as follows:

$$\rho\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n}}{\Delta t}+\rho u\frac{\phi_{i,j}^{n}-\phi_{i-1,j}^{n}}{\Delta x}+\rho v\frac{\phi_{i,j}^{n}-\phi_{i,j-1}^{n}}{\Delta y}=$$

$$\Gamma\frac{\phi_{i+1,j}^{n}-2\phi_{i,j}^{n}+\phi_{i-1,j}^{n}}{\Delta x^2}+\Gamma\frac{\phi_{i,j+1}^{n}-2\phi_{i,j}^{n}+\phi_{i,j-1}^{n}}{\Delta y^2}$$

By applying von Neumann analysis method to show that the stability condition is :

$$\Delta t\le\frac{1}{\frac{2a}{\Delta x^2}+\frac{2a}{\Delta y^2}+\frac{u}{\Delta x}+\frac{v}{\Delta y}},\ a=\frac{\Gamma}{\rho}$$

Problem 7-3

Discuss the transportive property for the 2nd-order and 3rd-order upwind schemes.

Problem 7-4

Judge the conservative property of the following two discretized equations of continuity for incompressible fluid flow:

$$(1)\quad \frac{u_{i+1,j}+u_{i+1,j-1}-u_{i,j}-u_{i,j-1}}{2\Delta x}+\frac{v_{i+1,j}-v_{i+1,j-1}}{\Delta y}=0;$$

$$(2)\quad \frac{u_{i+1,j}-u_{i,j-1}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}=0;$$

Problem 7-5

Demonstrate that following fully implicit scheme for 1-D transient heat conduction is unconditionaly stable:

$$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=a\frac{\phi_{i+1}^{n+1}-2\phi_i^{n+1}+\phi_{i-1}^{n+1}}{\Delta x^2}$$

本组网页地址：http://nht.xjtu.edu.cn 欢迎访问！

Teaching PPT will be loaded on ou website

同舟共济
渡彼岸！

People in the same boat help each other to cross to the other bank, where....