Numerical Heat Transfer

(数值传热学)

Chapter 5 Solution Methods for Algebraic Equations

(Chapter 7 in the textbook)

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5.1 Introduction to Solution Methods of Algebraic Eqs.

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5.1 Introduction to Solution Methods of ABEqs

5.1.1 Matrix feature of multi-dimensional discretized equation of HT and FF problems

For 2-D, 3-D flow and heat transfer problems, the discretized equations with 2nd order accuracy:

2-D $a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b$

3-D $a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + a_F\phi_F + a_B\phi_B + b$

For a 2D case with L1× M1unknown variables, the general algebraic equation of kth variable is:

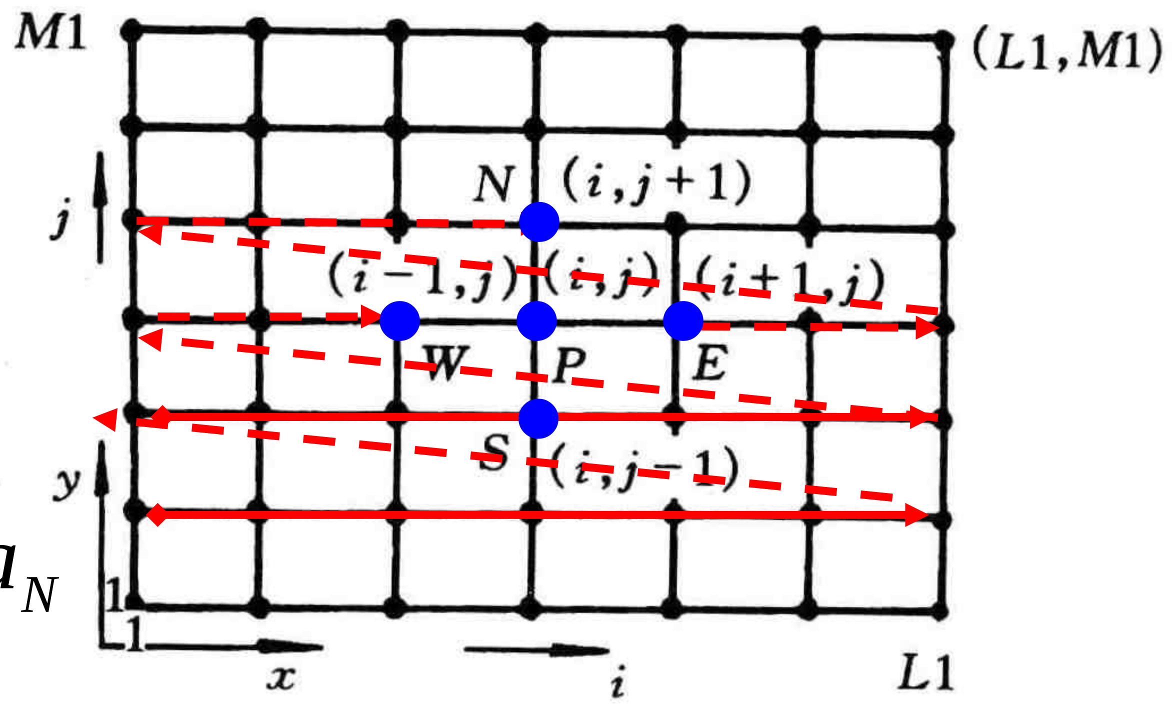
$$a_{k,1}\phi_{,1} + a_{k,2}\phi_2 + ... + a_{k,k-L1}\phi_{k-L1} + a_{k,k-L1+1}\phi_{k-L1+1} + ... + a_{k,k-1}\phi_{k-1}$$

$$+ a_{k,k}\phi_k + a_{k,k+1}\phi_{k+1} + ... + a_{k,k+L1}\phi_{k+L1} + ... + a_{k,L1\bullet M1}\phi_{L1\bullet M1} = b_k$$

For 2-D problem with 2nd order accuracy there are only five coefficients at the left hand side are not equal to zero, and the matrix is of quasi (准)five-diagonal, a large scale sparse matrix (大型稀疏矩阵).

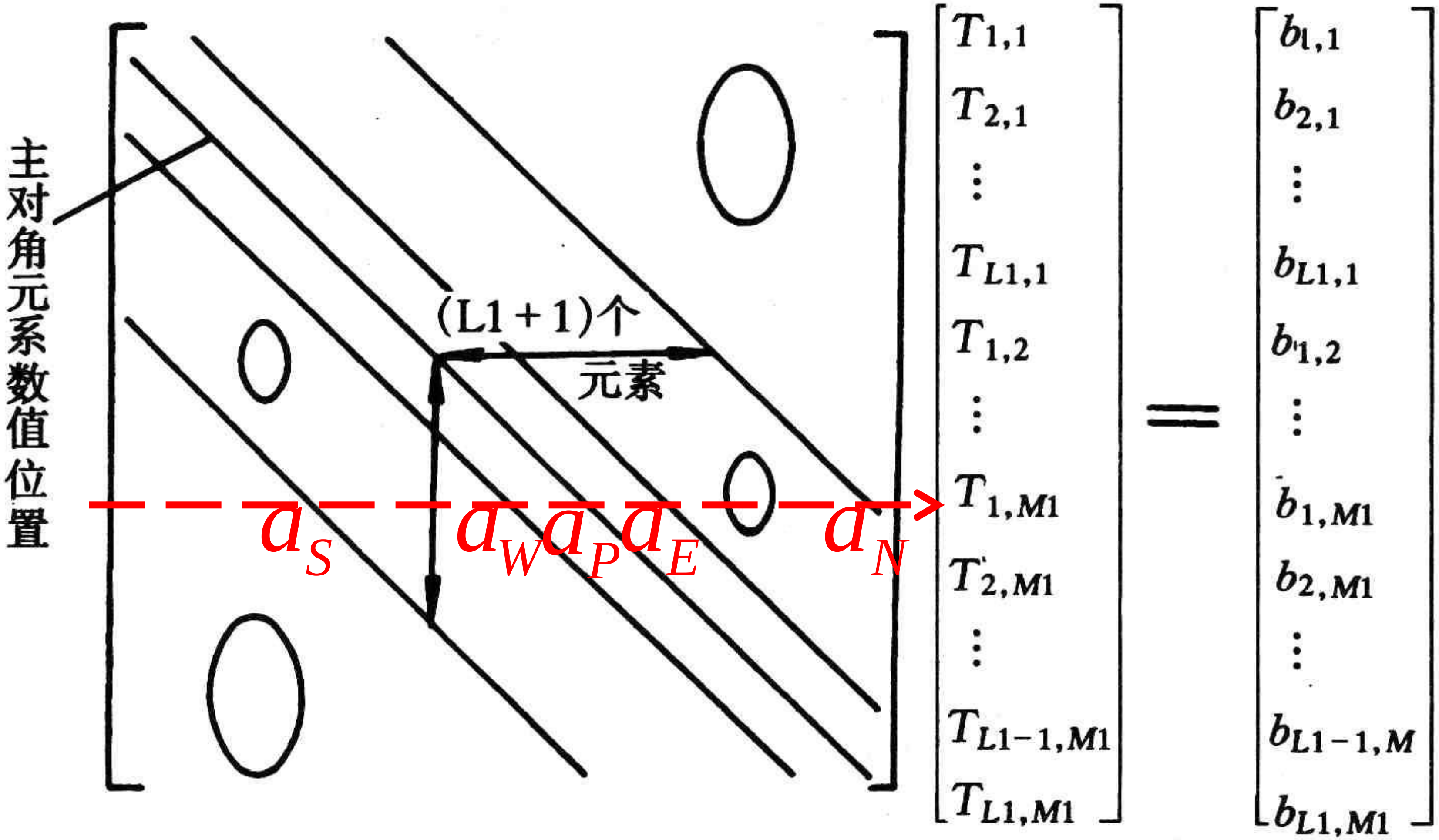
If the 1-D storage of the coefficients is conducted as shown right, then the order of coefficients in one line are:

$$a_S, ..0..a_W, a_P, a_E, ..0..a_N$$



$$\cancelto{0}{a_{k,1}}\phi_{,1} + \cancelto{0}{a_{k,2}}\phi_2 + ... + \underline{a_{k,k-L1}\phi_{k-L1}} + \cancelto{0}{a_{k,k-L1+1}}\phi_{k-L1+1} + ... + \underline{a_{k,k-1}\phi_{k-1}}$$

$$\underline{+a_{k,k}\phi_k} + \underline{a_{k,k+1}\phi_{k+1}} + ... + \underline{a_{k,k+L1}\phi_{k+L1}} + ... + \cancelto{0}{a_{k,L1\times M1}}\phi_{L1\times M1} = b_k$$



Features of the algebraic equations (ABEqs.) of discretized multi-dimensional flow and heat transfer problems:

1) For conduction of constant properties without source in uniform grid: The matrix is symmetric and positive definite (对称、正定)；

2) For other cases: matrix is neither symmetric nor positive definite.

The ABEqs. of large scale sparse matrix （大型稀疏矩阵） are usually solved by iteration methods.

5.1.2 Direct method and iterative method for solving ABEqs.

1.Direct method(直接法)

Accurate solution can be obtained via a finite times of operations if there is no round-off error (舍入误差), such as TDMA，PDMA.

2. Iterative method（迭代法)

From an initial field the solution is progressively(逐渐地) improved via the ABEqs. and terminated（终止） when a pre-specified (预先设定) criterion is satisfied.

The ABEqs. of fluid flow and heat transfer problems usually are solved by the iteration methods :

1) Due to the non-linearity of the problems， the coefficients of the ABEqs need to be updated（更新） . There is no need to get the true solution for the temporary (临时的） coefficients;

2) The operation times of direct method is proportional to $N^{2.5\sim3}$, where N is the number of unknown variables. When N is very large the operation times becomes very large, often unmanageable（难易处理）！（20241112）

5.1.3 Major Idea and Key Issues of Iteration Methods

1. Major idea

In the matrix form, the ABEqs. is :$\vec{A}\vec{\phi}=\vec{b}$. Its solution is $\vec{\phi}=(\vec{A})^{-1}\vec{b}$, $(\vec{A})^{-1}$ is the inverse matrix (逆矩阵).

The major idea of the iteration method is as follows: Constructing a series of $\vec{\phi}^{k}$ in multi-dimensional space R (where the dimension of the space equals the number of unknowns) such that when $k\to\infty$, $\vec{\phi}^{(k)}\to(\vec{A})^{-1}\vec{b}$

For the kth iteration $\vec{\phi}^{(k)}=f(\vec{A},\vec{b},\vec{\phi}^{(k-1)})$

2. Key issues of iteration methods

1) How to construct the iteration series of $\vec{\phi}^{k}$?

2) Is the series converged?

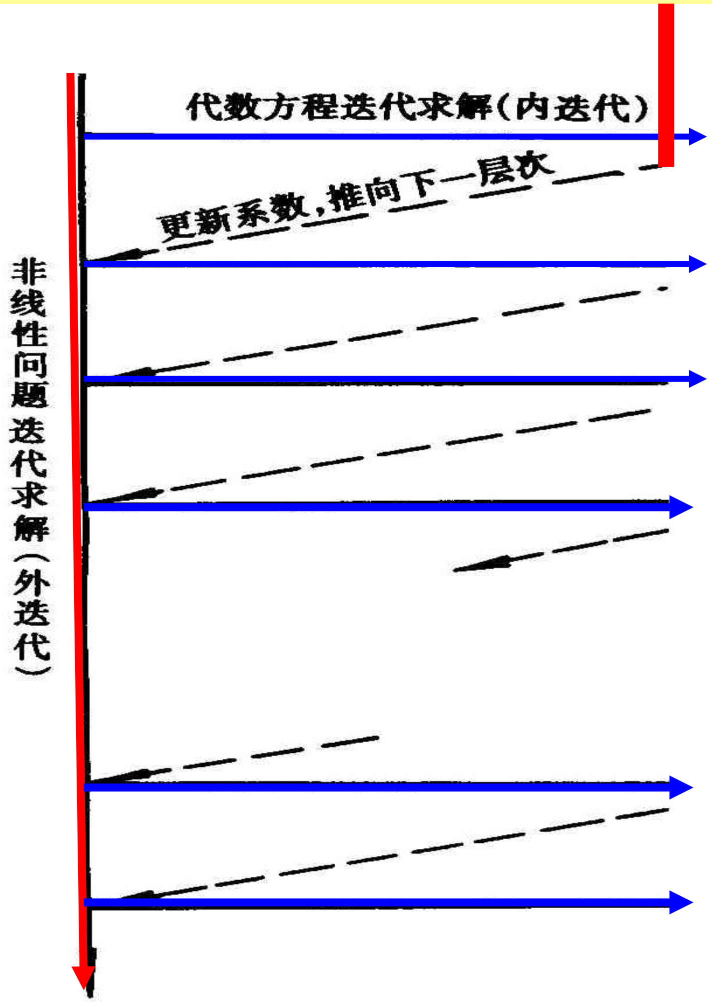
3) How to accelerate the convergence speed?

5.1.4 Criteria for terminating (inner) iteration

(1) Specifying iteration times;

(2) Specifying relative change of variable less than a small value;

$$\left|\frac{\phi^{(k+1)}-\phi^{(k)}}{\phi_{\max}^{(k+1)}}\right|_{\max}\le\varepsilon;\quad \left|\frac{\phi^{(k+1)}-\phi^{(k)}}{\phi^{(k+1)}+\varepsilon_0}\right|_{\max}\le\varepsilon$$



(3) Specifying the relative norm of residual (余量的范数) less than a certain small value. Detailed discussion will be presented in Chapter 6.

5.2 Construction of Iteration series of $\overrightarrow{\phi^{k}}$ for solving Linear Algebraic Equations

5.2.1 Point (explicit) iteration

5.2.2 Block (implicit) iteration

5.2.3 Alternative direction iteration－ADI

5.2 Construction of Iteration Methods of Linear Algebraic Equations.

5.2.1 Point (explicit) iteration

The variable updating (更新) is conducted from node to node; After every node has been visited a cycle (轮) of iteration is finished; The updated value at each node is **explicitly** related to the others.

1. Jakob iteration

In the updating of every node value the previous cycle values of neighboring nodes are always used; The convergence speed is independent of iteration direction.

2. Gauss－Seidel iteration

Always the present values **(当前值)** are used for updating.

3. SOR/SUR iteration (Successive over/under relaxation)

$$\phi^{(k+1)} = \phi^{(k)} + \alpha(\phi^{(k+1)} - \phi^{(k)}) \begin{cases} \alpha < 1 \text{ Under-relaxation} \\ (0 \le \alpha \le 2) \\ \alpha > 1 \text{ Over-relaxation} \end{cases}$$

Remarks: This relaxation is for solving the linear ABEqs., Not for the non-linearity of the problem studied.

5.2.2 Block (implicit) iteration （块隐式）

1. Basic idea

Dividing the solution domain into several regions, within each region direct solution method is used, while from block to block iteration is used， also called implicit iteration. Implicit means within each region all unknowns are solved simultaneously!

2. **Line iteration**（线迭代)-the most fundamental block iteration

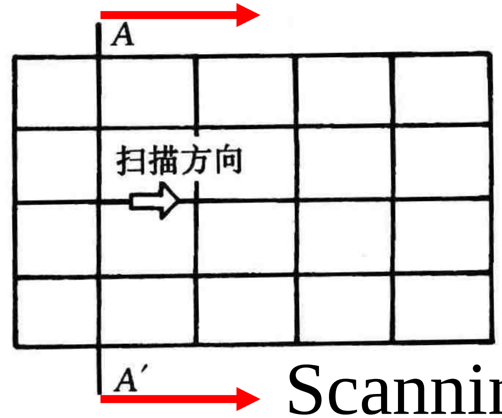
The smallest block is a line: At the same line TDMA is used for direct solution, from line to line iterative method is used. For example:

Solving in N-S direction and scanning (扫描) in E-W direction：

Jakob: $$a_P\phi_P^{(k+1)} = a_N\phi_N^{(k+1)} + a_S\phi_S^{(k+1)} + \underline{[a_E\phi_E^{(k)} + a_W\phi_W^{(k)} + b]}$$

G-S: $$a_P\phi_P^{(k+1)} = a_N\phi_N^{(k+1)} + a_S\phi_S^{(k+1)} + [\underline{a_E\phi_E^{(k)} + a_W\phi_W^{(k+1)} + b}]$$

New b term, b'

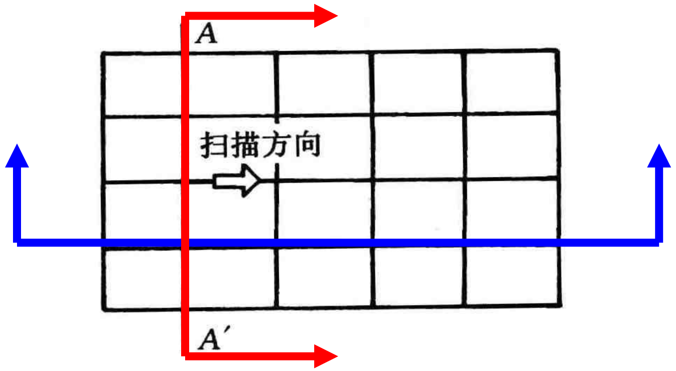


Scanning （扫描） in E-W direction

5.2.3 Alternative direction iteration(交替方向迭代)-ADI

1. Basic idea

First conducting direct solution for each row(行)（or column 列）， then doing direct solution for each column（or row)；The combination of the two updating operations of the entire domain consists of one iteration cycle :

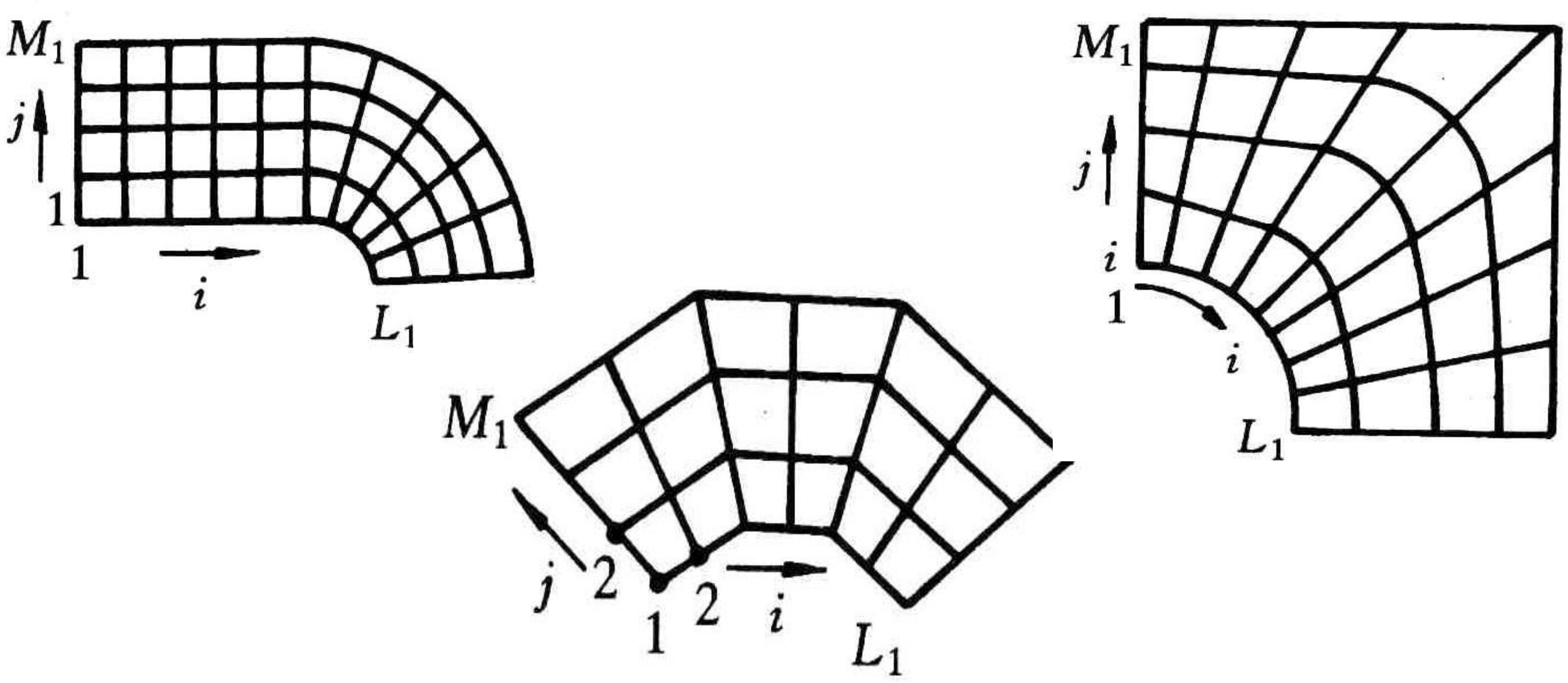


Alternative direction iteration (ADI) vs. alternative direction implicit (ADI)

It can be shown that: one-time step forward of unsteady (transient) problem is equivalent to one cycle iteration for steady problem.

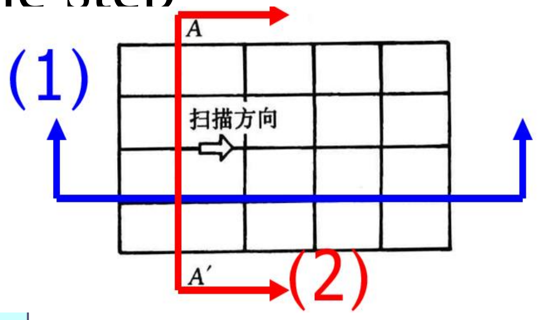
2. ADI-line iteration is widely adopted in the numerical solution of flow and heat transfer problem.

The ABEqs. generated on structured grid system can be solved by ADI---each line has the same number of unknowns



5.2.4 ADI-iteration（交替方向迭代）is identical to ADI-implicit (交替方向隐式)

ADI-iteration is identical to the **ADI-Implicit** of solving multidimensional unsteady problem for one time step



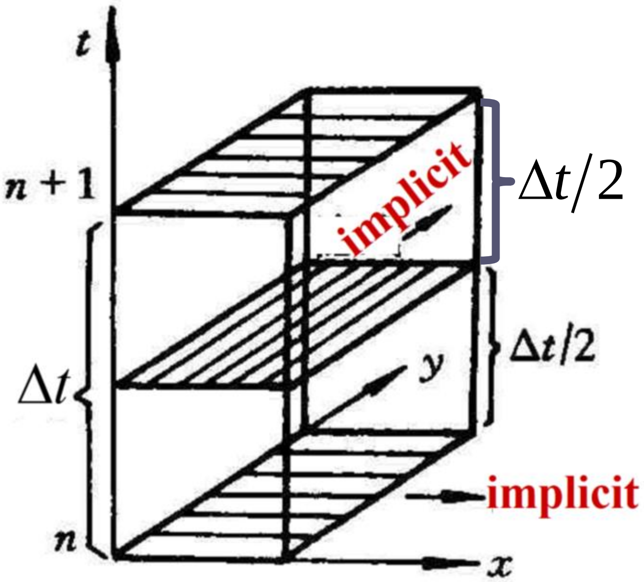
ADI-Jakob iteration can be expressed as：：

$$a_P\phi_P^{(k+1/2)} = a_E\phi_E^{(k+1/2)} + a_W\phi_W^{(k+1/2)} + \underbrace{[a_N\phi_N^{(k)} + a_S\phi_S^{(k)} + b]}_{b^{(k+1/2)}}$$

$$a_P\phi_P^{(k+1)} = a_N\phi_N^{(k+1)} + a_S\phi_S^{(k+1)} + \underbrace{[a_E\phi_E^{(k+1/2)} + a_W\phi_W^{(k+1/2)} + b]}_{b^{(k+1)}}$$

This expression is very similar to Peaceman-Rachford ADImplicit method for 2 D transient problem：

2-D Peaceman-Rachford method



2-D AD Implicit

Dividing Δt into two sub-periods.

In the 1st sub-period $\Delta t/2$

x-direction is implicit, y-direction is explicit;

In the 2nd $\Delta t/2$ y-direction is implicit, and x is explicit.

Let $\phi^{(k+1/2)}$ represent temporary values at middle time

$\delta_x^2\phi_{i,j}^k$ represent CD for 2nd-order x-direction derivative at time level k ; then we have:

1st sub-period：

$$\frac{\phi_{i,j}^{k+1/2}-\phi_{i,j}^{k}}{\Delta t/2}=a(\delta_x^2\phi_{i,j}^{k+1/2}+\delta_y^2\phi_{i,j}^{k}) \quad (1)$$

2nd sub-period：

$$\frac{\phi_{i,j}^{k+1}-\phi_{i,j}^{k+1/2}}{\Delta t/2}=a(\delta_x^2\phi_{i,j}^{k+1/2}+\delta_y^2\phi_{i,j}^{k+1}) \quad (2)$$

Eq.(1): $\delta_x^2\phi_{i,j}^{k+1/2}=\dfrac{\phi_{i+1,j}^{k+1/2}-2\phi_{i,j}^{k+1/2}+\phi_{i-1,j}^{k+1/2}}{\Delta x^2}$; $\quad \delta_y^2\phi_{i,j}^{k}=\dfrac{\phi_{i,j+1}^{k}-2\phi_{i,j}^{k}+\phi_{i,j-1}^{k}}{\Delta y^2}$

ADImplicit

$$\underbrace{(1+\frac{a\Delta t}{\Delta x^2})}_{a_P}\phi_{i,j}^{k+1/2}=\underbrace{(\frac{a\Delta t}{2\Delta x^2})}_{a_E,\,a_W}(\phi_{i+1,j}^{k+1/2}+\phi_{i-1,j}^{k+1/2})+\underbrace{(\frac{a\Delta t}{2\Delta x^2})}_{a_S,\,a_N}(\phi_{i,j+1}^{k}+\phi_{i,j-1}^{k})+\underbrace{(1-\frac{a\Delta t}{\Delta x^2})\phi_{i,j}^{k}}_{b}$$

ADIteration

$$a_P\phi_P^{(k+1/2)}=a_E\phi_E^{(k+1/2)}+a_W\phi_W^{(k+1/2)}+[a_N\phi_N^{(k)}+a_S\phi_S^{(k)}+b]$$

The same can be shown for the 2nd sub-time period. Thus one-time step forward of transient problem is equivalent to one cycle iteration for steady problem.

5.3 Convergence Conditions and Acceleration Methods for Solving Linear ABEqs.

5.3.1 Sufficient condition for iteration convergence of Jakob and G-S iteration

5.3.2 Analysis of factors influencing iteration convergence speed

5.2.3 Methods for accelerating transferring boundary condition influence into solution domain

5.3 Convergence Conditions and Acceleration Methods for Solving Linear ABEqs.

5.3.1 Sufficient condition for iteration convergence of Jakob and G-S iteration

1. Sufficient condition－Scarborough criterion

Coefficient matrix is non-red**u**cible (不可约， discussed later), and is diagonally pred**o**minant (对角占优)：

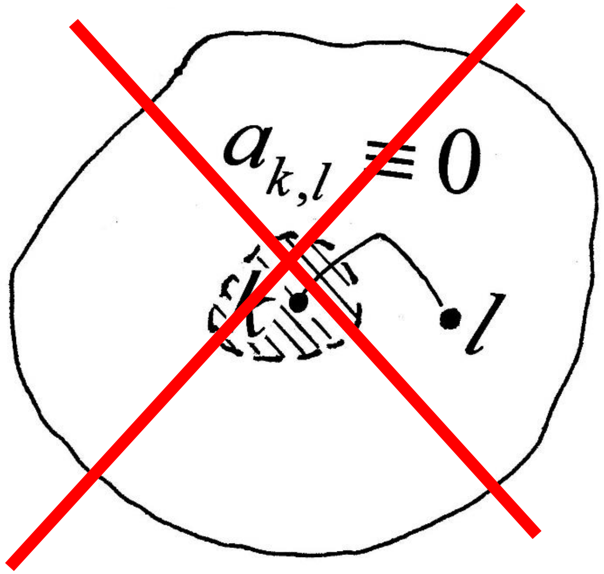
$$\frac{\sum |a_{nb}|}{|a_P|} \le 1 \begin{cases} \le 1 & \text{for all equations} \\ < 1 & \text{at least for one equation} \end{cases}$$

2. Analysis of coefficients of discretized diffusion-convection equation by the recommended method

1) **Matrix is non-reducible**—If matrix is reducible then the set（集合） of coefficients subscript (矩阵下标) ,W , can be divided into two non-empty (非空) sub-sets, R and S， W＝R＋S， and for any element from R and S, say *k* and *l* respectively， we must always have： $a_{k,l} \equiv 0$ ； If such condition does not exist, then the matrix is called non-re**du**cible (不可约)

Analysis： Coefficient of discretized equation represents the influence of neighboring nodes. For nodes in elliptic region (coordinate) any one must has its effects on its neighbors; If matrix is reducible it implies that the computational domain can be divided into two regions which do not affect each other---physically totally impossible .

Non-reducible matrix is determined by the physical fact that neighboring parts in flow and heat transfer are affected each other.



2) Diagonally predominant—Coefficients constructed in the present course must satisfy this condition:

(1) Transient and fully implicit scheme with source term

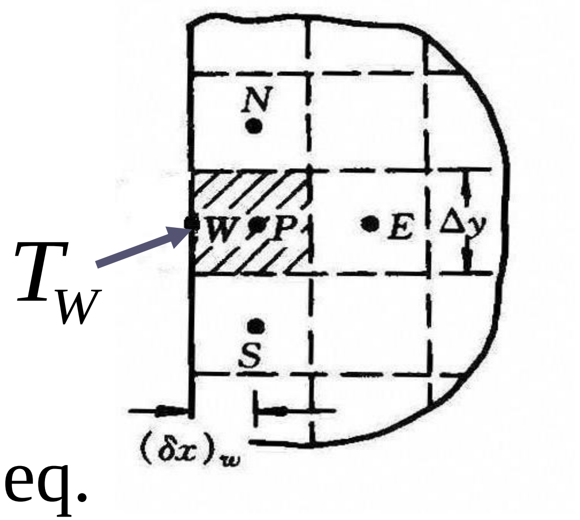
$$a_P = \sum a_{nb} + a_P^0 - S_P \Delta V, a_P^0 > 0, -S_P \geq 0 \text{ , } a_P > \sum a_{nb}$$

(2) Steady problem with non-constant source term

$$-S_P > 0 \text{ , } a_P > \sum a_{nb}$$

(3) Steady problem without source term

For inner grids: $a_P = \sum a_{nb}$



At least one node in the boundary can be found to satisfy : $a_P > \sum a_{nb}$

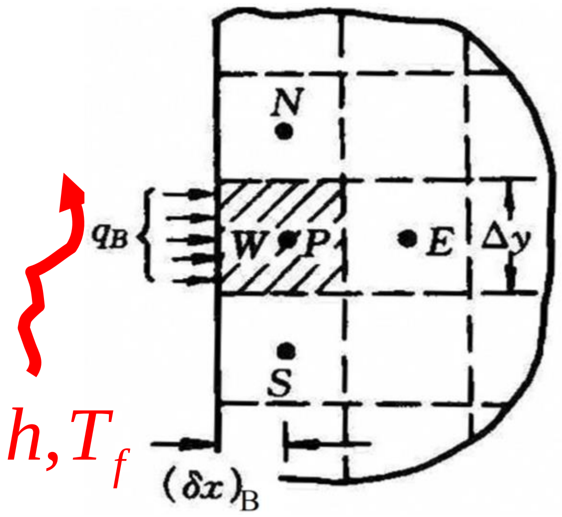
1)Assuming that T_w is known, then when the eq.

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b$$

is solved for control volume P, it becomes:

$$a_P T_P = a_E T_E + 0 + a_N T_N + a_S T_S + (b + a_W T_W)$$

Hence here: $a_P = \sum a_{nb} > a_E + 0 + a_N + a_S$



2) For 3rd kind boundary condition,

Additional source term helps

$$-S_{P,\ ad} > 0\ ,\ a_P = \sum a_{nb} - \left(-\left|S_{P,ad}\right|\right) > \sum a_{nb}$$

It is impossible that all boundary nodes are of 2nd type, at least one node is of 1st or 3rd type. Otherwise there is no definite solution!

Thus numerical methods recommended by the present course must satisfy this sufficient condition

5.3.2 Analysis of factors influencing iteration convergence speed

1. Transferring effects of B.C. into domain---View Point 1

The steady state heat conduction with constant properties is governed by Laplace equation, $\nabla^2\phi = 0$, for which a uniform field satisfies. However, it is not the solution because B.C. is not satisfied.

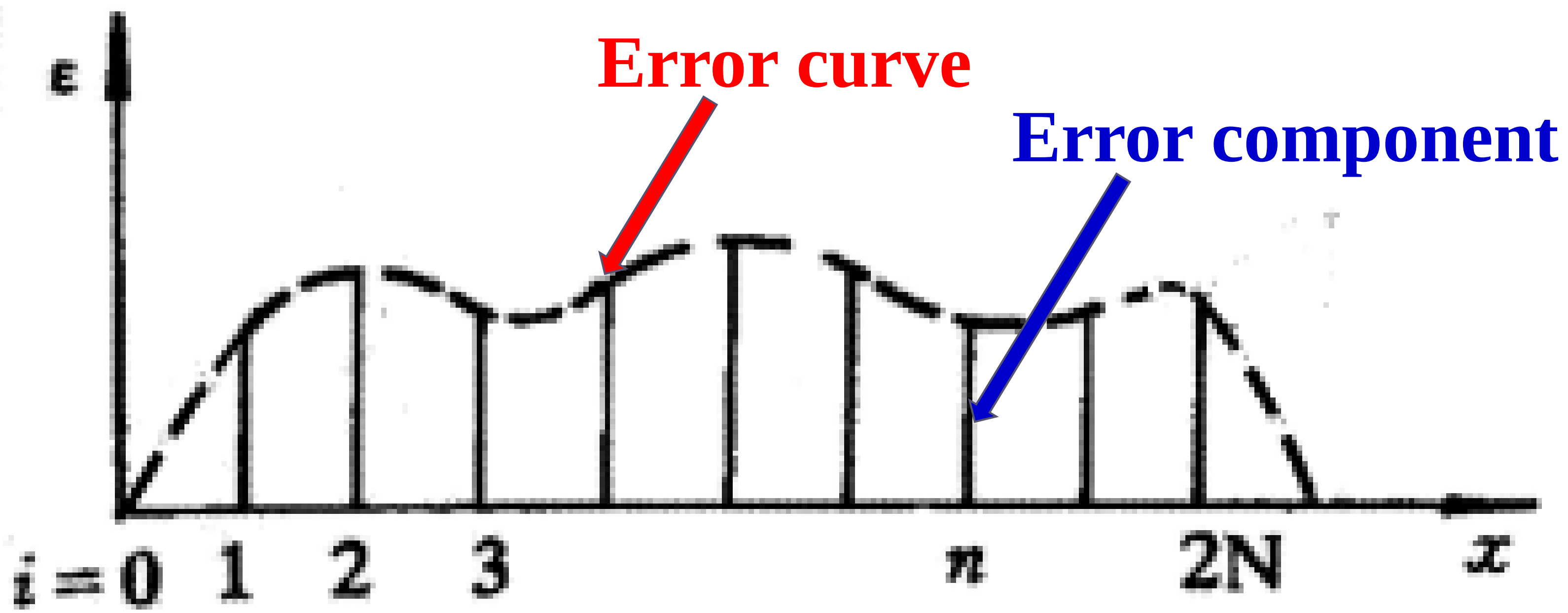
Thus the transferring speed of the boundary effects into the solution domain influences the convergence speed!

2. Satisfaction of conservation condition---View Point 2

For a problem with 1st kind boundary condition, it is possible to incorporate all the known boundary values into the initial field, but such an initial field does not satisfy conservation condition. Thus techniques which is in favor of (有利于) satisfying conservation condition can accelerate convergence speed;

3. Attenuation (衰减) of error vector---View Point 3

The initial assumption has some error. The error vector is attenuated during iteration. Error vector is composed of compo-nents of different frequency（不同频率的分量）. Techniques which can uniformly attenuate different components can accelerate convergence speed.



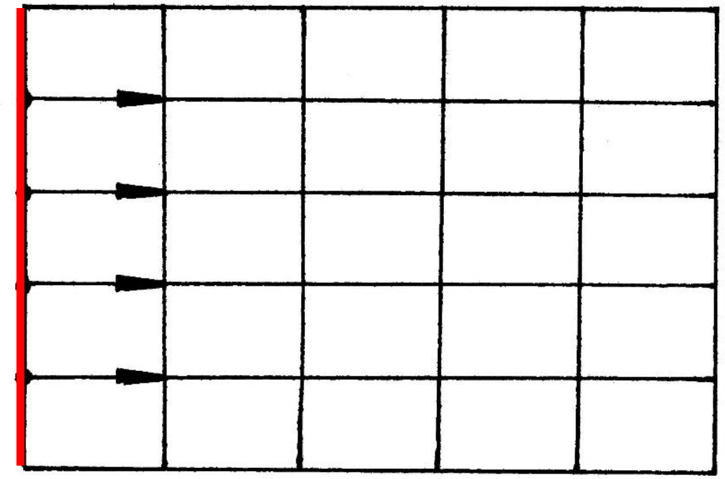
Taking the numerical error of each node as a component of a vector, then all the error components consist a vector, called error vector.

Mathematically, the error curve can be decomposed by a number of sine/cosine components with different frequencies. If the different frequency components can be attenuate uniformly the convergence speed will be accelerated.

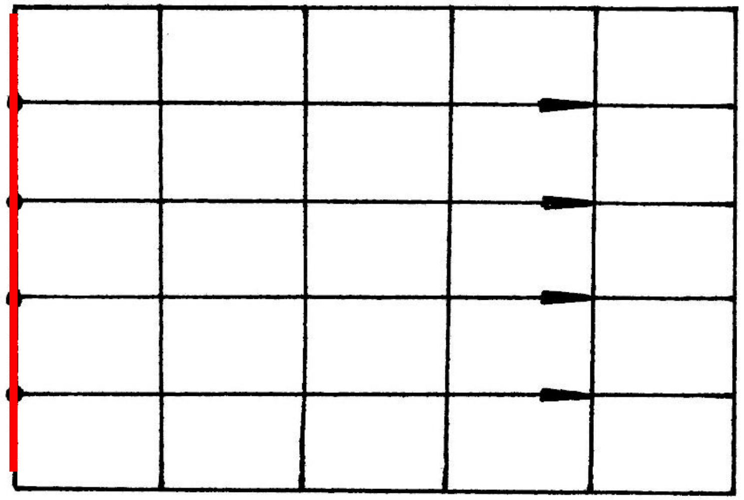
4. Increasing percentage of direct solution---View Point 4

Direct solution is the most strong technique that both conservation and boundary condition can be satisfied. Thus appropriately increasing direct solution proportion is in favor of （有利于） accelerating convergence speed.

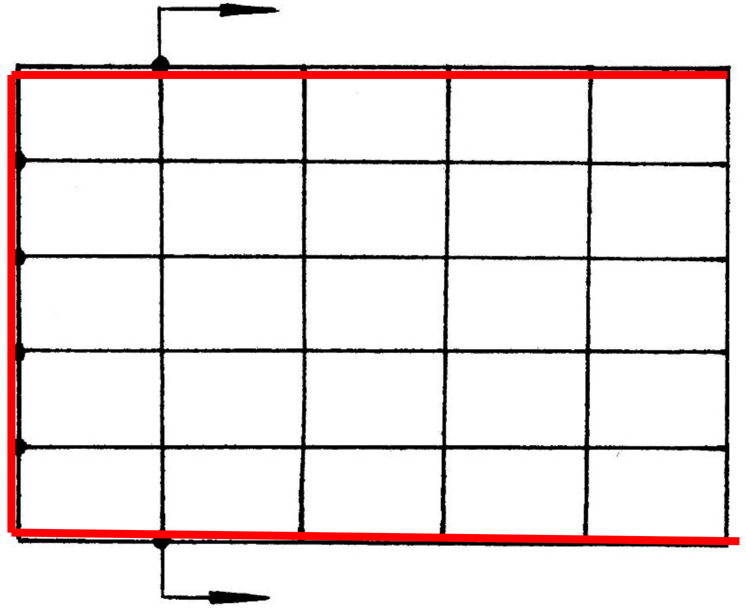
5.3.3 Techniques for accelerating transferring boundary condition's effects



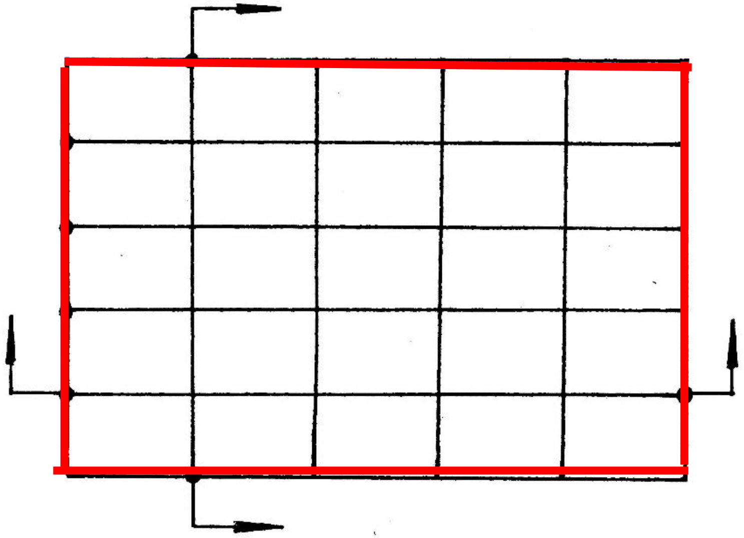
Jakob iteration： In each cycle the effect of boundary points can transfer into inner region by one space step. Very low convergence speed.



G-S iteration: In each cycle, the effects of the iteration starting boundary are transferred into the entire domain; convergence speed is accelerated.



Line iteration: In each cycle the effects of iteration starting boundary and the related two end boundaries are all transferred into the entire domain; convergence speed is further accelerated.



ADI line iteration： In every cycle iteration effects of all the boundaries are transferred into the entire domain. It has the fastest convergence speed.

ADI line iter.>Line iter.>G-S iter.>Jakob iter.

Jakob iteration has the slowest convergence speed. That is the change between two successive iterations is the smallest; This feature is in favor of iteration convergence for highly non-linear problems when iteration cycle number is specified. In the SIMPLEST algorithm (discussed later), Jakob iteration is used for the convective part of the ABEqs.

5.4 Block Correction Method –Promoting Satisfaction of Conservation

5.4.1 Necessity for block correction technique

5.4.2 Basic idea of block correction

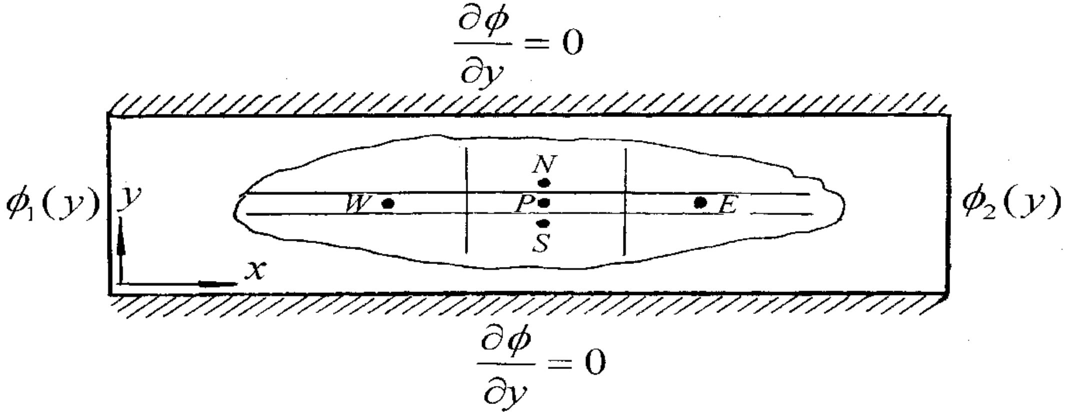
5.4.3 Single block correction and the boundary condition

5.4.4 Remarks of application of B.C. Technique

5.4 Block Correction Method –Promoting Satisfaction of Conservation

5.4.1 Necessity for block correction technique

For 2-D steady heat conduction shown below when ADI is used to solve the ABEqs. convergence speed is very low： E-W boundaries have the strongest effect because of 1st kind boundary, but the influencing coefficient is small ； N-S boundary is adiabatic, no definite information can offer, but has larger coefficient－Thus to accelerate convergence of solving the ABEqs., a special method is needed



a_E,a_W are much less than a_N,a_S

5.4.2 Basic idea of block correction

Physically, iteration is a process for satisfying conservation condition； In one cycle of iteration, a correction, ϕ', is added to previous solution, ϕ^* (which does not satisfy conservation condition), such that $(\phi^*+\phi')$ can satisfy conservation condition better. The process of solving ABEqs. of ϕ is the process of getting the solution of ϕ!

For 2-D problem, corrections are also of 2-D; In order that only 1-D corrections are solved, corrections are somewhat averaged for one block, denoted by $\bar{\phi}_i'$ or $\bar{\phi}_j'$, and it is required that $(\phi_{i,j}^* + \bar{\phi}_i')$ or $(\phi_{i,j}^* + \bar{\phi}_j')$ satisfies the conservation condition for one row or column, respectively.

5.4.3 Single block correction and the boundary condition

1.Equation for correction：

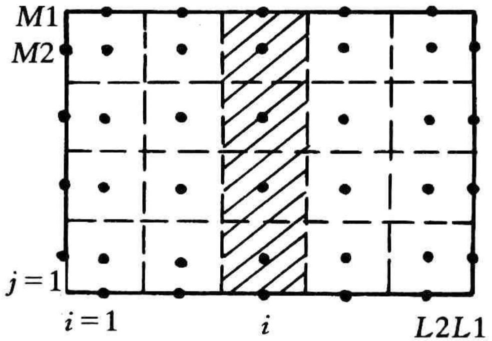
It is required that： $(\phi^*_{i,j} + \bar{\phi}'_i)$ satisfy following eq.

$$\sum_j AP(\phi^*_{i,j} + \bar{\phi}_i) = \sum_j AIP(\phi^*_{i+1,j} + \bar{\phi}_{i+1}) + \sum_j AIM(\phi^*_{i-1,j} + \bar{\phi}_{i-1})$$

$$+ \sum_j (AJM)(\phi^*_{i,j-1} + \bar{\phi}_i)$$

$$+ \sum_j (AJP)(\phi^*_{i,j+1} + \bar{\phi}_i) + \sum_j CON$$

$$(i = IST, \dots L2)$$



AP - - - - a_P

AIP - - - - a_E

AIM - - - a_W

AJP - - - - a_N

AJM - - - a_S

IST-solution starting subscript in X-direction； L2-last but one.

Here *AP*,*AIP*,*AIM*, etc. are the symbles adopted in teaching code.

Rewrite into ABEqs. of $\bar{\phi}'_{i-1}, \bar{\phi}'_i, \bar{\phi}'_{i+1}$:

$$(BL)\underline{\bar{\phi}'_i} = (BLP)\underline{\bar{\phi}'_{i+1}} + (BLM)\underline{\bar{\phi}'_{i-1}} + BLC, i = IST, ...L2$$

where
$$BL = \sum_{j=JST}^{M2}(AP) - \sum_{j \neq M2}(AJP) - \sum_{j \neq JST}(AJM)$$

$$BLP = \sum_{j=JST}^{M2}(AIP); \quad BLM = \sum_{j=JST}^{M2}(AIM)$$

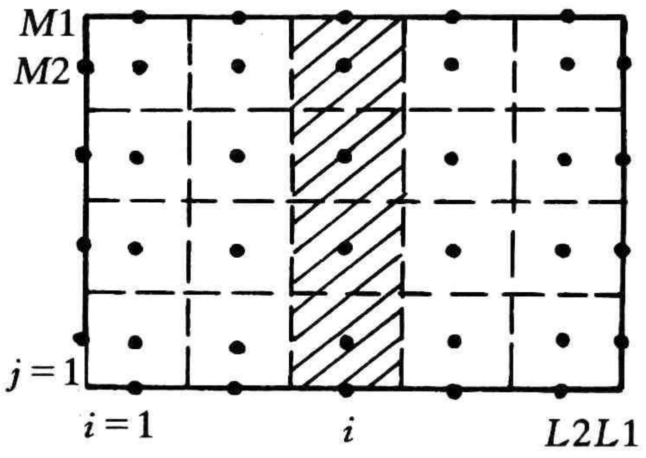
$$BLC = \sum_{j=JST}^{M2} CON + \sum_{j=JST}^{M2}(AJP)\phi^*_{i,j+1} + \sum_{j=JST}^{M2}(AJM)\phi^*_{i,j-1}$$

$$+\sum_{j=JST}^{M2}(AIP)\phi^*_{i+1,j} + \sum_{j=JST}^{M2}(AIM)\phi^*_{i-1,j} - \sum_{j=JST}^{M2}(AP)\phi^*_{i,j}$$

BL, BLP, BLM, BLC are coefficients and *b*-term for $\bar{\phi}'_i$

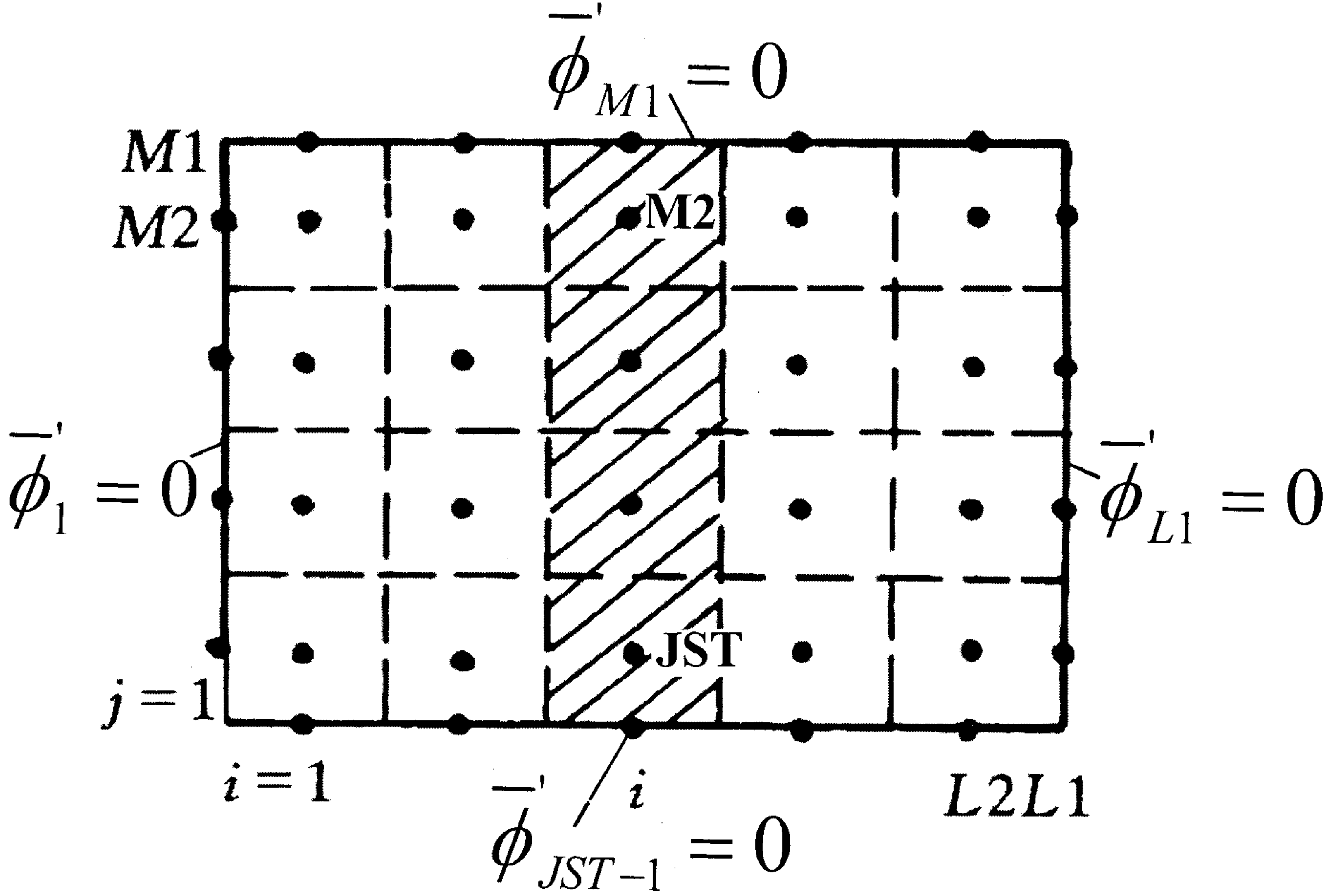
$$BL = \sum_{j=JST}^{M2} (AP) - \sum_{j \neq M2} (AJP) - \sum_{j \neq JST} (AJM)$$

ASTM is adopted to deal with 2nd and 3rd kind boundary condition, this is equivalent to that all boundaries are of 1st kind, and the correction for boundary nodes is zero; Thus when summation is conducted in y-direction the 1st term and the last term corrections are zero. Hence, for AJM term JST is not needed, and for AJP term M2 is not needed.



Question: Why in the expression of *BLC* , for *AJP and AJM* (*J=JST* and *M2*) are included?

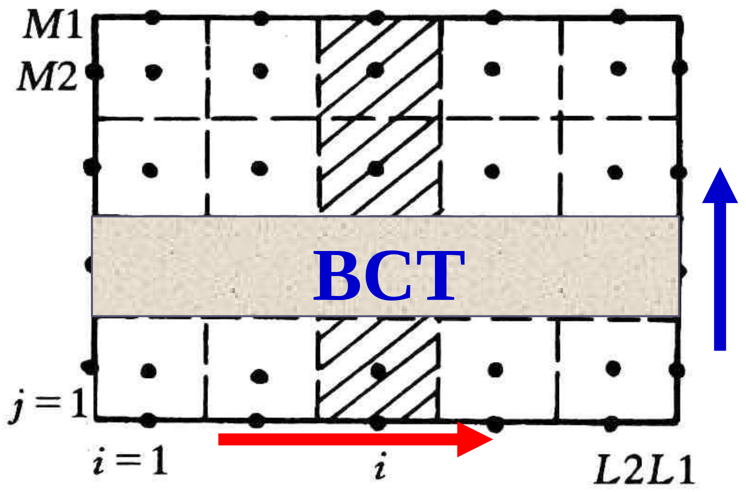
2. Boundary condition for the correction ---zero



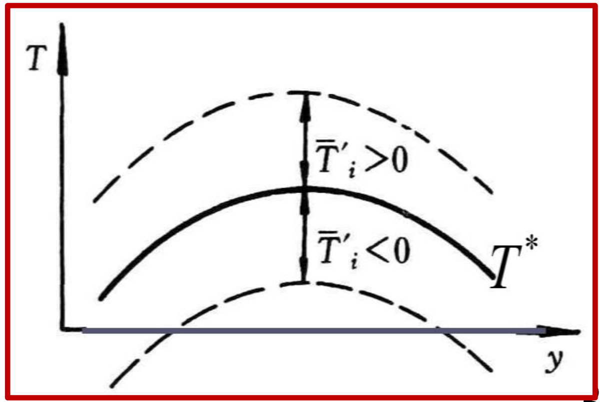
5.4.4 Remarks of application of block correction technique

1.Block correction technique (BCT) is not an independent solution method. It should be combined with some other method, such as ADI;

2. For further accelerating convergence ADI block correction may be used;



3. For variables of physically larger than zero values (such as turbulent kinetic energy, component of a mixed gas) the BCT should not be used. Because BCT adds or subtracts a constant correction within the entire block, which may lead to negative values.



5.5 Multigrid Techniques –Promoting Simultaneous Attenuation of Different Wave-length Components

5.5.1 Error vector is attenuated(衰减) in the iteration process of solving ABEqs.

5.5.2 Basic idea and key issue of multigrid technique

5.5.3 Transferring solutions between different grid systems

5.5.4 Cycling patterns between different grid systems

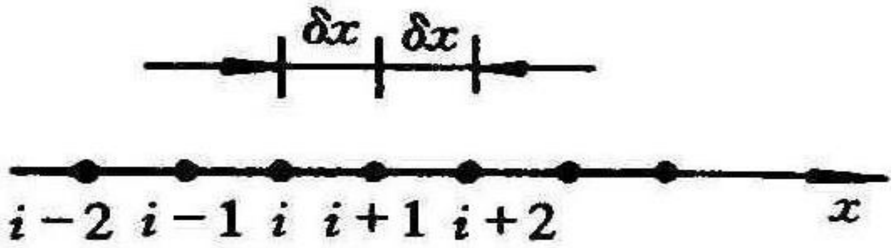
5.5 Multigrid Techniques –Promoting Simultaneous Attenuation of Different Wave-length Components

5.5.1 Error vector is attenuated in the iteration process of solving ABEqs

1. How error vector is attenuated during iteration?

Taking 1-D steady heat conduction problem as an example to analyze how error vector is attenuated:

$$\frac{d^2T}{dx^2} + f(x) = 0$$



Discretizing it at a uniform grid system, yielding:

$$T_{i-1} - 2T_i + T_{i+1} = -(\delta x)^2 f_i$$

Adopting G-S iteration method from left to right, for point i:

$$T_{i-1}^{(k)} - 2T_i^{(k)} + T_{i+1}^{(k-1)} = -(\delta x)^2 f_i$$

In the kth cycle iteration error vector is denoted by $\vec{\varepsilon}^{(k)}$ and its component is denoted by $\varepsilon_i^{(k)}$, then we have:

$$T_i = T_i^{(k)} + \varepsilon_i^{(k)}$$

Substituting this expression to the above equation we20241014 can get following variation of error with iteration

$$T_{i-1}^{(k)} - 2T_i^{(k)} + T_{i+1}^{(k-1)} = -(\delta x)^2 f_i$$

$$T_{i-1}^{(k)} = T_{i-1} - \varepsilon_{i-1}^{(k)} \quad T_i^{(k)} = T_i - \varepsilon_i^{(k)} \quad T_{i+1}^{(k)} = T_{i+1} - \varepsilon_{i+1}^{(k)}$$

$$\cancel{T_{i-1}} - \varepsilon_{i-1}^{(k)} - 2(\cancel{T_i} - \varepsilon_i^{(k)}) + \cancel{T_{i+1}} - \varepsilon_{i+1}^{(k)} = \cancel{-(\delta x)^2 f_i}$$

Since $T_{i-1} - 2T_i + T_{i+1} = -(\delta x)^2 f_i$ **T is the exact solution!**

Then we have:

$$\varepsilon_{i-1}^{(k)} - 2\varepsilon_i^{(k)} + \varepsilon_{i+1}^{(k-1)} = 0$$

This equation presents the transfer of error with iteration（20241014）.

2. **Analysis of attenuation of harmonic components**

It will be shown later that $\varepsilon_i^{(k)}$ can be expressed as: $\psi(k)e^{Ii\theta}$ where $\psi(k)$ is the amplitude (振幅) and θ is the phase angle (相角), by substituting this expression to the above equation and after rearrangement，yielding

$$\frac{\psi(k)}{\psi(k-1)} = \frac{e^{I\theta}}{2 - e^{-I\theta}} = \mu$$

Amplifying factor（增长因子）

$I = \sqrt{-1}$

Analyzing amplifying factor for different phase angles：

Euler equation: $e^{I\theta} = \cos\theta + I\sin\theta$

For $\theta = \pi$,

$$|\mu| = \frac{|\cos\pi + I\underbrace{\sin\pi}_{0}|}{|2-\cos\pi + I\underbrace{\sin\pi}_{0}|} = \frac{1}{2+1} = \frac{1}{3},$$

Iteration of 5 times $0.333^5 = 4.09\times10^{-3}$

For $\theta = \pi/2$,

$$|\mu| = \frac{\left|\underbrace{\cos\frac{\pi}{2}}_{0} + I\sin\frac{\pi}{2}\right|}{\left|2-\underbrace{\cos\frac{\pi}{2}}_{0} + I\sin\frac{\pi}{2}\right|} = \frac{1}{\sqrt{2^2+1}} = \frac{1}{\sqrt{5}},$$

Iteration of 5 times $0.447^5 = 0.0178$

For $\theta = \pi/10$,

$$|\mu| = \frac{\left|\cos\frac{\pi}{10} + I\sin\frac{\pi}{10}\right|}{\left|2-\cos\frac{\pi}{10} + I\sin\frac{\pi}{10}\right|} = \frac{|0.9510+0.3090I|}{|2-(0.9510+0.3090I)|} = \frac{1}{1.094},$$

Iteration of 5 times $0.914^5 = 0.658$

We will show later that : $\theta = k_x \Delta x = \dfrac{2\pi}{\lambda}\Delta x$

where λ is the wave length. At a fixed space step, short wave has a large phase angle, and is attenuated (衰减)very fast; while long wave component has small phase angle and attenuated very slowly.

From above calculation it can be seen that phase angle can be an **i**ndicator(标志) for short/long wave components.

Generally for components with phase angle within following range it is regarded as short wave ones:

$$\pi \le \theta \le \pi / 2$$

This phase angle is dependent on space step length $\theta = k_x \Delta x = 2\pi \Delta x / \lambda$. If after several iterations the length step Δx is amplified then originally long wave component may behave as a short wave and can be attenuated very fast at that grid system.

In such a way by **a**mplifying space step (放大空间步长） several times during iteration all the error components may be quite uniformly attenuated and the entire ABEqs. may be converged much faster than iteration just at a single grid system.

This is the major concept of multigrid technique (多重网格方法） for solving ABEqs.

5.5.2 Major idea and key problem of multigrid technique

1. Major idea－Solving ABEqs. is conducted at several grid systems with different space step length such that error components with different frequencies can be attenuated simultaneously.

2. Key problems－

(1) How to transfer solutions at different grid systems?

(2) How to cycle (轮转） the solutions between several grid systems?

5.5.3 Transferring solutions between two gird systems

Basic concept： The solution transferred between different grid system **is the solution of the finest grid.**

Taking two grid systems, one coarse (k-1) and one fine (k), as an example to show the transferring of solutions.

1.From fine grid to coarse grid

$$\vec{A}^{(k-1)}\phi^{(k-1)}=\vec{b}^{(k-1)}+I_k^{k-1}(\vec{b}^{(k)}-\vec{A}^{(k)}\phi^{(k)})$$

Matrix at (k-1)th grid determined from solution of kth grid.

Solution of the finest grid exhibited at (k-1)th grid(展现在k-1重网格上的最密网格的解)

Source term at (k-1)th grid determined from solution of kth grid

Operator for transferring from kth grid to (k-1)th grid

Residual （余量）of fine grid

2. Transferring from coarse grid to fine grid

$$\phi_{rev}^{(k)} = \phi_{old}^{k} + I_{k-1}^{k}(\phi^{(k-1)} - I_{k}^{k-1}\phi_{old}^{(k)})$$

Correction

Revised solution at fine grid

Original solution at fine grid

Operator for transferring correction part of solution at (k-1) th grid to kth grid

New solution of fine grid obtained at (k-1)th grid

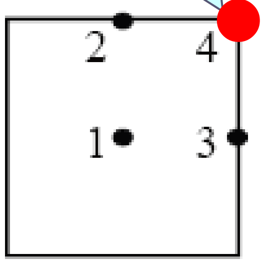
Solution of kth grid expressed at (k-1)th grid

3. Restriction and prolongation operators

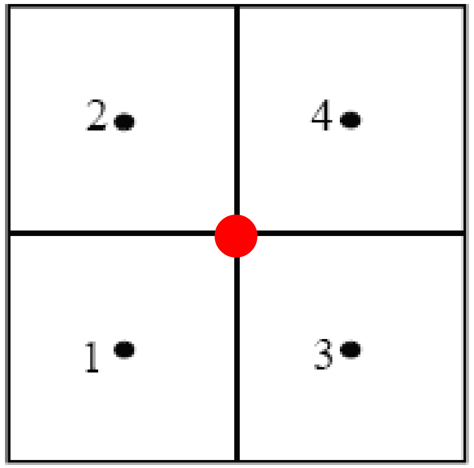
1) Restriction operator(限定算子)
(From fine to coarse)

$$I_k^{k-1}\begin{cases}\text{Direct injection(直接注入)}\\ \text{Nearby average(就近平均), Linear interpolation}\end{cases}$$

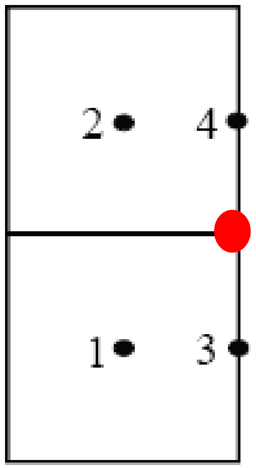
For node 4 direct injection from fine to coarse



Nearby average



Averaged from 1,2,3,4



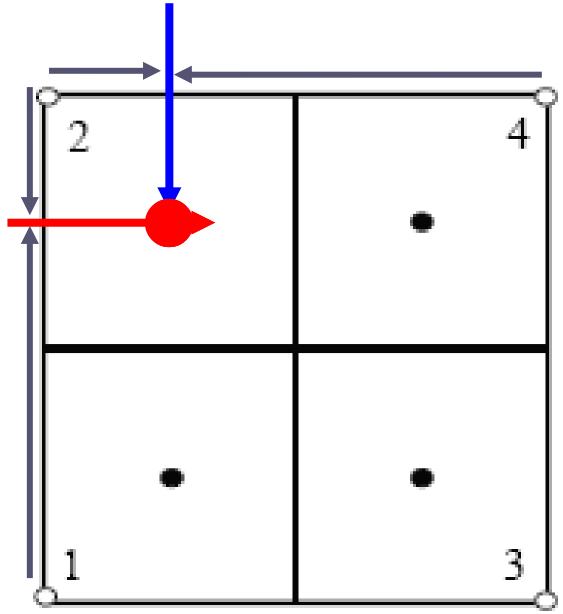
Averaged from 3,4

• —fine → coarse

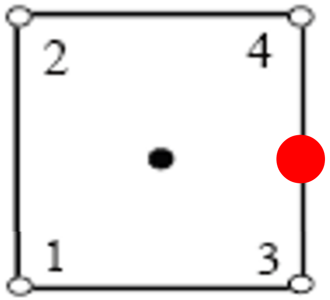
2) Prologation operator (延拓算子)

(From coarse to fine)

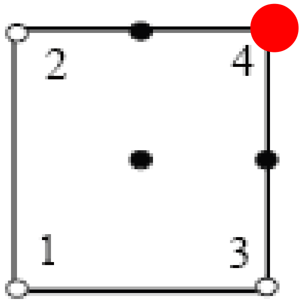
I_{k-1}^{k}
- Direct injection
- Linear interpolation
- Quadratic interpolation (二次插值)



Quadratic Interpolation from 1,2,3,4



Linear interpolation between nodes 3，4



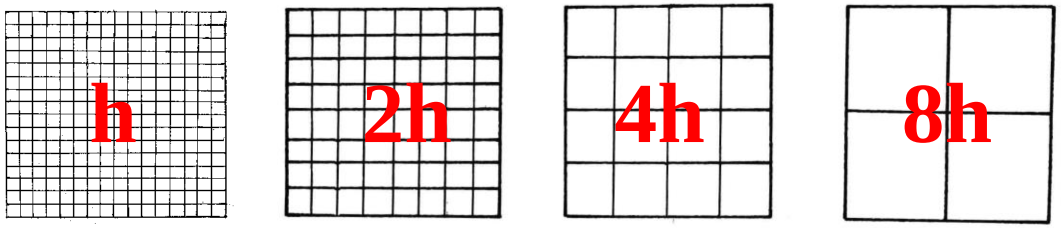
Node 4-Direct injection

1,2,4,3 coarse

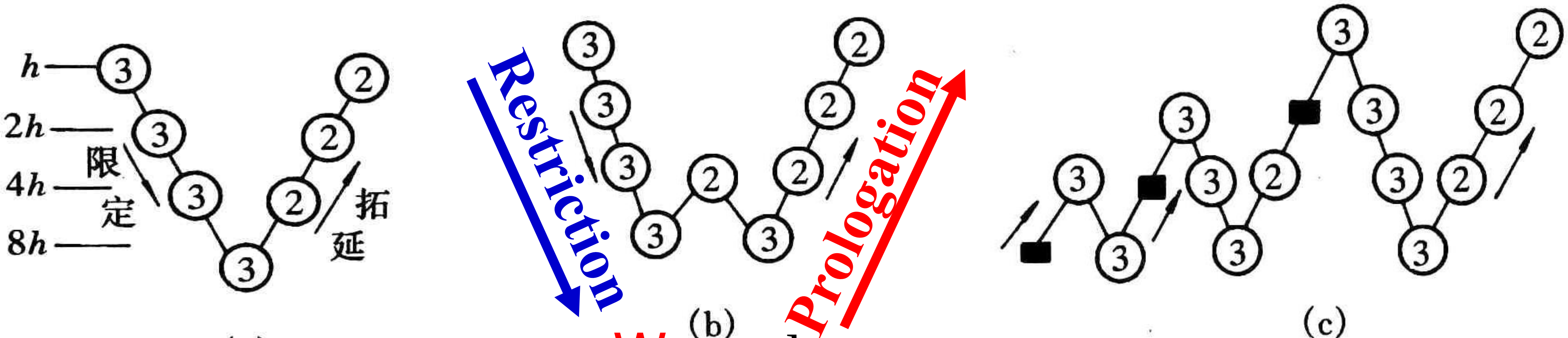
● – fine grids

⊙ Coarse → ● – Fine

5.5.4 Cycling method between several grid systems



Three cycling patterns(三种轮转的模式):



V-cycle W -cycle FMG-cycle

Number in the circle shows times of iteration. Black symbol represents converged solution. FMG cycle is widely adopted in fluid flow and heat transfer problems.

Home Work 5（2024-2025）

Please finish your homework independently !!!

Please hand in on Oct.21, 2024

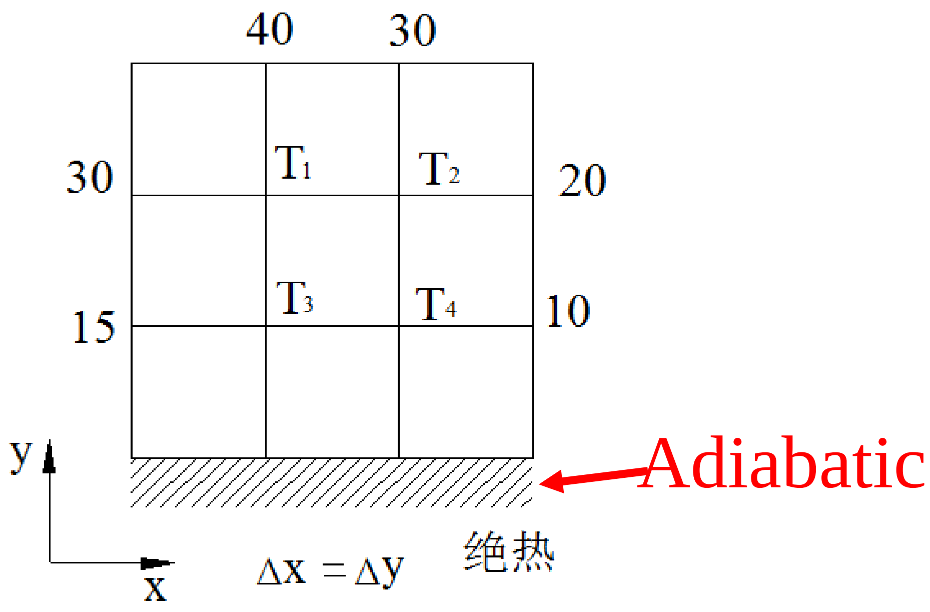
Problem 5-1 (Problem 7-2 of Textbook)

Try to show that following equations are convergent for G-S iteration method ,while divergent for Jacobi iteration,

$$5x_1+3x_2+4x_3=12;$$
$$3x_1+6x_2+4x_3=13;$$
$$4x_1+4x_2+5x_3=13$$

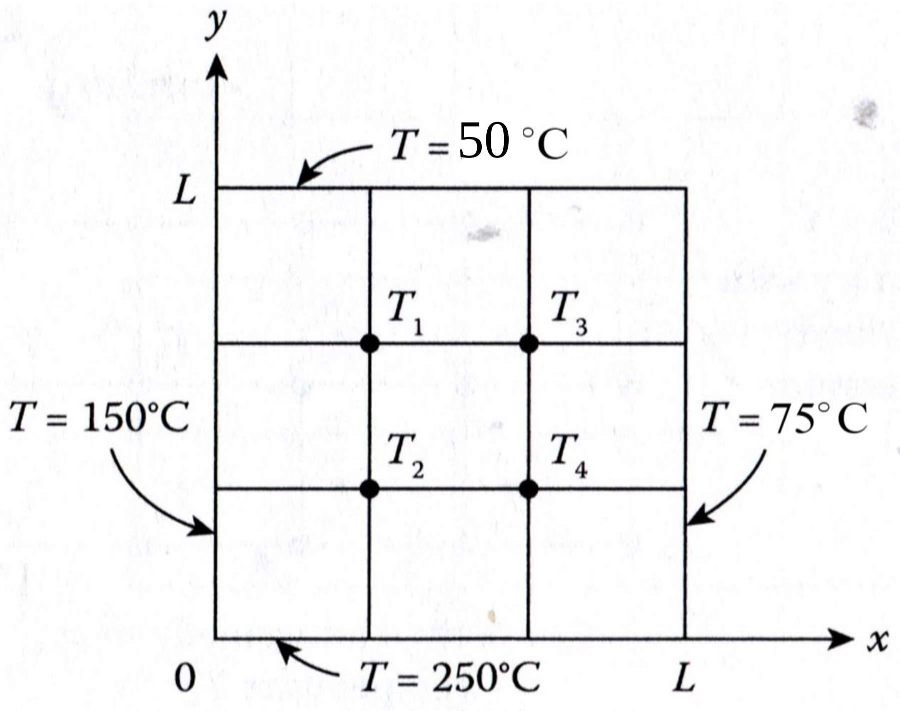
Problem 5-2

The bottom of a square region shown in the figure is adiabatic, and the other boundary conditions are shown in the figure. Further more , $S=0.015T^2, \Delta x=\Delta y=0.1, \lambda=4.0$. Determine the temperatures of the inner four node T_1, T_2, T_3 and T_4 for the steady state situation



Problem 5-3

For a steady heat conduction problem without source term shown in the figure, write the 2nd order discretized equations for nodes 1,2,3,4. Take $\Delta x = \Delta y = (1/3)L$. Express the resulting equations in matrix form for the four unknown node temperature T_1, T_2, T_3 and T_4 and determine their values



Problem 5-4

A sufficient condition for GS and Jacobi iteration convergence is that the algebraic equation coefficient matrix should satisfy following condition for either every row or every column (“strictly diagonally predominant condition”) :

$$\frac{\sum |a_{nb}|}{|a_P|} < 1$$

For the following equations, show that (1) The strictly diagonally predominant condition is satisfied;(2) By numerical calculation of several steps show that the errors are gradually reduced with the iteration.

$6x_1 - x_2 + x_3 = 4$,construction the iteration equation for x_1;

$x_1 + 4x_2 + 2x_3 = 9$,construction the iteration equation for x_2;

$-x_1 + 2x_2 + 5x_3 = 2$,construction the iteration equation for x_3.

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Teaching PPT will be loaded on ou website

同舟共济
渡彼岸！

People in the same boat help each other to cross to the other bank, where....