

# Numerical Heat Transfer

## (数值传热学)

### Chapter 4 Discretized Schemes of Diffusion and Convection Equation (1)



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## Chapter 4 Discretized diffusion – convection equation

4.1 Two ways of discretization of convection term

4.2 CD and UD of the convection term

4.3 Hybrid and power-law schemes

4.4 Characteristics of five three-point schemes

4.5 Discussion on false diffusion

4.6 Methods for overcoming or alleviating effects of false diffusion

4.7 Discretization of multi-dimensional problem and B.C. treatment

## 4.1 Two ways of discretization of convection term

### 4.1.1 Importance of discretized scheme of convection term

1. Accuracy

2. Stability

3. Economics

### 4.1.2 Two ways for constructing discretization schemes of convective term

### 4.1.3 Relationship between the two ways

## 4.1 Two ways of discretization of convection term

### 4.1.1 Importance of discretization scheme (离散格式)

Mathematically convective term is only of 1<sup>st</sup> order derivative, while its physical meaning ( strong directional) makes its discretization one of the hot spots (热点) of numerical simulation:

#### 1. It affects the numerical accuracy(精确性).

When a scheme of the convection term with 1<sup>st</sup>-order is used the solution involves severe numerical error.

#### 2. It affects the numerical stability(稳定性).

The schemes of CD, TUD(三阶迎风) and QUICK are only conditionally stable.

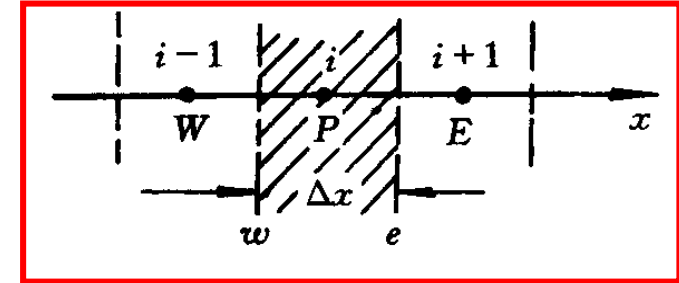
#### 3. It affects numerical economics (经济性).

## 4.1.2 Two ways for constructing(构建) schemes

### 1. Taylor expansion – providing the FD form at a point

Taking CD as an example:

$$\left(\frac{\partial \phi}{\partial x}\right)_P = \frac{\phi_E - \phi_W}{2\Delta x} = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}, O(\Delta x^2)$$



### 2. CV integration – providing average value within the domain

By assuming a profile for the interface variable

$$\frac{1}{\Delta x} \int_w^e \frac{\partial \phi}{\partial x} dx = \frac{\phi_e - \phi_w}{\Delta x} \xrightarrow[\text{Uniform grids}]{\text{Piecewise linear}} = \frac{(\phi_E + \phi_P)/2 - (\phi_P + \phi_W)/2}{\Delta x} = \frac{\phi_E - \phi_W}{2\Delta x}, O(\Delta x^2)$$

### 4.1.3 Relationship between the two ways

1. For the same scheme they have the same order of the T.E.
2. For the same scheme, the coefficients of the 1<sup>st</sup> term in T.E. are different. The absolute value of FVM is usually less than that of FD.
3. Taylor expansion provides the FD form at a point while CV integration gives the average value by integration within the domain

$$\frac{1}{\Delta x} \int_w^e \frac{\partial \phi}{\partial x} dx = \frac{\phi_e - \phi_w}{\Delta x}$$

## 4.2 CD and UD of the convection term

### 4.2.1 Analytical solution of 1-D model equation

### 4.2.2 CD discretization of 1-D diffusion-convection equation

### 4.2.3 Up wind scheme of convection term

#### 1. Definition of CV integration

#### 2. Compact form

#### 3. Discretization equation with UD of convection and CD of diffusion

## 4.2 CD and UD of convection term

### 4.2.1 Analytical solution of 1-D model eq. without source term (diffusion and convection eq.)

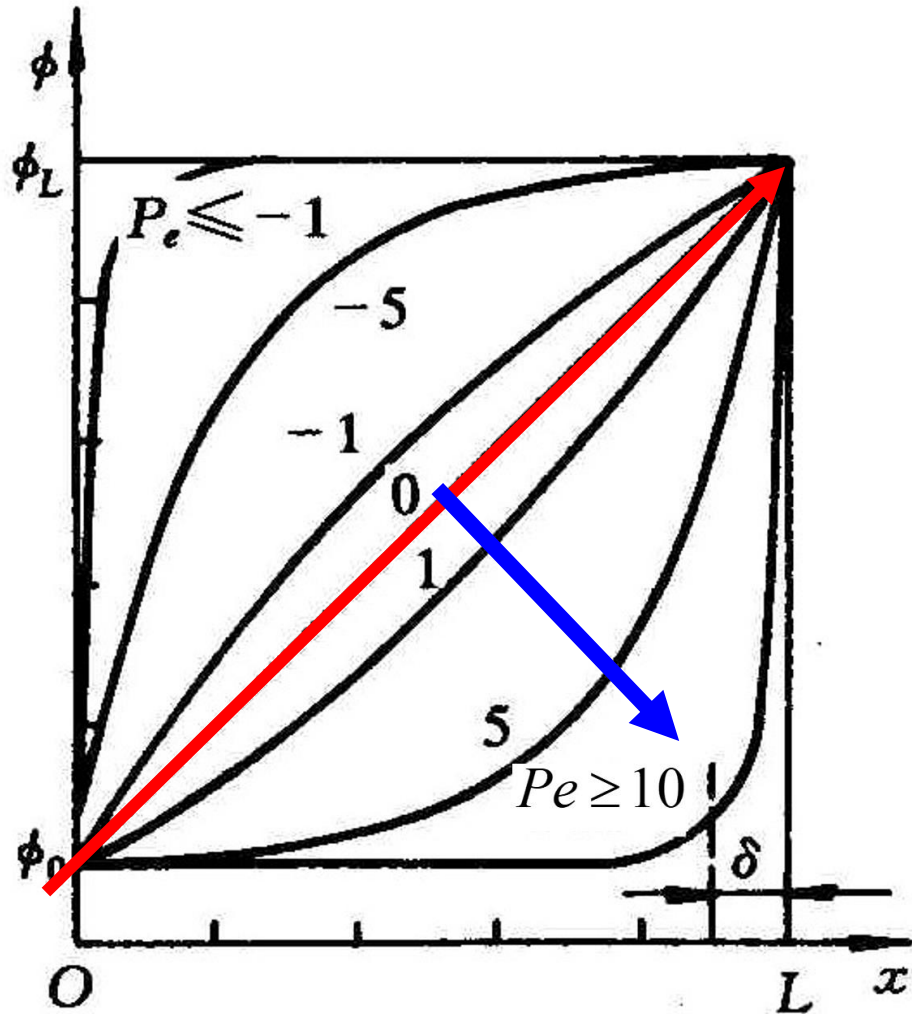
$$\left\{ \begin{array}{l} \frac{d(\rho u \phi)}{dx} = \frac{d}{dx} \left( \Gamma \frac{d\phi}{dx} \right), \\ x = 0, \phi = \phi_0; \quad x = L, \phi = \phi_L \end{array} \right. \quad \begin{array}{l} \text{Physical properties and} \\ \text{velocity are known constants} \end{array}$$

The analytical solution of this ordinary differential equation:

$$\frac{\phi - \phi_0}{\phi_L - \phi_0} = \frac{\exp(\rho u x / \Gamma) - 1}{\exp(\rho u L / \Gamma) - 1} = \frac{\exp\left(\frac{\rho u L}{\Gamma} \frac{x}{L}\right) - 1}{\exp(\rho u L / \Gamma) - 1} = \frac{\exp(Pe \frac{x}{L}) - 1}{\exp(Pe) - 1}$$



## Solution Analysis



$Pe = 0$ , pure diffusion, linear Distribution;

With increasing  $Pe$ , distribution curve becomes more and more convex downward (下凸);

When  $Pe = 10$ , in the most region from  $x=0-L$

$$\phi = \phi_0$$

Only when  $x$  is very close to  $L$ ,  $\phi$  increases dramatically and

when  $x=L$ ,  $\phi = \phi_L$ .

The above variation trend with Peclet number is consistent(**协调的**) with the physical meaning of **Pe**

$$Pe = \frac{\rho u L}{\Gamma} = \frac{\rho u}{\Gamma / L} \quad \frac{\text{Convection}}{\text{Diffusion}}$$

**When Pe is small – Diffusion dominated**, linear distribution ;

**When Pe is large – Convection dominated**, i.e., upwind(**上游**) effect dominated, upwind information is transported downstream, and when  $Pe \geq 100$ , axial conduction can be totally neglected.

It is required in some sense that the discretized scheme of the convective term has some similar physical characteristics.

# 4.2.2 CD discretization of 1-D diffusion-convection equation

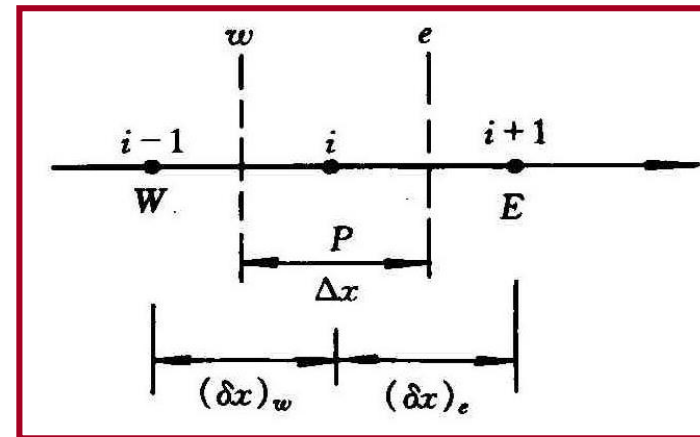
## 1. Integration of 1-D model equation

Adopting the linear profile, integration over a CV yields:

$$\phi_P \left[ \underbrace{\frac{1}{2}(\rho u)_e + \frac{\Gamma_e}{(\delta x)_e}}_{a_P} - \underbrace{\frac{1}{2}(\rho u)_w + \frac{\Gamma_w}{(\delta x)_w}}_{a_W} \right] = \underbrace{\phi_E \left[ \frac{\Gamma_e}{(\delta x)_e} - \frac{1}{2}(\rho u)_e \right]}_{a_E} + \underbrace{\phi_W \left[ \frac{\Gamma_w}{(\delta x)_w} + \frac{1}{2}(\rho u)_w \right]}_{a_W}$$

Thus:

$$a_P \phi_P = a_E \phi_E + a_W \phi_W$$



## 2. Relationship between coefficients

Rewriting  $a_P$  as follows:

$$a_P = \frac{1}{2}(\rho u)_e + \frac{\Gamma_e}{(\delta x)_e} - \frac{1}{2}(\rho u)_w + \frac{\Gamma_w}{(\delta x)_w} =$$

$$\frac{\frac{1}{2}(\rho u)_e - (\rho u)_e + (\rho u)_e}{2} + \frac{\Gamma_e}{(\delta x)_e} - \frac{\frac{1}{2}(\rho u)_w + (\rho u)_w - (\rho u)_w}{2} + \frac{\Gamma_w}{(\delta x)_w} =$$

$$-\frac{1}{2}(\rho u)_e + \frac{\Gamma_e}{(\delta x)_e} + \frac{1}{2}(\rho u)_w + \frac{\Gamma_w}{(\delta x)_w} + [(\rho u)_e - (\rho u)_w] = a_E + a_W + [(\rho u)_e - (\rho u)_w]$$

$a_E$

$a_W$

Defining diffusion  
Conductance:

$$\frac{\Gamma}{\delta x} = D,$$

Interface flow rate:

$$\rho u = F$$

The discretized form of 1-D steady diffusion and convection equation is:

$$a_P \phi_P = a_E \phi_E + a_W \phi_W \quad a_E = D_e - \frac{1}{2} F_e \quad a_W = D_w + \frac{1}{2} F_w$$
$$a_P = a_E + a_W + \underline{(F_e - F_w)}$$

If in the iterative process the mass conservation is satisfied then

$$F_e - F_w = 0$$

In order to guarantee the convergence of iterative process, it is always required:

$$a_P = a_E + a_W$$

**Hence, it is demanded(要求) that at any iteration level mass must be conserved, i.e., mass conservation should be satisfied!**

### 3. Analysis of discretized diffusion-convection eq. by CD

From  $a_P \phi_P = a_E \phi_E + a_W \phi_W$  it can be obtained:

$$\phi_P = \frac{a_E \phi_E + a_W \phi_W}{a_E + a_W} = \frac{(D_e - \frac{1}{2} F_e) \phi_E + (D_w + \frac{1}{2} F_w) \phi_W}{(D_e - \frac{1}{2} F_e) + (D_w + \frac{1}{2} F_w)}$$

**Uni.grid**

**Const property**

$$\phi_P = \frac{(1 - \frac{1}{2} \frac{F}{D}) \phi_E + (1 + \frac{1}{2} \frac{F}{D}) \phi_W}{(D + D) / D} \longrightarrow \frac{(1 - \frac{1}{2} P_\Delta) \phi_E + (1 + \frac{1}{2} P_\Delta) \phi_W}{2}$$

$P_\Delta$  is the grid Peclet number,  $P_\Delta = \frac{\rho u (\delta x)}{\Gamma}$

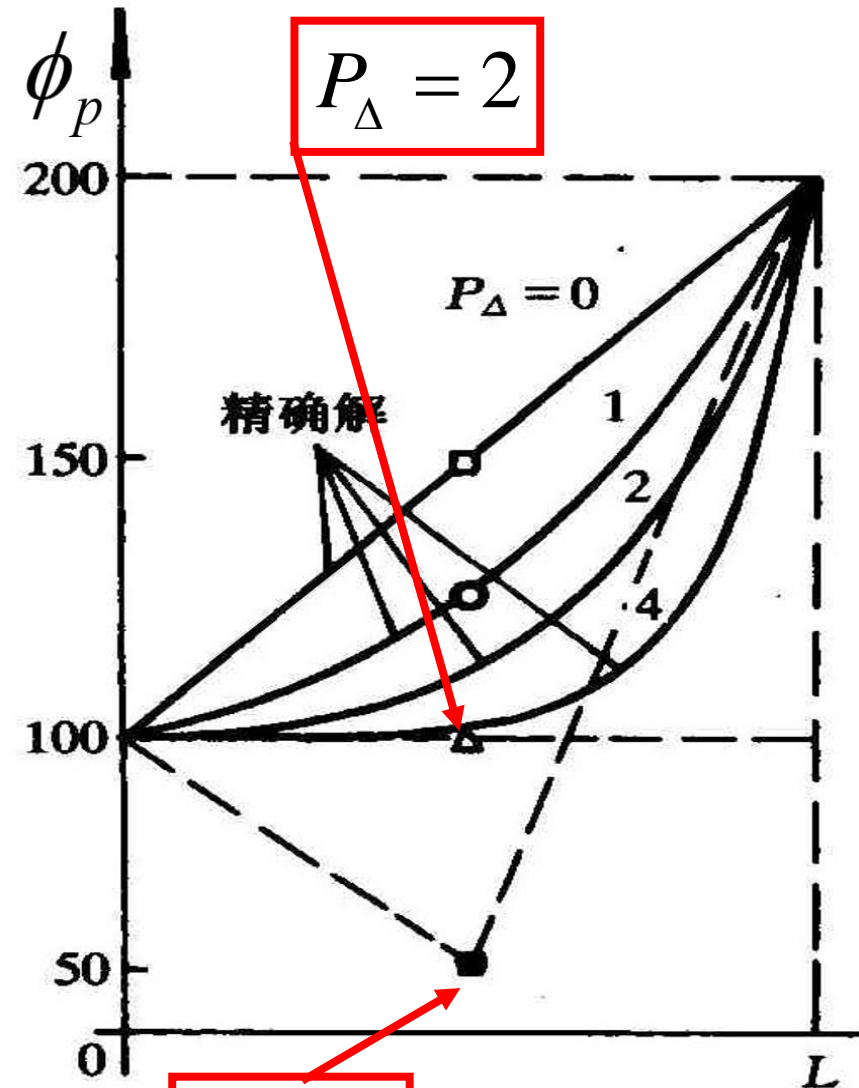
With the given  $\phi_E$  and  $\phi_W$ ,  $\phi_P$  can be determined.

Given  $\phi_W = 100, \phi_E = 200$   
for  $P_\Delta = 0, 1, 2, 4$   
the calculated results  $\phi_p$  are  
shown in the figure.

**Physically and according  
to the analytical solution**

$$\frac{\phi - \phi_0}{\phi_L - \phi_0} = \frac{\exp\left(\frac{\rho u L}{\Gamma} \frac{x}{L}\right) - 1}{\exp\left(\frac{\rho u L}{\Gamma}\right) - 1}$$

**the value of  $\phi_p$  should be  
larger than zero.**



$$P_\Delta = 4$$

$$\frac{\rho u L}{\Gamma} = Pe = 2 P_\Delta$$

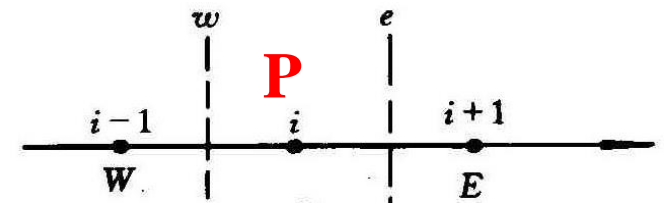
Thus when  $P_\Delta$  is larger than 2, numerical solutions are unrealistic (不现实的) :  $\phi_P$  is less than its two neighboring grid values, which is not possible for the case without source.

The reason is  $P_\Delta > 2 \Rightarrow a_E = \frac{1}{2}(1 - \frac{1}{2}P_\Delta) < 0$ , i.e. the east influencing coefficient is negative, which is physically meaningless.

### 4.2.3 First order upwind (FUD) of convection term

#### 1. Definition in FD –

$$\left(\frac{\partial \phi}{\partial x}\right)_i = \frac{\phi_i - \phi_{i-1}}{\Delta x}, u > 0; \left(\frac{\partial \phi}{\partial x}\right)_i = \frac{\phi_{i+1} - \phi_i}{\Delta x}, u < 0$$



2. Definition in FV – interpolation of interface always takes upstream grid value



$$\phi_e = \begin{cases} \phi_P, u_e > 0 \\ \phi_E, u_e < 0 \end{cases} O(\Delta x) \quad \phi_w = \begin{cases} \phi_W, u_w > 0 \\ \phi_P, u_w < 0 \end{cases}$$

## 2. Compact form (紧凑形式)

For the convenience of discussion, combining interface value  $\phi_e$  with flow rate

$$(\rho u \phi)_e = F_e \phi_e = \phi_P \max(F_e, 0) - \phi_E \max(-F_e, 0)$$

Patankar proposed a special symbol as follows

MAX:  $[[X, Y]]$ , then:  $(\rho u \phi)_e = \phi_P [[F_e, 0]] - \phi_E [[-F_e, 0]]$

Similarly:

$$(\rho u \phi)_w = \phi_W [[F_w, 0]] - \phi_P [[-F_w, 0]]$$

## 3. Discretized form of 1-D model equation with FUD for convection term and CD for diffusion term

$$a_P \phi_P = a_E \phi_E + a_W \phi_W$$

$$a_E = D_e + (-F_e), 0 \quad a_W = D_w + F_w, 0$$

$$a_P = a_E + a_W + (F_e - F_w)$$

Because  $a_E \geq 0, a_W \geq 0$  **FUD can always obtained physically plausible solution** (物理上看起来合理的解).

Because of this important **feature**(特点), it was widely used in the past **decades** (十年) since its was proposed in 1950s.

However, because of its severe **numerical errors** (severe false diffusion, 严重的假扩散), it is now not recommended for the final solution.

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## 4.3 Hybrid and Power-Law Schemes

4.3.1. Relationship between  $a_E, a_W$  of 3-point schemes

4.3.2. Hybrid scheme

4.3.3. Exponential scheme

4.3.4. Power-law scheme

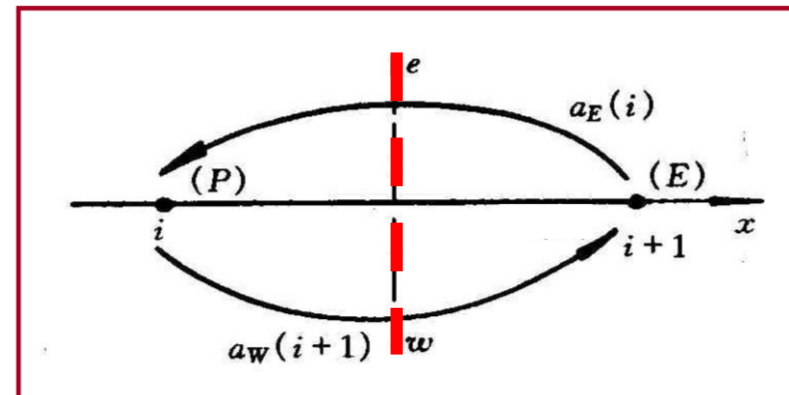
4.3.5. Expressions of coefficients of five 3-point schemes and their plots

## 4.3 Hybrid and Power-Law Schemes

### 4.3.1. Relationship between coefficients $a_E, a_W$ of 3-point schemes

- 3-point scheme** — interface interpolation is conducted by using two points at the two sides of the interface  
 With such scheme the coefficients of 1-D problem leads to tri-diagonal matrix, and 2-D to penta-diagonal (五对角) matrix.
- Relationship between  $a_E, a_W$**   
 East or West interfaces are relative to the grid position.

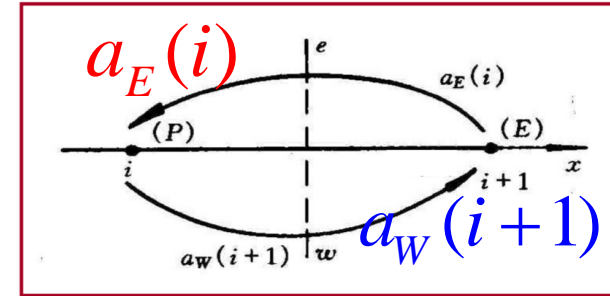
For the same interface shown by the red line:  
 it is East for point P,  
 while West for E.



$a_E(i)$  and  $a_W(i+1)$  share (共享) the same interface, the same conductivity and the same absolute flow rate, hence they must have some interrelationship (内在关系).

For CD:

$$a_E = D_e \left(1 - \frac{1}{2} P_{\Delta e}\right) a_W = D_w \left(1 + \frac{1}{2} P_{\Delta w}\right)$$



At the same interface  $P_{\Delta e} = P_{\Delta w} = P_{\Delta}$   $D_e = D_w = D$

$$\frac{a_W(i+1)}{D} - \frac{a_E(i)}{D} = 1 + \frac{1}{2} P_{\Delta} - \left(1 - \frac{1}{2} P_{\Delta}\right) = P_{\Delta}$$

**Meaning:** for diffusion problem,  $P_{\Delta} = 0$ ,  $a_E(i) = a_W(i+1)$

For convection if ( $u > 0$ ), node  $i$  has effect on  $(i+1)$ ,  $P_{\Delta}$  while  $(i+1)$  has no convection effect on  $i$ ;  $a_E(i)$  has no convection effect on grid  $i$ , while  $a_W(i+1)$  has some convection effect on grid  $(i+1)$

For **FUD**:  $a_E = D_e (1 + -P_{\Delta e}, 0)$   $a_W = D_w (1 + P_{\Delta w}, 0)$

$$\frac{a_W(i+1)}{D} - \frac{a_E(i)}{D} = 1 + \underline{P_{\Delta,0}} - (1 + \underline{-P_{\Delta,0}}) \longrightarrow$$

$$P_{\Delta,0} - -P_{\Delta,0} = P_{\Delta}$$

Therefore for  $a_E$  or  $a_W$  once one of them is known, the other can be obtained.

Thus defining a scheme can be conducted just by defining one coefficient. We will define the E-coefficient.

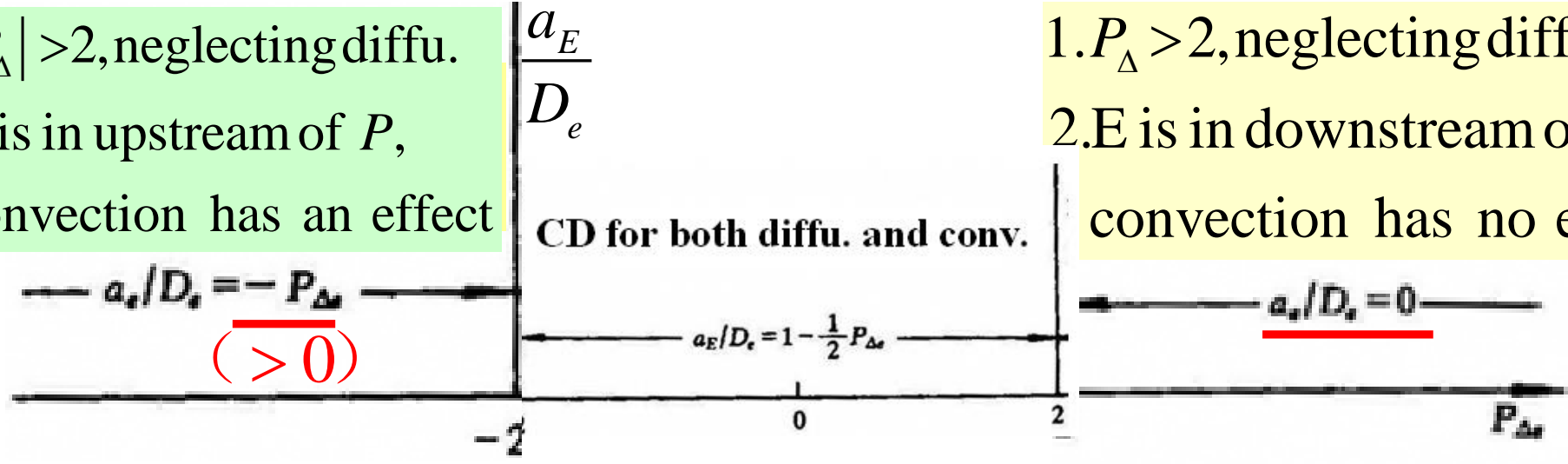
### 4.3.2 Hybrid scheme (混合格式)

#### 1. Graph(图形) definition

**Spalding** proposed: taking  $P_\Delta$  as abscissa (横坐标)  
and  $a_E / D_e$  as ordinate (纵坐标)

1.  $|P_\Delta| > 2$ , neglecting diffu.  
2. E is in upstream of P,  
convection has an effect

1.  $P_\Delta > 2$ , neglecting diffu.  
2. E is in downstream of P,  
convection has no effect



$$\frac{a_E}{D_e} = \begin{cases} 0, & P_\Delta > 2 \\ 1 - \frac{1}{2} P_\Delta, & |P_\Delta| \leq 2 \\ -P_\Delta, & P_\Delta < -2 \end{cases} \quad \text{Hybrid scheme of Spalding}$$

**2. Compact definition**



$$\frac{a_E}{D_e} = \left[ -P_{\Delta e}, 1 - \frac{1}{2} P_{\Delta e}, 0 \right]$$

### 4.3.3. Exponential scheme (指数格式)

**Definition:** the discretized of this scheme form identical to the analytical solution of the 1-D model equation.

**Method:** rewriting the analytical solution in the form of algebraic equation of  $\phi$  at three neighboring grid points.

#### 1. Total flux $J$ (总通量) of diffusion and convection

Define  $J = \rho u \phi - \Gamma \frac{d\phi}{dx}$ , then 1-D model eq. can be rewritten as  $\frac{dJ}{dx} = 0$ , or  $J = \text{const}$

For CV. P:  $J_e = J_w$

## 2. Analytical expression for total flux of diffu. and conv.

Substituting the analytical solution of  $\phi$  into  $J$  :

$$\phi = \phi_0 + (\phi_L - \phi_0) \frac{\exp(Pe \frac{x}{L}) - 1}{\exp(Pe) - 1}$$

$$J = \rho u \phi - \Gamma \frac{d\phi}{dx} = \rho u \left[ \phi_0 + (\phi_L - \phi_0) \frac{\exp(Pe \frac{x}{L}) - 1}{\exp(Pe) - 1} \right] - \Gamma \left[ (\phi_L - \phi_0) \frac{\frac{Pe}{L} \exp(Pe \frac{x}{L})}{\exp(Pe) - 1} \right]$$

$Pe = \frac{\rho u L}{\Gamma}$

$\rho u \phi$

$\Gamma d\phi / dx$

Hence:  $J = F \left[ \phi_0 + \frac{\phi_0 - \phi_L}{\exp(Pe) - 1} \right]$

$$\frac{\Gamma}{L} Pe = \frac{\Gamma}{L} \frac{\rho u L}{\Gamma} = \rho u$$

## 2. Expressions of total flux for $e, w$ interfaces

For  $e$ :  $\phi_0 = \phi_P, \phi_L = \phi_E, L = (\delta x)_e : J_e = F_e \left[ \phi_P + \frac{\phi_P - \phi_E}{\exp(P_{\Delta e}) - 1} \right]$

For  $w$ :  $\phi_0 = \phi_W, \phi_L = \phi_P, L = (\delta x)_w : J_w = F_w \left[ \phi_W + \frac{\phi_W - \phi_P}{\exp(P_{\Delta w}) - 1} \right]$

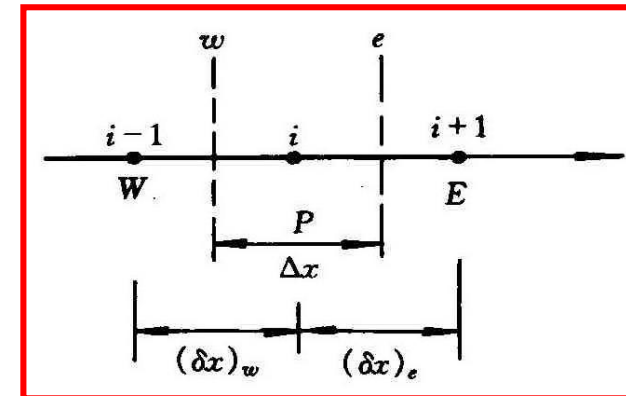
Substituting the two expressions into  $J_e = J_w$  and

rewrite into algebraic equation among  $\phi_W, \phi_P, \phi_E$

yields:  $a_P \phi_P = a_W \phi_W + a_E \phi_E$

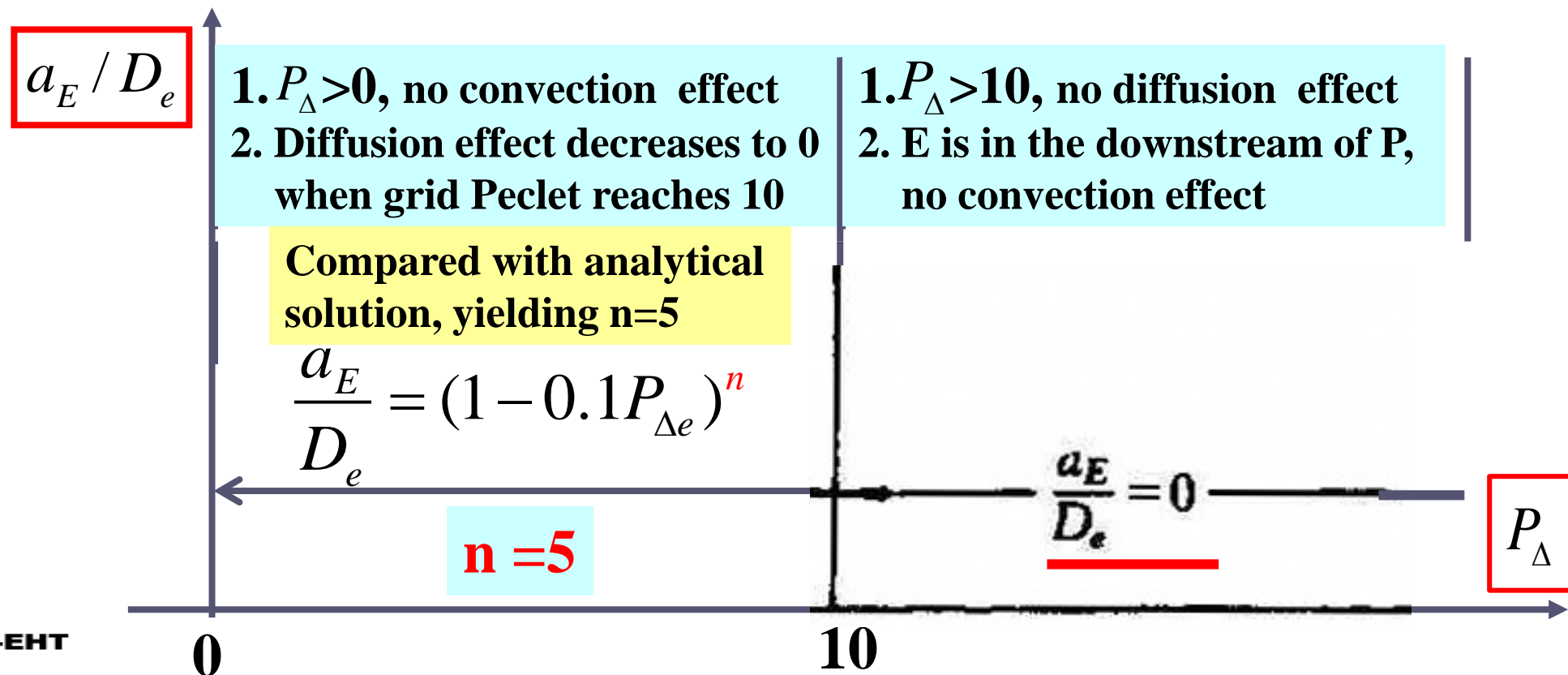
$$a_E = \frac{F_e}{\exp(P_{\Delta e}) - 1}, a_W = \frac{F_w \exp(P_{\Delta w})}{\exp(P_{\Delta w}) - 1}$$

$$a_P = a_E + a_W + (F_e - F_w)$$



### 4.3.4. Power-law scheme (乘方格式)

Exponential scheme is computationally very expensive (昂贵). Patankar proposed the power-law scheme, which is very close to the exponential scheme and computationally much cheaper. Its graphical definition: For  $P_{\Delta} > 0$ :



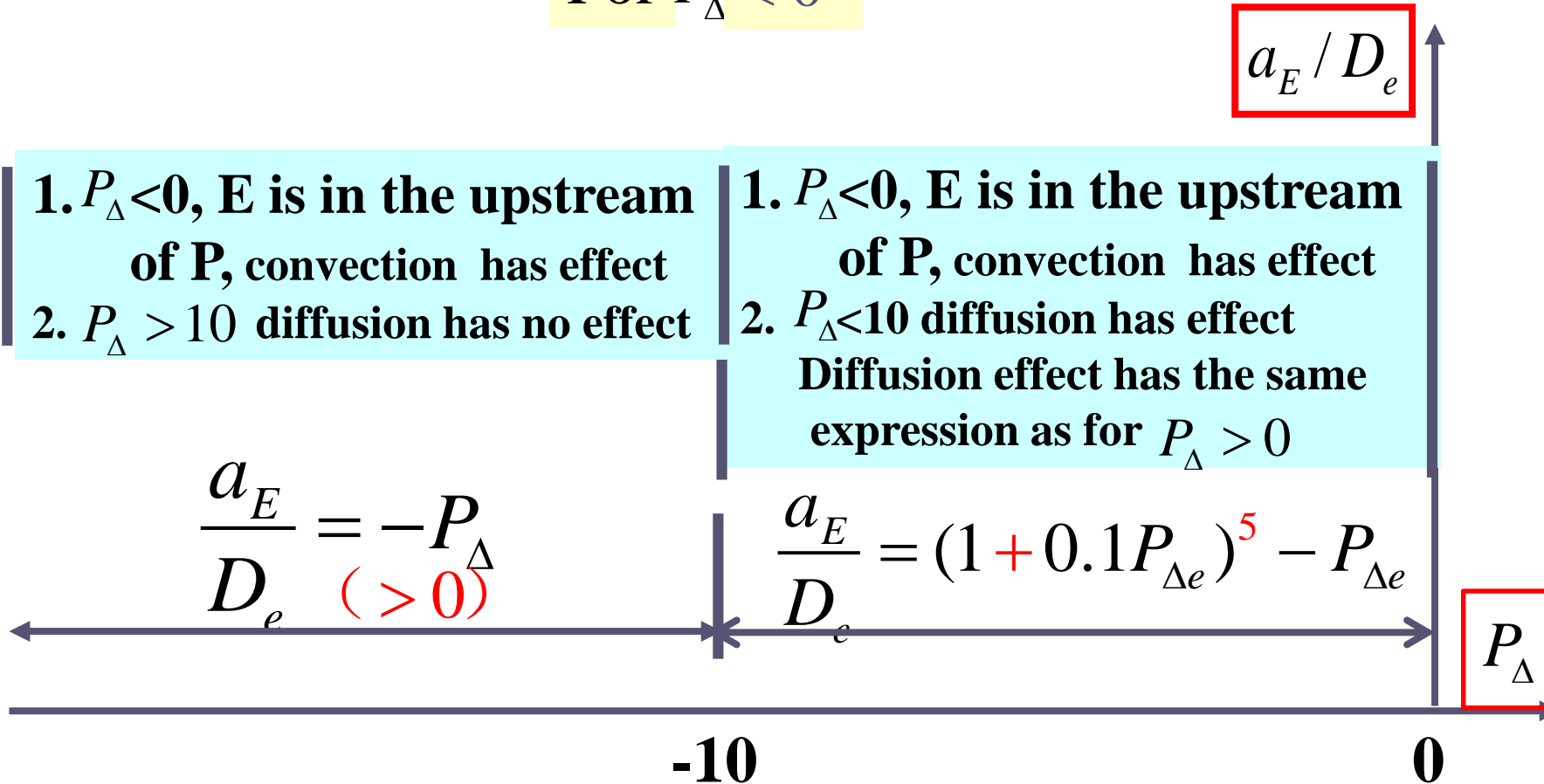
For  $P_{\Delta} < 0$

1.  $P_{\Delta} < 0$ , E is in the upstream of P, convection has effect
2.  $P_{\Delta} > 10$  diffusion has no effect

$$\frac{a_E}{D_e} = -P_{\Delta} \quad (> 0)$$

1.  $P_{\Delta} < 0$ , E is in the upstream of P, convection has effect
2.  $P_{\Delta} < 10$  diffusion has effect  
Diffusion effect has the same expression as for  $P_{\Delta} > 0$

$$\frac{a_E}{D_e} = (1 + 0.1P_{\Delta e})^5 - P_{\Delta e}$$



Compact form of the power-law scheme

$$\frac{a_E}{D_e} = \left[ \left[ 0, (1 - 0.1|P_{\Delta e}|)^5 \right] + 0, -P_{\Delta e} \right]$$

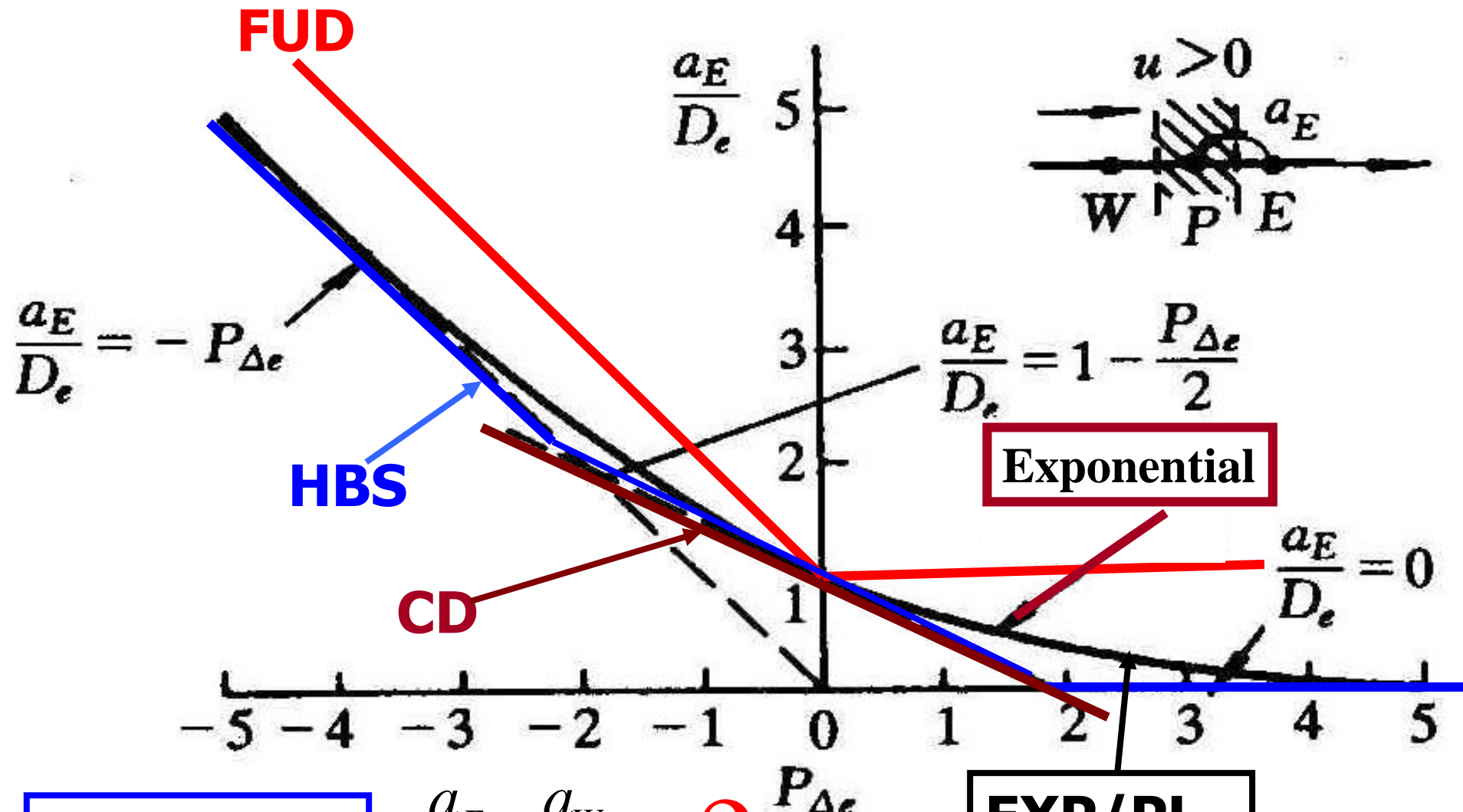
Diffusion effect

Convection effect

### 4.3.5. $a_E / D_e$ coefficient expressions of five schemes and their graph illustration (说明)

Scheme	Central difference	Upwind difference
Definition	$1 - 0.5 P_{\Delta e}$	$1 + -P_{\Delta e}, 0$
Hybrid	Power-law	Exponential
$\left[ -P_{\Delta e}, 1 - \frac{1}{2} P_{\Delta e}, 0 \right]$	$\left[ 0, (1 - 0.1P_{\Delta e})^5 \right] + 0, -P_{\Delta e}$	$\frac{P_{\Delta e}}{\exp(P_{\Delta e}) - 1}$

$$a_E / D_e = 1 + \left\| -P_{\Delta e}, 0 \right\|$$



$$a_E, a_W \dots J$$

$$\frac{a_E}{D_E}, \frac{a_W}{D_W} \dots ?$$

EXP/PL

## 4.4 Characteristics of five three-point schemes

4.4.1  $J^*$  flux definition and its discretized form

4.4.2 Relationship between coefficients A and B

4.4.3 Important conclusions from coefficient characters

4.4.4 General expression for coefficients  $a_E, a_W$

4.4.5 Discussion



## 4.4 Characteristics of five three-point schemes

### 4.4.1 $J^*$ flux definition and its discretized form

#### 1. $J^*$ definition (analytical expression)

$J$  flux is correspondent to the discretized equation  $a_P \phi_P = a_W \phi_W + a_E \phi_E$ , while flux correspondent to coefficient  $a_E / D_e$  is called  $J^*$ , which is defined by:

$$J^* = \frac{J}{D} = \frac{1}{\Gamma / \delta x} (\rho u \phi - \Gamma \frac{d\phi}{dx}) = \left( \frac{\rho u \delta x}{\Gamma} \right) \phi - \frac{d\phi}{d\left(\frac{x}{\delta x}\right)} =$$

$$J^* = P_{\Delta} \phi - \frac{d\phi}{dX} \quad P_{\Delta} = \frac{\rho u \delta x}{\Gamma} \quad X = \frac{x}{\delta x}$$

## 2. Discretized form of $J^*$

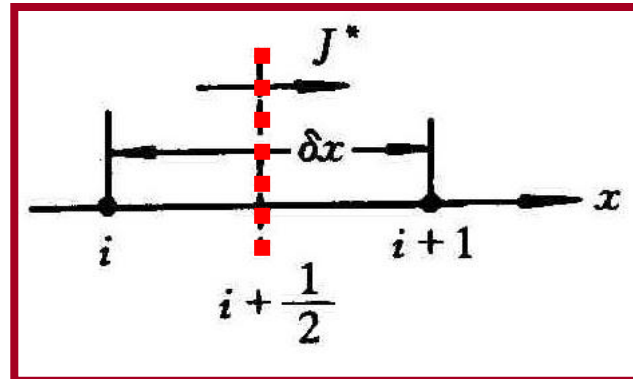
For the three-point scheme  $J^*$  at interface can be expressed by a combination of variables at nearby two grids.

For interface  $(i+1/2)$ ,

$$\text{Let } J^* = B\phi_i - A\phi_{i+1}$$

Ahead of the interface (在界面之前)

Behind of the interface (在界面之后)



Viewed from positive direction of coordinate (从坐标正向看)

Coefficients  $A$ ,  $B$  are dependent on grid Peclet,  $P_{\Delta}$

## 4.4.2 Analysis of relationship between $A$ and $B$

Analysis is based on fundamental physical and mathematical concepts.

### 1. Summation-subtraction character (和差特性)

For a uniform field, there is no diffusion at all. Then  $J^*$  is totally caused by convection

From the analytical expression of  $J^*$ :

$$J^* = \left( P_{\Delta} \phi - \frac{d\phi}{dX} \right)_i = \left( P_{\Delta} \phi - \frac{d\phi}{dX} \right)_{i+1} = P_{\Delta} \phi_i = P_{\Delta} \phi_{i+1}$$

From its discretized expression:

$$J^* = B\phi_i - A\phi_{i+1} = (B - A)\phi_i = (B - A)\phi_{i+1}$$

Analytical =  
Discretized!

$$(B - A) \cancel{\phi_{i+1}} = P_{\Delta} \cancel{\phi_i} = P_{\Delta} \cancel{\phi_{i+1}} \longrightarrow$$

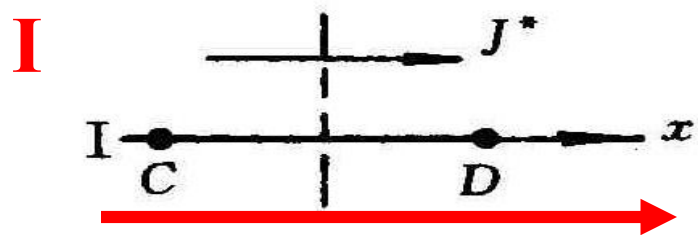
$$B - A = P_{\Delta} \quad \text{Summation-subtraction(和差特性)}$$

## 2. Symmetry character

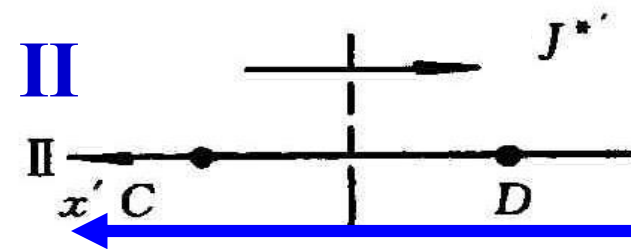
For the same process its mathematical formulation is expressed in two coordinates. The two coordinates are I, II , and their positive directions are opposite (相反的). Two points C,D are located at the two sides of an interface

**Viewed from coordinate positive direction**

**C-behind/D-ahead**



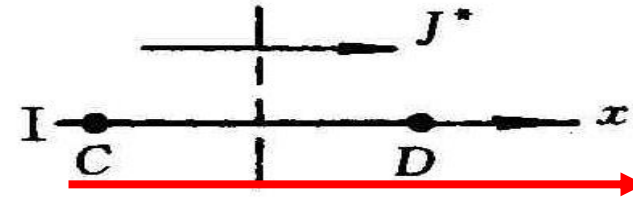
**C-ahead/D-behind**



For the same flux, in coordinate I it is denoted by  $J^*$ , while in II denoted by  $J^{*'}$ , then we have

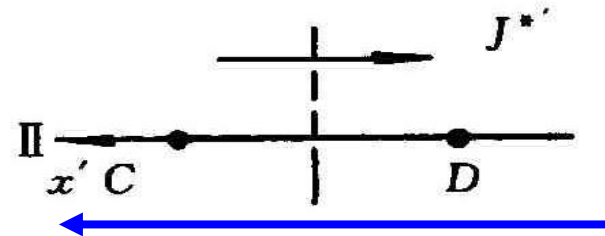
For I **C-behind/D-ahead**

$$J^* = B(P_\Delta) \phi_C - A(P_\Delta) \phi_D$$



For II **D-behind/C-ahead**

$$J^{*' } = B(-P_\Delta) \phi_D - A(-P_\Delta) \phi_C$$



The flux is the same so:  $J^* = -J^{*'}$

$$B(P_{\Delta})\phi_C - A(P_{\Delta})\phi_D = -[B(-P_{\Delta})\phi_D - A(-P_{\Delta})\phi_C]$$

Merging (合并) the terms according to  $\phi_D, \phi_C$

$$[B(P_{\Delta}) - A(-P_{\Delta})]\phi_C = [A(P_{\Delta}) - B(-P_{\Delta})]\phi_D$$

$\phi_D, \phi_C$  can take any values. In order that above eq.

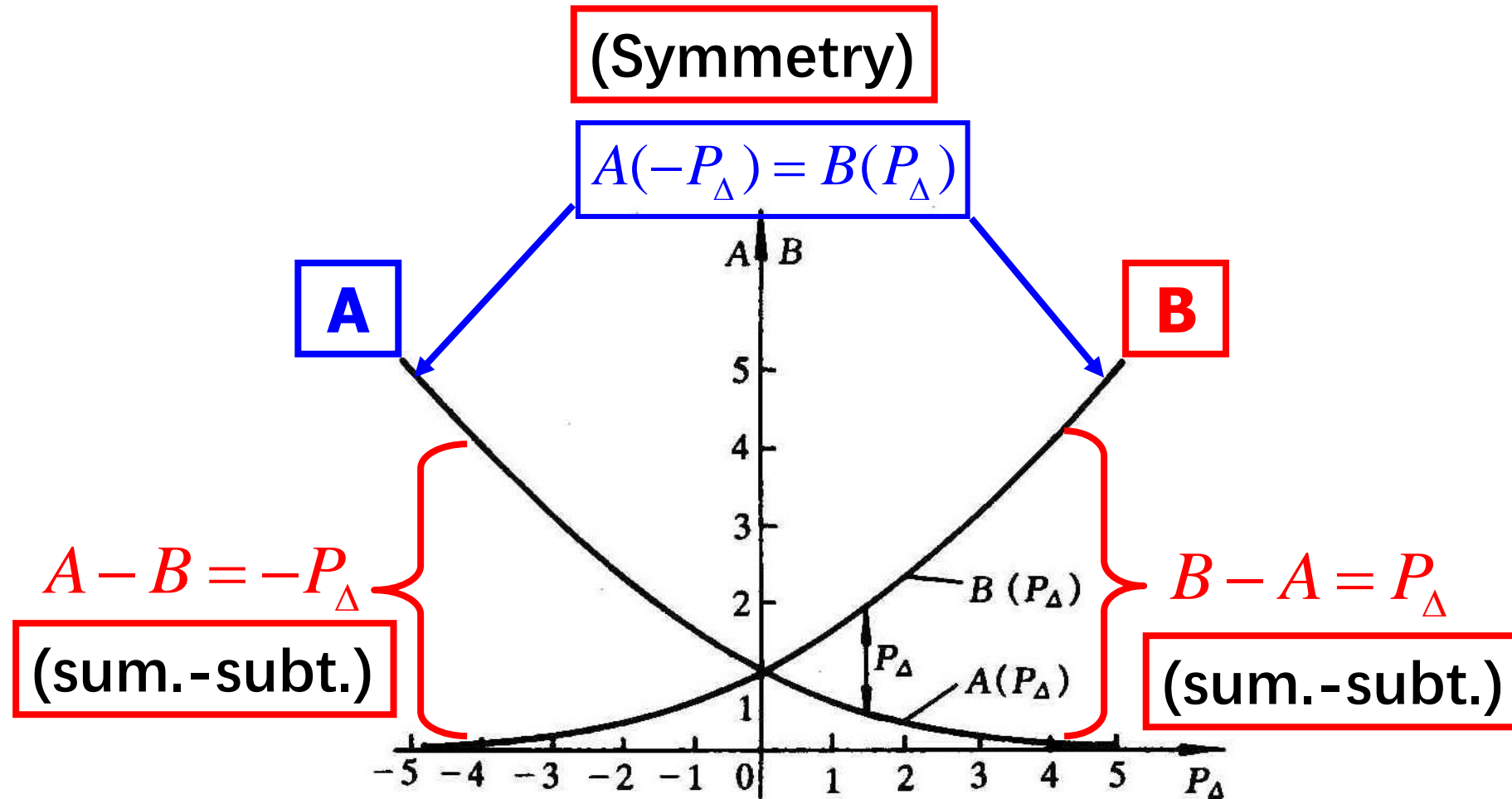
is valid for any  $\phi_D, \phi_C$ , **the only solution is:**

$$B(P_{\Delta}) - A(-P_{\Delta}) = 0 \quad A(P_{\Delta}) - B(-P_{\Delta}) = 0$$

i.e.,:  $B(P_{\Delta}) = A(-P_{\Delta}); \quad A(P_{\Delta}) = B(-P_{\Delta})$

**Symmetry character (对称特性)**

Taking  $P_{\Delta} = 0$  as the symmetric axis, their plots (图) are:



These are basic features of  $A$  and  $B$  of the five 3-point schemes. (20230927)

### 4.4.3 Important conclusions from the two features

For the five 3-point schemes **if and only if** (当且仅当) the function of  $A(P_\Delta)$  is known for  $P_\Delta \geq 0$ , then in the entire range of  $-|P_\Delta| \leq P_\Delta \leq |P_\Delta|$ , the analytical expressions are known for both  $A(P_\Delta)$  and  $B(P_\Delta)$ .

**[Proving]** 1. First we show that this is correct for  $A(P_\Delta)$ .

(1) For case of  $P_\Delta \geq 0$   $A(|P_\Delta|)$  is given in the conditions.

(2) For case of  $P_\Delta < 0$  We have

$$A(P_\Delta) \xrightarrow{\text{Sum-sub}} B(P_\Delta) - P_\Delta \xrightarrow{\text{Symmet}} A(-P_\Delta) - P_\Delta$$

$$\xrightarrow{P_\Delta \leq 0} A(|P_\Delta|) + |P_\Delta|$$



Therefore either  $P_{\Delta} > 0$  or  $P_{\Delta} < 0$

$$A(P) = \left\{ \begin{array}{l} A(P_{\Delta}), P \geq 0 \\ A(|P_{\Delta}|) + |P_{\Delta}|, P_{\Delta} < 0 \end{array} \right\} \quad A(P_{\Delta}) = A(|P_{\Delta}|) + \begin{array}{l} -P_{\Delta}, 0 \end{array}$$

2. Then we show that for  $B(P_{\Delta})$  above statement is also valid.

$$B(P_{\Delta}) \xrightarrow{\text{Sum.-subt.}} A(P_{\Delta}) + P_{\Delta} \xrightarrow{\text{From } A(P) \text{ expression}} A(|P_{\Delta}|) + \underbrace{-P_{\Delta}, 0}_{\text{ }} + P_{\Delta} \longrightarrow A(|P_{\Delta}|) + P_{\Delta}, 0$$

Thus  $B(P_{\Delta}) = A(|P_{\Delta}|) + P_{\Delta}, 0$

**Verification (证明) is finished!**

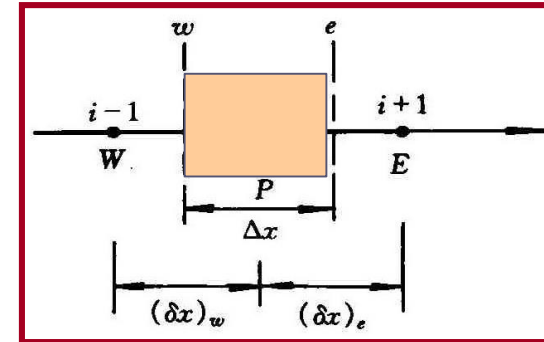
## 4.4.4 Derivation of general expression for $a_E, a_W$ of three-point schemes from coefficient characters

### Basic idea

- (1) For CV. P writing down diffusion/convection flux balance equation for its two interfaces;

$$J_e = J_w$$

$$J_e^* D_e = J_w^* D_w$$



- (2) Expressing  $J^*$  via  $A, B$  and the related grid value;
- (3) Expressing  $A, B$  via  $A(|P_\Delta|)$  ;
- (4) Then rewrite above eq. in terms of  $\phi_W, \phi_P, \phi_E$  ;

(5) Comparing the above-resulted eq. with the standard form

$$a_P \phi_P = a_E \phi_E + a_W \phi_W$$

The general expressions of coefficients of the discretized equation of five 3-point schemes can be obtained:

$$a_E = D_e A(|P_{\Delta e}|) + \quad -F_e, 0$$

$$a_W = D_w A(|P_{\Delta w}|) + \quad F_w, 0$$

$$a_P = a_E + a_W + (\cancel{F_e} - \cancel{F_w})$$

**See the appendix for the detailed derivation.**

## Expressions of $A(|P_\Delta|)$

Scheme	$A( P_\Delta )$
CD	$1 - 0.5  P_\Delta $
FUD	1
Hybrid	$[0, 1 - 0.5  P_\Delta ]$
Exponential	$ P_\Delta  / (\exp( P_\Delta ) - 1)$
Power-law	$[0, (1 - 0.1  P_\Delta )^5]$

To select one scheme of the three-point just define the expression of  $A(|P_\Delta|)$ .

## 4.4.5 Discussion

1. Extend (推广) from 1-D to multi-D:

Regarding every coordinate as 1-D coordinate and constructing the influencing coefficients by the way as shown above;

2. For the five 3-point schemes, by selecting  $A(|P_{\Delta}|)$  the discretized scheme for convection is set up (设置).

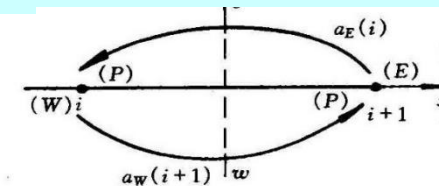
3. Relationship between  $a_w(i+1)$  and  $a_E(i)$

can be used to simplify computation

$$a_w(i+1) = \{ D_w A(|P_{\Delta w}|) + F_w, 0 \}_{i+1}$$

$$a_E(i) = \{ D_e A(|P_{\Delta e}|) + -F_e, 0 \}_i$$

$$a_w(i+1) - a_E(i) = F, 0 - -F, 0 = F$$



$$(D_w)_{i+1} = (D_e)_i$$

$$(F_w)_{i+1} = (F_e)_i$$

$$(P_{\Delta w})_{i+1} = (P_{\Delta e})_i$$

## Appendix 1 of Section 5-4

$$J_e^* D_e = J_w^* D_w$$

$$D_e [B(P_{\Delta e}) \phi_P - A(P_{\Delta e}) \phi_E] = D_w [B(P_{\Delta w}) \phi_W - A(P_{\Delta w}) \phi_P]$$

$$\phi_P \underbrace{[D_e B(P_{\Delta e}) + D_w A(P_{\Delta w})]}_{a_P} = \underbrace{[D_e A(P_{\Delta e})]}_{a_E} \phi_E + \underbrace{[D_w B(P_{\Delta w})]}_{a_W} \phi_W$$

Expressing  $A$ ,  $B$  via  $A(|P_{\Delta}|)$

$$A(P_{\Delta w}) = A(|P_{\Delta w}|) + \|-P_{\Delta w}, 0\| \quad B(P_{\Delta w}) = A(|P_{\Delta w}|) + \|P_{\Delta w}, 0\|$$

$$A(P_{\Delta e}) = A(|P_{\Delta e}|) + \|-P_{\Delta e}, 0\| \quad B(P_{\Delta e}) = A(|P_{\Delta e}|) + \|P_{\Delta e}, 0\|$$

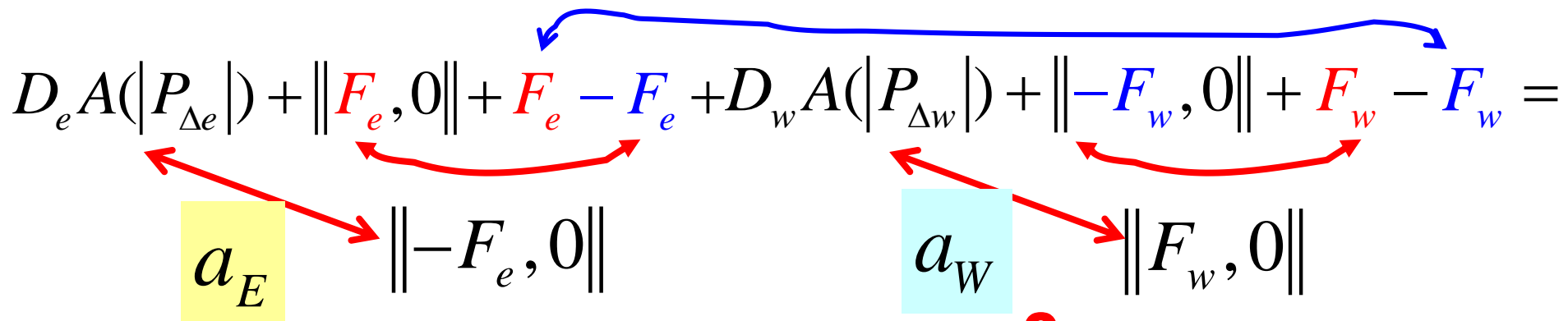
$$a_E = D_e A(P_{\Delta e}) = D_e \{A(|P_{\Delta e}|) + \|-P_{\Delta e}, 0\|\} \quad \longrightarrow$$

$$a_E = D_e A(|P_{\Delta e}|) + \|-F_e, 0\| \quad a_W = D_w A(|P_{\Delta w}|) + \|F_w, 0\|$$

$$a_P = D_e \underline{B(P_{\Delta e})} + D_w \underline{A(P_{\Delta w})} \text{ can be transformed as}$$

$$D_e [A(|P_{\Delta e}|) + \|P_{\Delta e}, 0\|] + D_w [A(|P_{\Delta w}|) + \|-P_{\Delta w}, 0\|] =$$

$$D_e A(|P_{\Delta e}|) + \|F_e, 0\| + D_w A(|P_{\Delta w}|) + \|-F_w, 0\| =$$

$$D_e A(|P_{\Delta e}|) + \|F_e, 0\| + F_e - F_e + D_w A(|P_{\Delta w}|) + \|-F_w, 0\| + F_w - F_w =$$


$$a_E \quad \|-F_e, 0\|$$

$$a_W \quad \|F_w, 0\|$$

$$a_P = a_E + a_W + (\cancel{F_e} - \cancel{F_w})$$

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