Numerical Heat Transfer

(数值传热学)

Chapter 3 Mathematical and Physical Characteristics of Discretized Equations

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3.1 Consistence, Convergence and Stability of Discretized Equations

3.1.1 Truncation error and consistence（相容性）

1. Analytical solution of discretized equations (离散方程的精确解)

Numerical solution without any round-off (舍入) error introduced in the solution procedure.

It is assumed that Taylor expansion can be applied to numerical solutions ϕ_i^n **,**

2. Differential vs. difference operators (算子)

（1）Differential operator (微分算子)-

Implementing(执行) some differential and/or arithmetic(算术) operations on function $\phi(i,n)$ at a point (*i*,*n*):

$$L(\phi)_{i,n} = (\rho \frac{\partial \phi}{\partial t} + \rho u \frac{\partial \phi}{\partial x} - \Gamma \frac{\partial^2 \phi}{\partial x^2} - S)_{i,n}$$

Then $L(\phi)_{i,n} = 0$ --1-D model equation.

（2）Difference operator(差分算子)

Implementing some difference and/or arithmetic operations on function ϕ_i^n at point (*i*,*n*)

$$L_{\Delta x,\Delta t}(\phi_i^n) = \rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} + \rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x} - \Gamma\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{\Delta x^2} - S_i^n$$

Then $L_{\Delta x,\Delta t}(\phi_i^n)=0$ **---discretized form of 1-D model equation: Forward time and central space —FTCS**

3. Truncation error (截断误差)of discretized equation

The difference between differential and difference operators (微分算子与差分算子的差)

(1) Definition – $\mathrm{T.E.} = L_{\Delta x,\Delta t}(\phi_i^n) - L(\phi)_i^n$

(2) Analysis—Expanding $\phi_i^{n+1}, \phi_{i\pm1}^{n}$ at point (i,n) by Taylor series (with respect to both space and time), and substituting the series into the discretized equation and rearranging into the form of two operators:

For1-D discretized model equation (FTCS):

$$\underline{\rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}+\rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}-\Gamma\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{\Delta x^2}-S_i^n}-\{\underline{\rho\frac{\partial\phi}{\partial t}+\rho u\frac{\partial\phi}{\partial x}-\Gamma\frac{\partial^2\phi}{\partial x^2}}-S\}_{i,n}=\underbrace{O(\Delta t,\Delta x^2)}_{T.E.}$$

How to get this result? Taking transient term as an example:

transient term

$$\rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=\rho\frac{\cancel{\phi_i^n}+\left(\frac{\partial\phi}{\partial t}\right)_{i,n}\Delta t+\left(\frac{\partial^2\phi}{\partial t^2}\right)_{i,n}\frac{\Delta t^2}{2!}+\ldots-\cancel{\phi_i^n}}{\Delta t}$$

i.e.

$$\rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}-\rho\left(\frac{\partial\phi}{\partial t}\right)_{i,n}=\frac{1}{2}\frac{\partial^2\phi}{\partial t^2}\Delta t+\ldots.=O(\Delta t)$$

Similarly, for the convection term:

$$\rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}=\rho u\left[\frac{\cancel{\phi_i^n}+\Delta x\frac{\partial\phi}{\partial x}+\frac{1}{2}\Delta x^2\cancel{\frac{\partial^2\phi}{\partial x^2}}+O(\Delta x^3)}{2\Delta x}-\frac{\left(\cancel{\phi_i^n}-\Delta x\frac{\partial\phi}{\partial x}+\frac{1}{2}\Delta x^2\cancel{\frac{\partial^2\phi}{\partial x^2}}+O(\Delta x^3)\right)}{2\Delta x}\right]$$

$$=\frac{2\rho u\frac{\partial\phi}{\partial x}+O(\Delta x^3)}{2\Delta x}=\rho u\frac{\partial\phi}{\partial x}+O(\Delta x^2)$$

Thus:

$$\rho u \frac{\phi_{i+1}^{n}-\phi_{i-1}^{n}}{2\Delta x}-\rho u\left(\frac{\partial \phi}{\partial x}\right)_{i,n}=O(\Delta x^{2})$$

Similarly:

$$\Gamma \frac{\phi_{i+1}^{n}-\phi_{i}^{n}+\phi_{i-1}^{n}}{\Delta x^{2}}-\Gamma \frac{d^{2}\phi}{dx^{2}}=O(\Delta x^{2})$$

Assuming that the source term does not introduce any truncation error, then:

The T.E. of FTCS scheme for 1-D model equation:

$$\boxed{O(\Delta t, \Delta x^{2})}$$

Its mathematical meaning is:

Existing two positive constants，K1，K2，when $\Delta t \to 0, \Delta x \to 0$ **the difference between the two operators will be less than** $(\mathrm{K}_1 \Delta t+\mathrm{K}_2 \Delta x^{2})$

4. Consistence (相容性)of discretized equations

If the T.E. of discretized equations approaches zero when $\Delta t \to 0, \Delta x \to 0$ then:

the discretized equations are said to be in consistence with the PDE.

When T.E. is in the form of $O(\Delta t^n, \Delta x^m)(n, m > 0)$

the discretized equations possess(具有) consistence;

However when T.E. contains $\Delta t / \Delta x$ only when the time step approaches zero much faster than space step , the consistence can be guaranteed (保证).

3.1.2 Discretization error and convergence

1. Discretization error(离散误差) ρ_i^n

$$\rho_i^n = \phi(i,n) - \phi_i^n$$

Analytical solution of PDE

Analytical solution of FDE

2. Factors affecting discretization error

（1） **T.E.**： The higher the order， the smaller the value of ρ_i^n for the same grid system;

（2） **Grid step**： For the same T.E., a finer grid system leads to less error.

For conventional engineering simulation:

Diffusion term—2nd order, convection term — 2nd or 3rd order .

3. Convergence (收敛性) of discretized equations

When $\Delta t \rightarrow 0, \Delta x \rightarrow 0$ **if** $\rho_i^n \rightarrow 0$ **then it is said:**

The discretized equations possess convergence.

Proving convergence is not easy.

It should be noted :that above descriptions of consistence and convergance are only qualitatively (定性地）, not in the strict mathematical sense. But enough for enginerring students.

Quantitatively----定量地

3.1.3 Round-off error(舍入误差) and stability of initial problems

1. Round-off error ε_i^n

$$\varepsilon_i^n = \phi_i^n - \phi_i^n$$

ϕ_i^n **--actual solution from computer**

2. Factor affecting round-off error

Length of computer word; Numerical solution method

3. Errors of numerical solutions

$$\phi(i,n) - \phi_i^n = \underline{\phi(i,n) - \phi_i^n} + \underline{\phi_i^n - \phi_i^n} = \rho_i^n + \varepsilon_i^n$$

For most engineering problems, generally ρ_i^n is predominant (占优).

4. Stability of initial problems

The solution procedure of an initial problem is of marching (步进)type; if errors introduced at any time level are enlarged(放大) in the subsequent （随后的） simulation such that the solutions become infinite（无限）, this scheme is called **unstable（不稳定）**; Otherwise the scheme for the initial problem is **stable**.

Stability is an inherent (固有的）character of a scheme，no matter what kind of error is introduced.

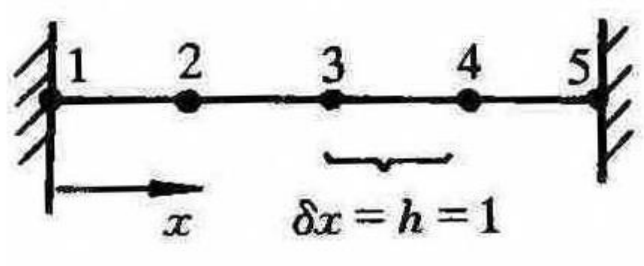
Scheme stability should not be confused (搞混） with the stability of solution procedure for algebriac equations.

3.1.4 Example

[Example 3－1] Effect of T.E. and grid number

$$\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}-2\phi=0$$

$$\phi(0)=0;\ \phi(4)=1$$



Find: Values of nodes 2, 3 and 4.

Solution: By FDM: replacing $\frac{d^2\phi}{dx^2}, \frac{d\phi}{dx}$ **by FDExp.**

For 2nd order scheme, FDEqs can be established For Node 2, 3, 4;

In addition for Node 3 fourth order scheme may be adopted:

The analytical solution

$$\phi = \frac{e^{x} - e^{2x}}{e^{4} - e^{-8}}$$

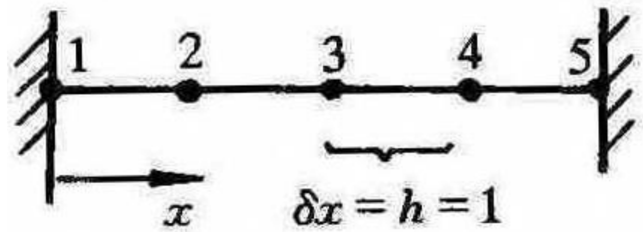


表 3-1　不同格式计算结果的比较

格　式	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
精确解	0	0.047 3	0.135 0	0.367 9	1
(i=2,3,4) 二阶格式	0	0.058 2	0.155 2	0.394 4	1
(i=3) 四阶格式	0	0.050 5	0.134 8	0.391 8	1

The fourth order scheme is only adopted at Node 3, while solution accuracy is greatly improved

表 3-2 网格疏密的影响(二阶格式)

区间数	4	8	16	32	64	精确解
$\phi_{x=1}$	0.058 2	0.050 2	0.048 0	0.047 5	0.047 3	0.047 3
$\phi_{x=2}$	0.155 2	0.140 4	0.136 4	0.135 3	0.135 0	0.135 0
$\phi_{x=3}$	0.394 4	0.375 2	0.369 7	0.368 3	0.367 9	0.367 9

Solution of 32 intervals (区间)may be regarded as grid-independent!

[Ex. 3-3] Instability of explicit scheme

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2},\ 0<x<1$$

$$t\le 0,\ T=2x,\ 0\le x\le 0.5; T=2(1-x), 0.5\le x\le 1$$

$$t>0,\ T(0,t)=T(1,t)=0$$

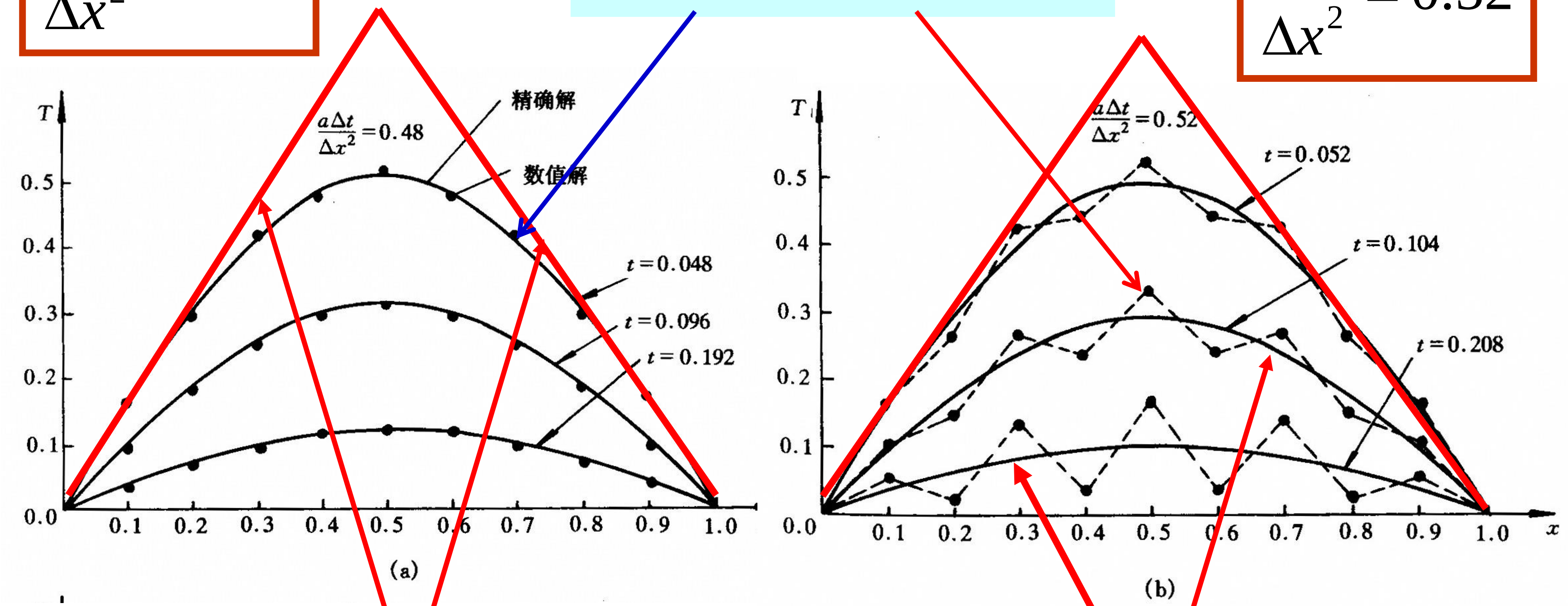
Solution: Numerical solutions are conducted for

$$\frac{a\Delta t}{\Delta x^2} = 0.48, \text{ and } 0.52$$

$$\frac{a\Delta t}{\Delta x^2} = 0.48$$

Numerical solution

$$\frac{a\Delta t}{\Delta x^2} = 0.52$$



Initial field

Analytical

3.2 von Neumann Method for Analyzing Stability of Initial Problems

3.2.1 Propagation of error vector with time

3.2.2 Discrete Fourier expansion

3.2.3 Basic idea of von Neumann analysis

3.2.4 Examples of von Neumann analysis

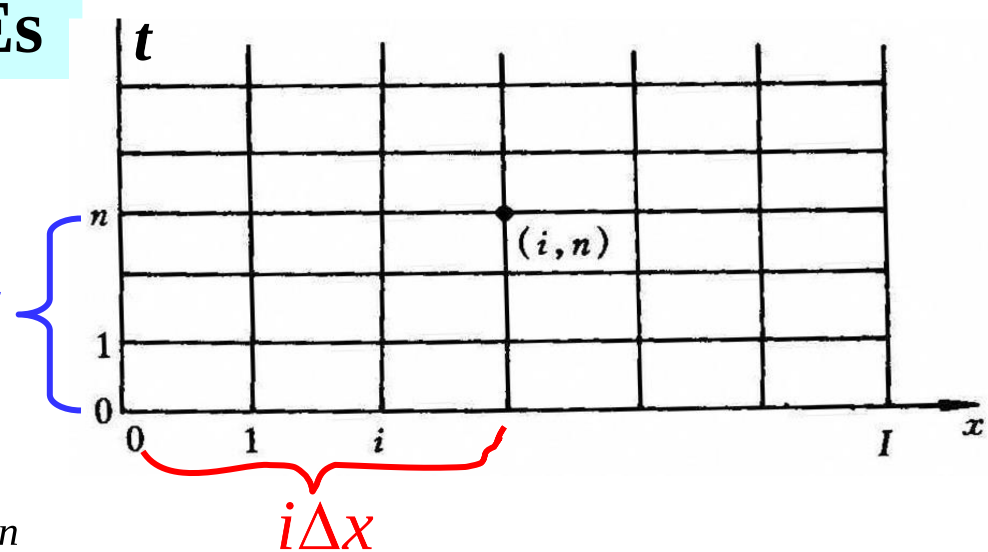
3.2.5 Discussion on von Neumann analysis

3.2 von Neumann Method for Analyzing Stability of Initial Problems

3.2.1 Propagation(传递) of error vector with time

1 Matrix expression of DEs

$$\begin{cases} \dfrac{\partial T}{\partial t} = a\dfrac{\partial^2 T}{\partial x^2}, \quad 0 < x < L \\ T(x,0) = F(x) \\ T(0,t) = f_1(t),\ T(L,t) = f_2(t) \end{cases}$$



$$\begin{cases} \dfrac{T_i^{n+1} - T_i^n}{\Delta t} = a\dfrac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}, \ i = 1, 2, , 3, ...(I-1) \\ T_i^0 = F(x_i), \ i = 0, 1, 2, 3.....I \\ T_0^n = f_1(n\Delta t), \ T_I^n = f_2(n\Delta t), \ n = 1, 2, ... \end{cases}$$

Set $\dfrac{a\Delta t}{\Delta x^2} = r,$ **The difference eqs. can be expressed as**

$$\boxed{T_i^{n+1} = T_i^n(1-2r) + r(T_{i+1}^n + T_{i-1}^n),\ i = 1, 2, \ldots.(I-1)}$$

For a fixed time level n,the above eqs. can be re-written for each inner point as follows:

$$\begin{cases} i = 1, & T_1^{n+1} = T_1^n(1-2r) + r(T_0^n + T_2^n) \\ i = 2, & T_2^{n+1} = T_2^n(1-2r) + r(T_1^n + T_3^n) \\ i = 3, & T_3^{n+1} = T_3^n(1-2r) + r(T_2^n + T_4^n) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ i = I-2, & T_{I-2}^{n+1} = T_{I-2}^n(1-2r) + r(T_{I-1}^n + T_{I-3}^n) \\ i = I-1, & T_{I-1}^{n+1} = T_{I-1}^n(1-2r) + r(T_I^n + T_{I-2}^n) \end{cases}$$

Expressed in matrix (矩阵)form:

$$\begin{bmatrix} T_1^{n+1}\\ T_2^{n+1}\\ T_3^{n+1}\\ \cdot\\ \cdot\\ \cdot\\ T_{I-2}^{n+1}\\ T_{I-1}^{n+1}\end{bmatrix} = \begin{bmatrix} (1-2r) & r & & & & \\ r & (1-2r) & r & & \mathbf{0} & \\ & r & (1-2r) & r & & \\ \cdot & & & & & \\ \cdot & & \mathbf{0} & & & \\ \cdot & & & & & \\ & & & r & (1-2r) & r \\ & & & & r & (1-2r)\end{bmatrix} \begin{bmatrix} T_1^{n}\\ T_2^{n}\\ T_3^{n}\\ \cdot\\ \cdot\\ \cdot\\ T_{I-2}^{n}\\ T_{I-1}^{n}\end{bmatrix} + \begin{bmatrix} rT_0^{n}\\ 0\\ 0\\ \cdot\\ \cdot\\ 0\\ rT_I^{n}\end{bmatrix}$$

$\vec{T}^{n+1}$ **Column vector(列矢量)** $\vec{A}$ **Matrix(矩阵)** $\vec{T}^{n}$ $\vec{g}$

Simply:

$$\left.\begin{aligned} \vec{T}^{n+1} &= \vec{A}\vec{T}^{n} + \vec{g} \\ \vec{T}^{0} &= \vec{F} \end{aligned}\right\} \qquad \text{(a)}$$

$\vec{A}$ represents a transformation(变换) from $\vec{T}^{n}$ to $\vec{T}^{n+1}$.

2 Propagation(传递) of error vector with time

Assuming that no error is introduced at the boundary, while it is introduced at the initial condition . Then the error components at each node form(形成) an error vector, denoted by $\vec{\varepsilon}^{0}$; For the exact solution:

$$\left.\begin{aligned}\vec{T}^{n+1} &= \vec{A}\vec{T}^{n} + \vec{g}\\ \vec{T}^{0} &= \vec{F}\end{aligned}\right\}\text{(a)}$$

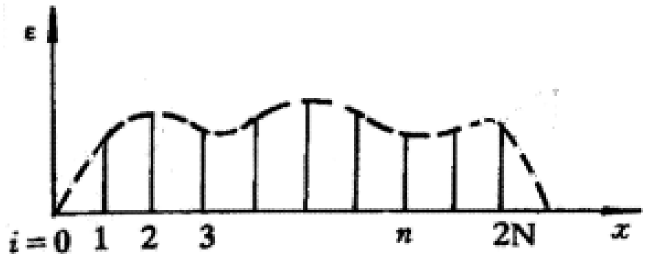
then

$$\left.\begin{aligned}\vec{\tilde{T}}^{n+1} &= \vec{A}\vec{\tilde{T}}^{n} + \vec{g}\\ \vec{\tilde{T}}^{0} &= \vec{F} + \vec{\varepsilon}^{0}\end{aligned}\right\}\text{(b)}$$

Denoting the solution with error by $\vec{\tilde{T}}$.

(b)−(a) →

$$\left\{\begin{aligned}\vec{\tilde{T}}^{n+1} - \vec{T}^{n+1} &= \vec{A}(\vec{\tilde{T}}^{n} - \vec{T}^{n})\\ \vec{\tilde{T}}^{0} - \vec{T}^{0} &= \vec{\varepsilon}^{0}\end{aligned}\right.$$



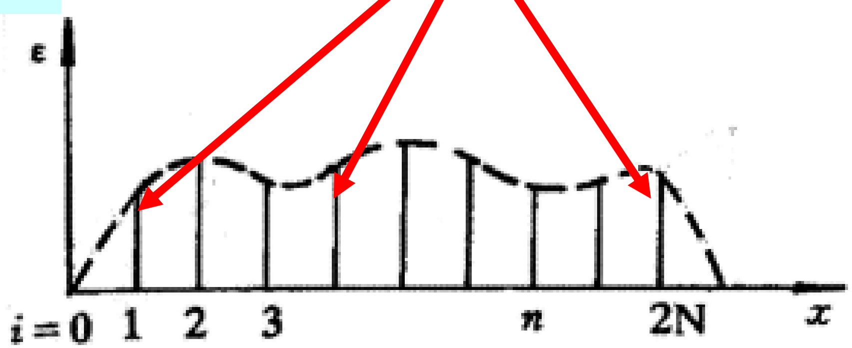
$\vec{\varepsilon}^{\,n+1} = \vec{A}\vec{\varepsilon}^{\,n}$, **with** $\vec{\varepsilon}^{\,0}$ **being specified** (给定) } **(c)**

Thus the propagation of error vector can be described by matrix $\vec{A}$ under the condition that:

No error is introduced at the boundary!

3 Expression of error vector (误差矢量的表示方法)

Discrete components

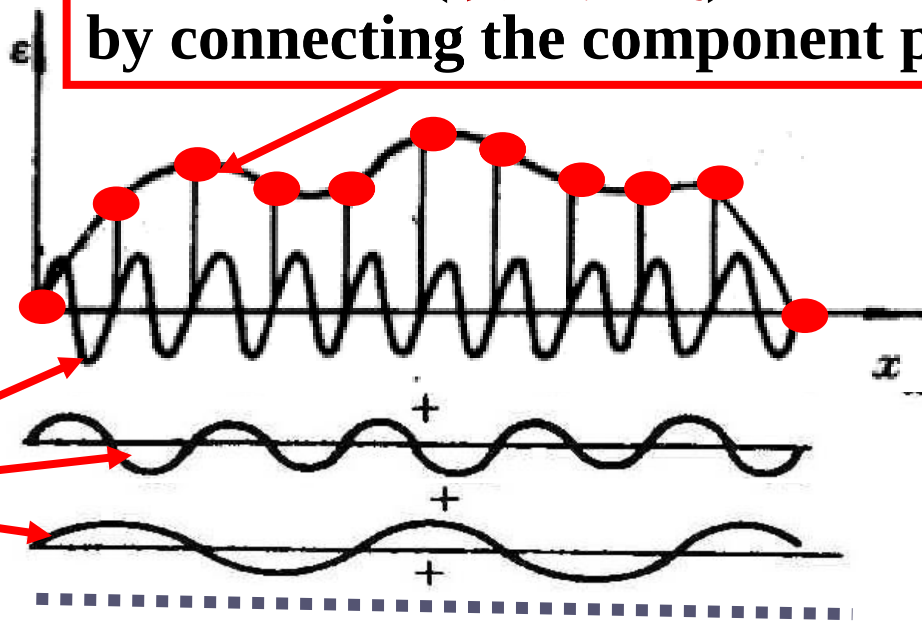


(1) Expressed by discrete components（离散分量）

Error curve (误差曲线)obtained by connecting the component po

(2) Expressed by harmonic components

(谐波分量)



Harmonic components

3.2.2 Discrete FOURIER expansion(离散傅里叶展开)

1 Expansion eq. Similar to Fourier expansion for continuous function within the region $[-l, l]$, (2N+1) pair of numbers (数对) , (x_i, y_i) ,can be expressed by a summation of harmonic components (谐波分量) ：

Continuous FOURIER exp.(连续傅里叶展开)

FOURIER exp. for finite pair of numbers(有限个数对的傅氏展开)

Continuous func.within $[-l, l]$

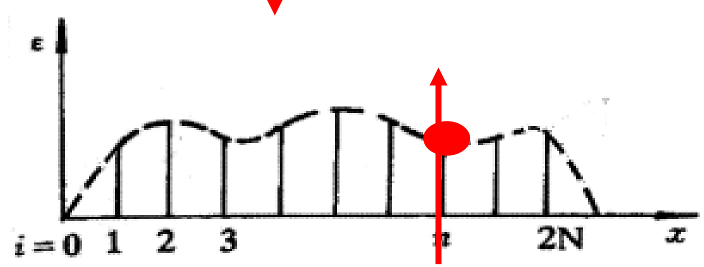
$$f(x) = y = \sum_{n=-\infty}^{\infty} C_n e^{I\left(\frac{2n\pi}{2l}\right)x}$$

$$I = \sqrt{-1}$$

(2N+1) pair of numbers

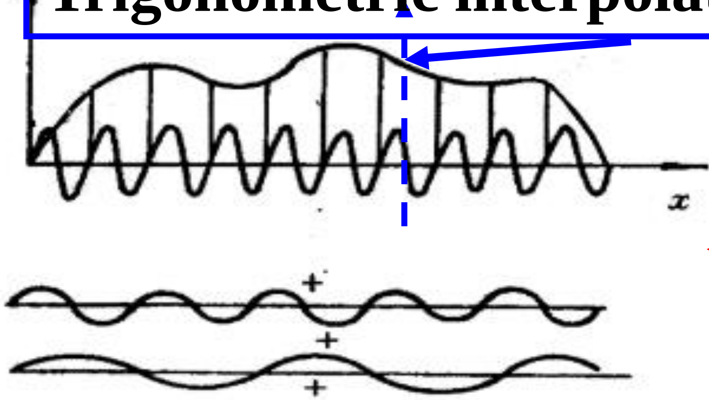
$$y_i = \sum_{k=-N}^{N} C_k e^{I\left(\frac{2k\pi}{2N+1}\right)x_i}$$

subscripts $i = 0, 1, 2, \ldots\ldots 2N$



Sum of harmonic components

Trigonometric interpolation



$$x = x_i, y = y_i;$$

When x is between x_i y is the interpolation by finite terms of trigonometric functions

2. Expression of harmonic component

Corresponding to the term $(\frac{2n\pi}{2l})x$ **in Fourier expansion**

$$(\frac{2k\pi}{2N+1})x_i = (\frac{2k\pi}{2N+1})i\Delta x = i(\frac{2k\pi}{2N+1})\Delta x = ik_x\Delta x = i\theta_k$$

Then $C_k e^{I(\frac{2k\pi}{2N+1})x_i} = C_k e^{Ii\theta_k}$

k_x **—wave number,** $k_x\lambda = 2\pi$, θ **—phase angle**

$C_k e^{Ii\theta_k}$ **harmonic component,** C_k **— amplitude(振幅)**

In transient problem it is a function of time, $\psi(t)$

The general expression of harmonic component is then

$\psi(t)e^{Ii\theta}$ **--Purpose of discussion on discrete Fo-Expansion**

3.2.3 Basic idea of von Neumann analysis

1. Basic idea

The numerical error is considered as a kind of disturbances(扰动)，which can be decomposed(分解) into a finite number of harmonic components; If some discretized scheme can guarantee that the amplitude of any component will be attenuated (衰减) or at least be kept unchanged in the calculation procedure then the scheme is stable; Otherwise it is unstable

2. Analysis method

How to implement (实施)this idea? Replacing the dependent variable by the expression of harmonic component , finding the ratio of amplitude of two subsequent time levels, and demanding (要求)that

$$\left|\frac{\psi(t+\Delta t)}{\psi(t)}\right| \le 1$$

The condition of this inequality is the criterion of scheme stability.

The ratio is called magnified factor(放大因子)

3.2.4 Examples

1. Stability analysis for FTCS of 1-D conduction eq.

Replacing *T* in the discretized eq. by $\varepsilon(t)=\psi(t)e^{Ii\theta}$

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=a\frac{T_{i+1}^n-2T_i^n+T_{i-1}^n}{\Delta x^2}$$

yields $\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}e^{Ii\theta}=a\psi(t)\frac{e^{I(i+1)\theta}-2e^{Ii\theta}+e^{I(i-1)\theta}}{\Delta x^2}$

Divided by $e^{Ii\theta}$ **and from Euler Eq.** $\boxed{e^{I\theta}=\cos\theta+I\sin\theta}$

Rearranging, $\frac{\psi(t+\Delta t)}{\psi(t)}=1-2\left(\frac{a\Delta t}{\Delta x^2}\right)(1-\cos\theta)$

$$\boxed{1-\cos\theta = 2(\sin\frac{\theta}{2})^2} \longrightarrow \boxed{\frac{\psi(t+\Delta t)}{\psi(t)} = 1-4(\frac{a\Delta t}{\Delta x^2})\sin^2(\frac{\theta}{2})}$$

Stability requires:

$$-1 \le \frac{\psi(t+\Delta t)}{\psi(t)} \le 1$$

i.e.,

$$-1 \le \underbrace{1-4(\frac{a\Delta t}{\Delta x^2})\sin^2(\frac{\theta}{2}) \le 1}_{\text{Automatically satisfied}}$$

Thus, it is required:

$$-1 \le 1-4(\frac{a\Delta t}{\Delta x^2})\sin^2(\frac{\theta}{2}) \longrightarrow \boxed{4(\frac{a\Delta t}{\Delta x^2})\sin^2(\frac{\theta}{2}) \le 2}$$

This requirement should be satisfied for all possible values of θ, the most severe case is $\sin^2(\frac{\theta}{2}) = 1$

$$\boxed{4\left(\frac{a\Delta t}{\Delta x^2}\right)\sin^2\left(\frac{\theta}{2}\right) \le 2} \quad \text{if } \sin^2\left(\frac{\theta}{2}\right) = 1 \quad \rightarrow \quad \frac{a\Delta t}{\Delta x^2} \le \frac{1}{2}$$

Discussion: **The above derived stability criterion can be applied only for internal nodes, because it is assumed that at the boundary no error is introduced. ; For the 2nd and 3rd kinds of B.C. the criterion may be obtained from the discretized equations obtained by balance method by requiring that the coefficient of neighbors must be possitive !**

It is called von Neumann method: concept is clear , and implementation is easy !

2. Stability criterion of FTCS scheme of 1-D model eq.

Replacing ϕ **by** $\varepsilon(t) = \psi(t)e^{Ii\theta}$ **in the discretized eq.**

$$\rho\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}+\rho u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}=\Gamma\frac{\phi_{i+1}^n-2\phi_i^n+\phi_{i-1}^n}{\Delta x^2}$$

$$\rho\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}e^{Ii\theta}+\rho u\psi(t)\frac{e^{I(i+1)\theta}-e^{I(i-1)\theta}}{2\Delta x}=$$

$$\Gamma\psi(t)\frac{e^{I(i+1)\theta}-2e^{Ii\theta}+e^{I(i-1)\theta}}{\Delta x^2}$$

$$\boxed{\frac{\Gamma}{\rho}=a}$$

Rearranging, yields:

$$\frac{\psi(t+\Delta t)}{\psi(t)}=\mu=1-\frac{1}{2}\left(\frac{u\Delta t}{\Delta x}\right)\underbrace{(e^{I\theta}-e^{-I\theta})}_{2I\sin\theta}+\left(\frac{a\Delta t}{\Delta x^2}\right)\underbrace{(e^{I\theta}-2+e^{-I\theta})}_{(2\cos\theta-2)}$$

Set $c=\dfrac{u\Delta t}{\Delta x}$ **(Courant number) and** $r=\dfrac{a\Delta t}{\Delta x^2}$

Courant the supervisor of Professor G J Zhu (朱公瑾)

《柯士微积分》

$$\frac{\psi(t+t)}{\psi(t)} = \underline{1-2r+2r\cos\theta} - \underline{Ic\sin}\theta$$ **Complex variable**

Stability requires: $\left|1-2r+2r\cos\theta - Ic\sin\theta\right| \le 1$

How to get criterion? Analysis and graphics. The later has advantages of clear concept and easy to be implemented.

The locus (轨迹) of the complex represents an E.C.

Graphics: $\mu = \underline{1-2r} + \underline{2r}\cos\theta - I\underline{c}\sin\theta$

Center | **Radius of long axis** | **Radius of short axis**

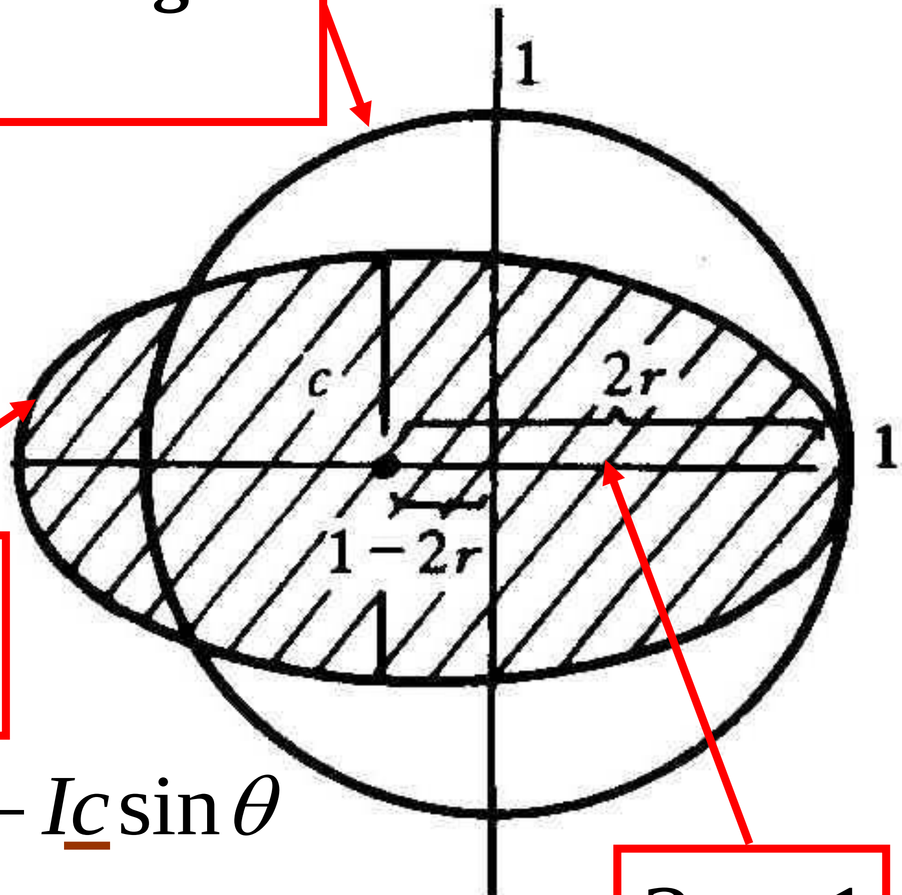
For $|\mu| \le 1$, the locus of the elliptic circle(椭圆) must be within the unit circle with its center at coordinate origin(原点).

Possible case 1

Circle with its center at origin and radius of 1

$(a)\ 2r > 1, c \le 1$

Locus of the elliptic circle



$$\mu = \underline{1 - 2r} + \underline{2r}\cos\theta - I\underline{c}\sin\theta$$

Center | **Radius of long axis** | **Radius of short axis**

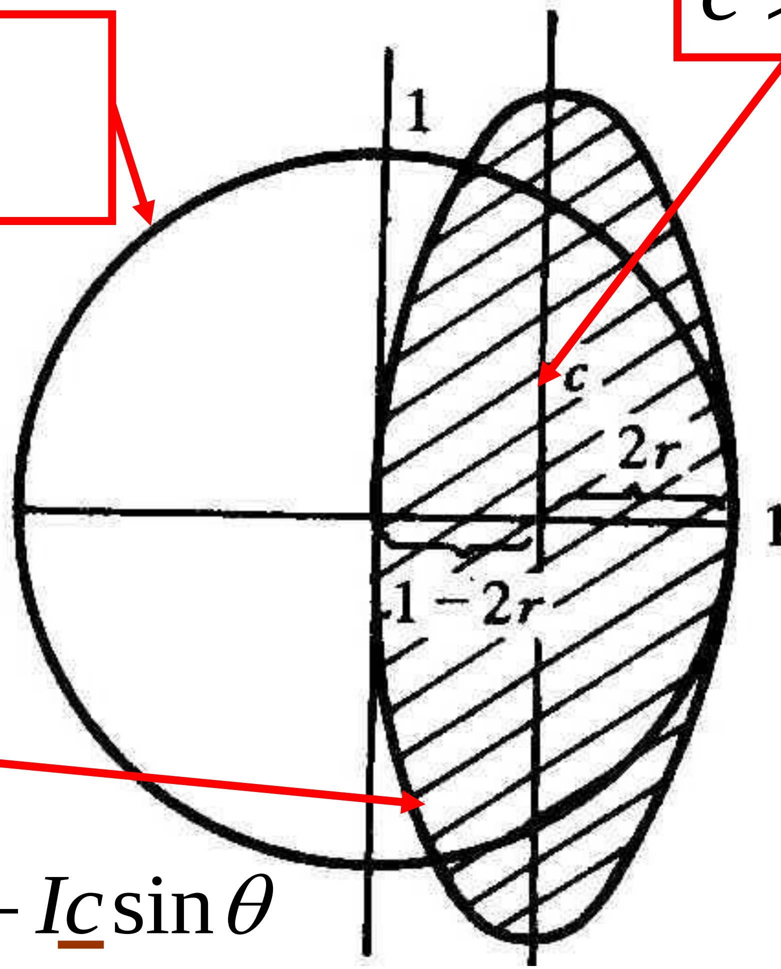
$2r > 1$

Possible case 2

$c > 1$

Circle with its center at origin and radius of 1

$\cancel{(b)\ 2r \le 1,\ c > 1}$



Locus of the elliptic circle

$$\mu = \underline{1 - 2r} + \underline{2r}\cos\theta - I\underline{c}\sin\theta$$

Center

Radius of long axis

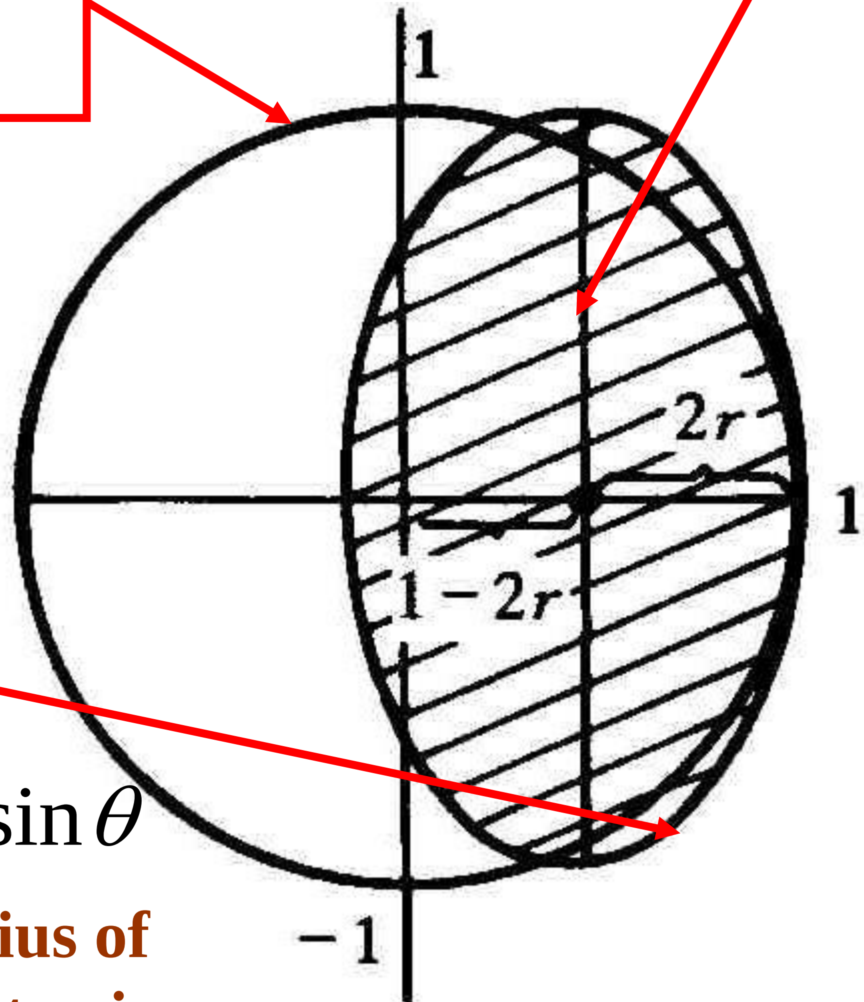
Radius of short axis

Possible case 3

$c < 1$

Circle with its center at origin and radius of 1

$\cancel{(c)\ 2r \le 1, c \le 1}$



Locus of the elliptic circle

$$\mu = \underline{1 - 2r} + \underline{2r}\cos\theta - I\underline{c}\sin\theta$$

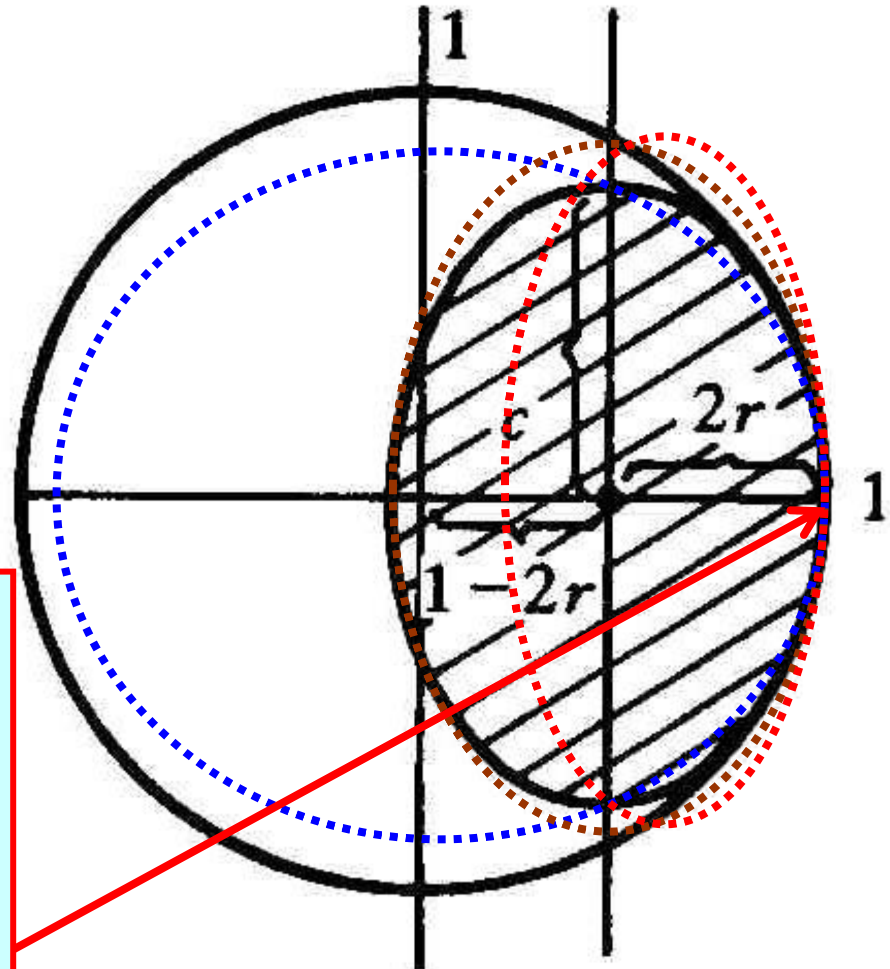
Center

Radius of long axis

Radius of short axis

Possible case 4

$$(d)\ 2r \le 1, c^2 \le 2r$$



The curvature(曲率) radius at the right end of the elliptical circle should be less than at most equal to 1.

Proof: **Magnified factor is:** $\mu = \underline{1 - 2r + 2r\cos\theta} - \underline{Ic\sin\theta}$

The parameter equation (参数方程) of the E.C. is

$$x = 1 - 2r + 2r\cos\theta,\ \ y = c\sin\theta$$

The curvature radius (曲率半径) is:

$$R = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\left|\dot{x}\ddot{y} - \ddot{x}\dot{y}\right|}$$

At the right end,

where $\theta = 0$ **, it is required that** $R \le 1$ **, yields:**

$$R = \frac{c^3}{2rc} \le 1 \rightarrow c^2 \le 2r$$

Thus the stability condition of FTCS for 1-D model eq.:

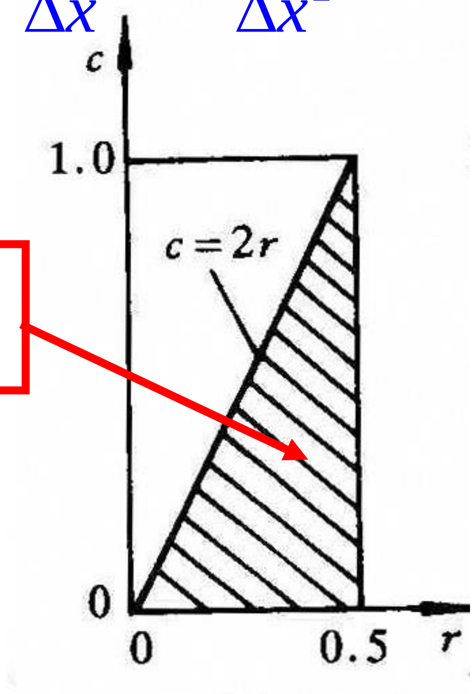
$$2r \le 1; \qquad c^2 \le 2r$$

Discussion: **Historically it was considered that:**

From $2r \le 1;\ c \le 1 \quad \longrightarrow \quad c \le 2r$

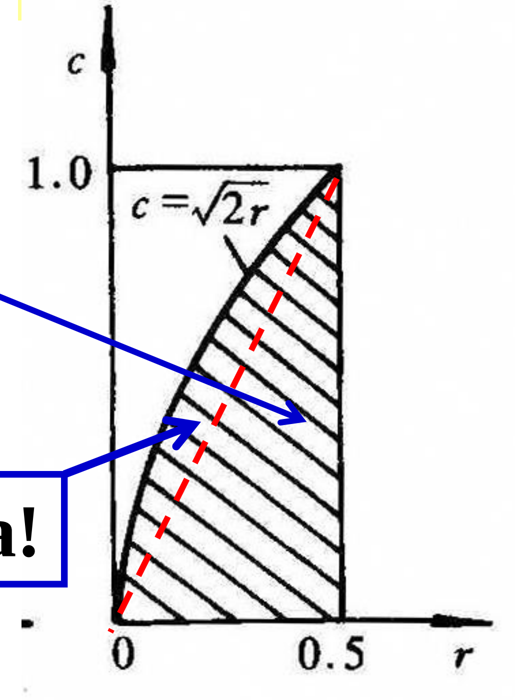
$$c \le 2r \rightarrow \frac{u\Delta t}{\Delta x} \le 2\frac{a\Delta t}{\Delta x^2} \rightarrow \frac{u\Delta x}{a} \le 2 \quad \textbf{or} \quad \mathrm{Re}_\Delta \le 2;\ \ \mathrm{Pe}_\Delta \le 2$$

$$\boxed{c \le 2r}$$



$$\boxed{c^2 \le 2r}$$

Extra area!



3.2.5 Application discussion**

1.It is applicable to linear transient problem, leading to the maximum allowable(允许的) time step;

2.For non-linear transient problems（transient NS Eqs.）locally linearized(局部线性化)approximation may be adopted : Analyzing the problem as it was linear and making a reduction of the resulting time step, say taking 80%;

3. It is a very useful analysis tool. It may be used to reveal the major concept of MG method(多重网格), to analyze the stability condition for iterative solution of algebraic equations.

Ni MJ, Tao WQ, Wang SJ .Stability analysis for discretized steady convective-diffusion equation. Numerical Heat Transfer, Part B,1999, 35 (3): 369-388

3.3 Conservation of Discretized Equations

3.3 Conservation of Discretized Equations

3.3.1 Definition and analyzing model

1. Definition

If the summation of a certain number of discretized equations over a finite volume satisfies conservation requirement , these discretized equations are said to possess conservation(具有守恒性).

2. Analyzing model---advection equation

It is easy to show that CD of diffusion term possesses conservation. Discussion is only performed for the equation which only has transient term and convective term (advection equation, 平流方程).

Advection equation

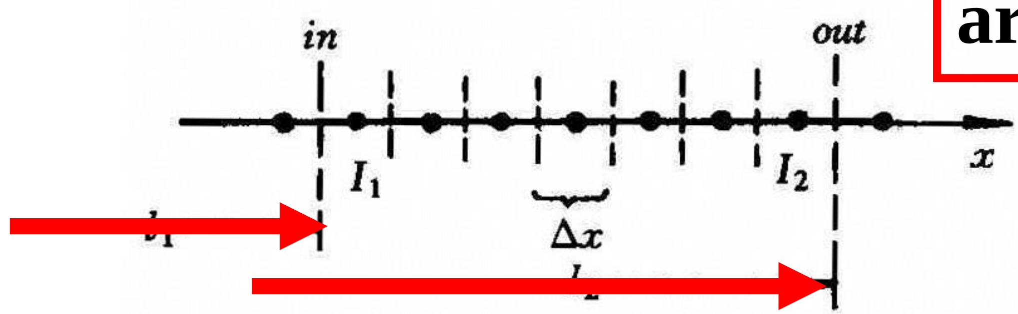
$$\begin{cases} \rho \dfrac{\partial \phi}{\partial t} + \rho \dfrac{\partial (u\phi)}{\partial x} = 0 & \textbf{(Conservative)} \\[2ex] \rho \dfrac{\partial \phi}{\partial t} + \rho u \dfrac{\partial \phi}{\partial x} = 0 & \textbf{(Non-conservative)} \end{cases}$$

3.3.2 Direct summation method

Summing up FTCS scheme of advection eq. of conservative form over the region of $[I_1, I_2]$:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -\frac{u_{i+1}\phi_{i+1} - u_{i-1}\phi_{i-1}}{2\Delta x}$$

Time level of the spatial terms are not shown



$$\sum_{I_1}^{I_2}\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=-\sum_{I_1}^{I_2}\frac{u_{i+1}\phi_{i+1}-u_{i-1}\phi_{i-1}}{2\Delta x}=-\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2\Delta x}$$

$$\underbrace{\sum_{I_1}^{I_2}(\phi_i^{n+1}-\phi_i^n)\Delta x}=\underline{-\Delta t\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2}}$$

Increment(增值) of ϕ within Δt and $[I_1, I_2]$

Is it equal to the net amount of ϕ entering the space region within the same time step?

Analyzing should be made for the right hand terms of the equation:

$$-\Delta t\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2}=\frac{\Delta t}{2}\underline{\sum_{I_1}^{I_2}[(u\phi)_{i-1}-(u\phi)_{i+1}]}$$

Directly summing up: for the region left end, we have:

$$i = I_1 \qquad (u\phi)_{I_1-1} \qquad \cancel{-(u\phi)_{I_1+1}}$$

$$i = I_1 + 1 \qquad (u\phi)_{I_1} \qquad \cancel{-(u\phi)_{I_1+2}}$$

$$i = I_1 + 2 \qquad \cancel{(u\phi)_{I_1+1}} \qquad \cancel{-(u\phi)_{I_1+3}}$$

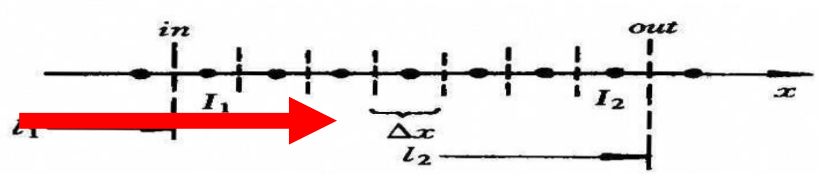
$$i = I_1 + 3 \qquad \cancel{(u\phi)_{I_1+2}} \qquad \cancel{-(u\phi)_{I_1+4}}$$

$$i = I_1 + 4 \qquad \cancel{(u\phi)_{I_1+3}}$$

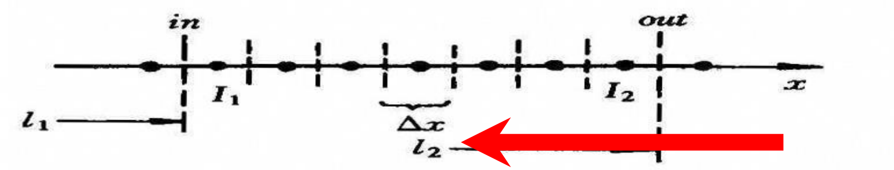
....

....

$$(u\phi)_{I_1-1} + (u\phi)_{I_1}$$



For the region right end:



$$
\begin{array}{llll}
i = I_2 - 3 & \cancel{(u\phi)_{I_2-4}} & \cancel{-(u\phi)_{I_2-2}} & \\
i = I_2 - 2 & \cancel{(u\phi)_{I_2-3}} & \cancel{-(u\phi)_{I_2-1}} & \\
i = I_2 - 1 & \cancel{(u\phi)_{I_2-2}} & -(u\phi)_{I_2} & \\
i = I_2 & \cancel{(u\phi)_{I_2-1}} & & -(u\phi)_{I_2+1}
\end{array}
$$

$$-[(u\phi)_{I_2} + (u\phi)_{I_2+1}]$$

Then: $\dfrac{\Delta t}{2}\sum_{I_1}^{I_2}[(u\phi)_{i-1} - (u\phi)_{i+1}]$

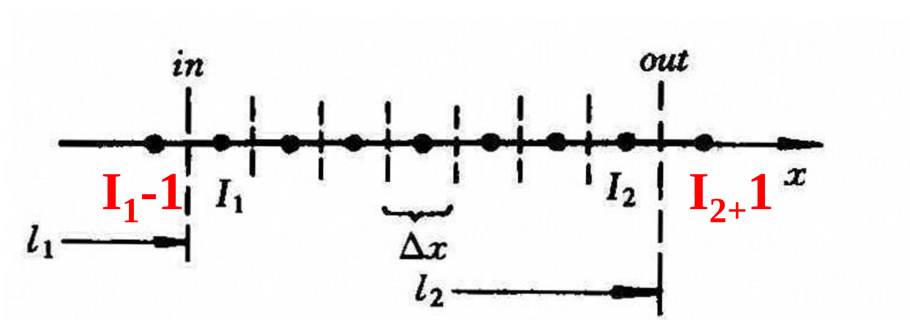
$$= \frac{\Delta t}{2}\{\underbrace{[(u\phi)_{I_1-1} + (u\phi)_{I_1}]}_{\text{Left end of domain}} - \underbrace{[(u\phi)_{I_2} + (u\phi)_{I_2+1}]}_{\text{Right end of domain}}\}$$

Further: $$\frac{\Delta t}{2}\{[(u\phi)_{I_1-1}+(u\phi)_{I_1}]-[(u\phi)_{I2}+(u\phi)_{I_2+1}]\}=$$

$$\Delta t\{[\frac{(u\phi)_{I_1-1}+(u\phi)_{I_1}}{2}]-[\frac{(u\phi)_{I2}+(u\phi)_{I_2+1}}{2}]\}$$

CD-uniform grid

$$=\Delta t(\phi\, flowin-\phi\, flowout)$$



Thus the central difference discretization of the convective term possesses conservative feature.

3.3.3 Conditions guaranteeing conservation

1.Governing equation should be conservative

For non-conservative form:

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$

Its FTCS scheme is $$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -u_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}$$

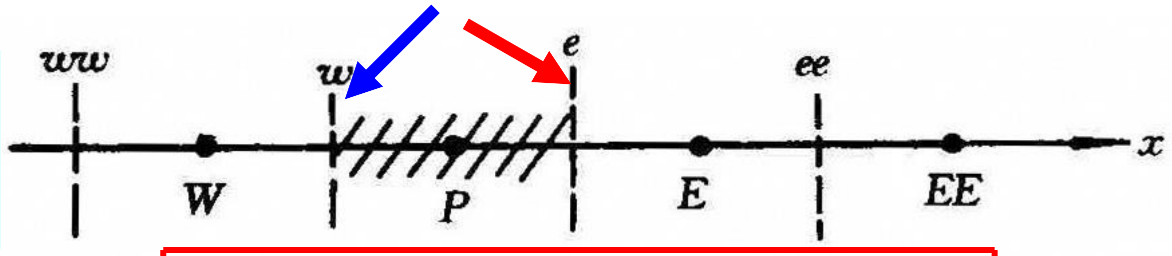
By direct summation, the results do not possess conservation because of no cancellation (抵消) can be made for the interface terms.

$$\boxed{u_i\phi_{i-1} - u_i\phi_{i+1} \neq 0}$$

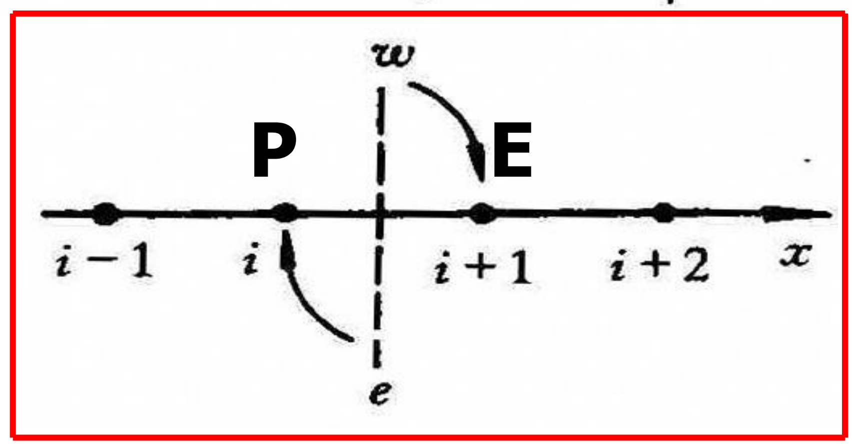
2. Dependent variable and its 1st derivative are continuous at interface

Meaning of “Continuous”

Different interfaces viewed from point P



The same interface viewed from two points, P and E



By “Continuous” we mean:

$$(\phi_e)_P = (\phi_w)_E ;$$

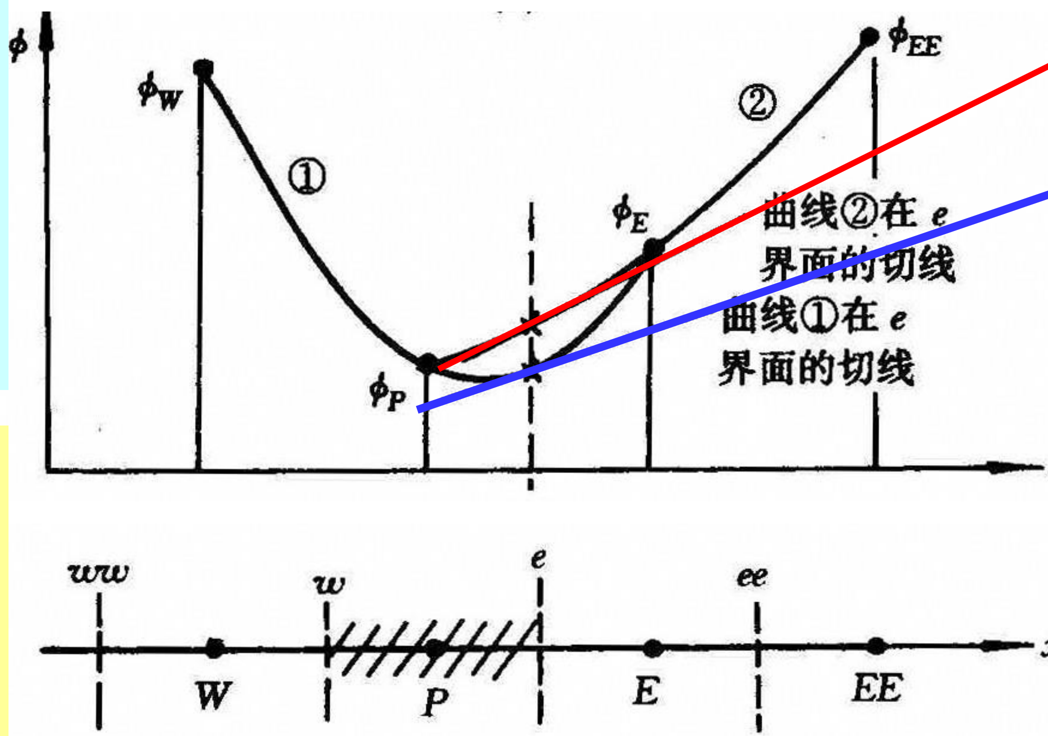
$$[(\frac{\delta\phi}{\delta x})_e]_P = [(\frac{\delta\phi}{\delta x})_w]_E$$

The piecewise linear profile can meet this condition.

Interface-biased quadratic (界面偏向的二次插值) can not satisfy such requirement

For west side of the interface, W, P and E are used for interpolation

For east side of the interface, P , E and EE are used for interpolation



Tangential to P-E-EE at interface

Tangential to W-P-E at interface

First order derivatives are not equal!

3.3.4 Discussion－expected but not necessary

Contents

3.4 Transportive (迁移)Character of Discretized Equations

3.4.1 Essential (基本的) difference between convection and diffusion

3.4.2 CD of diffusion term can propagate(传播) disturbance all around (四面八方) uniformly

3.4.3 Analysis of transport character of discretized scheme of convection term

3.4.4 Upwind scheme of convection term possesses transport character

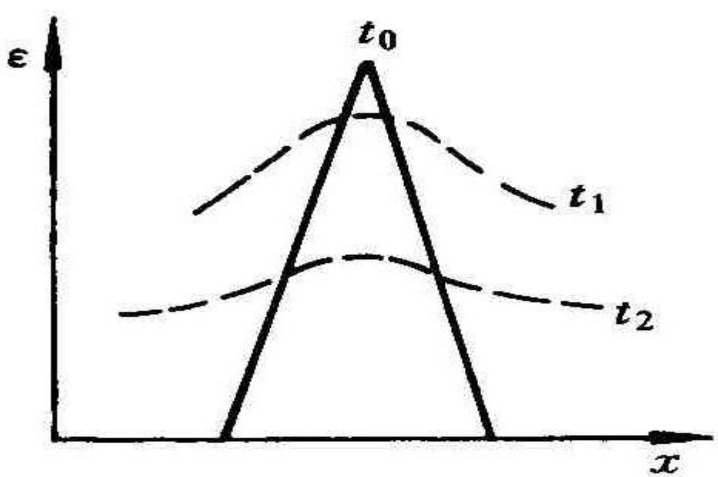
3.4.5 Discussion on transport character of discretized convection term

3.4 Transportive Property of Discretized Equations

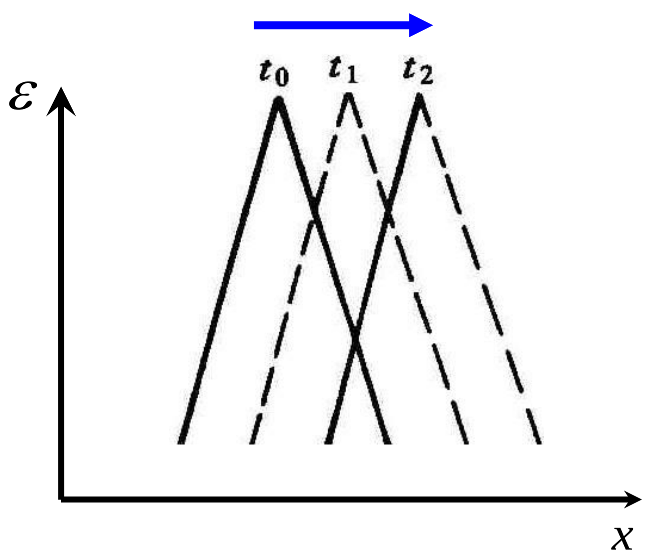
3.4.1 Essential difference between convection and diffusion

Diffusion—**Random thermal motions molecules, no bias(偏向) in direction;**

Convection—**Directional moving of fluid element, always from upstream to downstream(从上游到下游)**



(a)



(b)

3.4.2 CD of diffusion term can propagate disturbances all around (四面八方) uniformly

1. FTCS scheme of diffusion eq.

$$\frac{\partial \phi}{\partial t} = \Gamma \frac{\partial^2 \phi}{\partial x^2} \longrightarrow \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \Gamma \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}$$

$$\phi_i^{n+1} = \phi_i^n \left(1 - 2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right) + \frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\left(\phi_{i-1}^n + \phi_{n+1}^n\right)$$

2. Discrete disturbance analysis（离散扰动分析法）

（1） Assuming a uniform and zero initial field；

（2） Assuming that a disturbance ε occurs at a point, i, at some instant, n, while at all other points and subsequent time no any disturbances;

（3）Analyzing the transfer of the disturbance by the studied scheme.

3. Implementation of discrete disturbance analysis

For Point i at (n+1) instant:

Known: $\phi_i^n = \varepsilon,\ \phi_{i-1}^n = \phi_{i+1}^n = 0,$

$$\phi_i^{n+1} = \overset{=\varepsilon}{\phi_i^n}\left(1-2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right)+\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\left(\cancelto{0}{\phi_{i-1}^n}+\cancelto{0}{\phi_{n+1}^n}\right)$$

$$\phi_i^{n+1} = \varepsilon\left(1-2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right)\xrightarrow[\text{Stability requires}]{\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\le 0.5} 0<\phi_i^{n+1}<\varepsilon$$

Physically reasonable

For Point (i +1) at (n+1) instant:

$$\frac{\phi_{i+1}^{n+1}-\cancelto{0}{\phi_{i+1}^{n}}}{\Delta t}=\frac{\Gamma}{\rho}\frac{\cancelto{0}{\phi_{i+2}^{n}}-2\cancelto{0}{\phi_{i+1}^{n}}+\overset{=\varepsilon}{\phi_{i}^{n}}}{\Delta x^{2}}$$

$$\phi_{i+1}^{n+1}=\varepsilon\left(\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^{2}}\right)$$

Physically reasonable

For Point (i −1) at (n+1) instant:

$$\frac{\phi_{i-1}^{n+1}-\cancelto{0}{\phi_{i-1}^{n}}}{\Delta t}=\frac{\Gamma}{\rho}\frac{\overset{=\varepsilon}{\phi_{i}^{n}}-2\cancelto{0}{\phi_{i-1}^{n}}+\cancelto{0}{\phi_{i-2}^{n}}}{\Delta x^{2}}$$

$$\phi_{i-1}^{n+1} = \varepsilon\left(\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right)$$

$$\phi_{i+1}^{n+1} = \phi_{i-1}^{n+1}$$

Disturbance is transported onto two directions uniformly by diff.term

3.4.3 Analysis of transport character (迁移特性)of discretized convective term

1. **Definition－If a scheme can only transfer disturbance towards the downstream (下游) direction, then it possesses the transport character(具有迁移特性);**
2. **Analysis－Applying discrete disturbance analysis to advection equation with the studied scheme;**
3. **CD does not possess transport character.**

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} \longrightarrow \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -u\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x}$$

For Point (i+1) at (n+1) instant:

$$\frac{\phi_{i+1}^{n+1} - \cancelto{0}{\phi_{i+1}^n}}{\Delta t} = -u\frac{\cancelto{0}{\phi_{i+2}^n} - \overset{=\varepsilon}{\phi_i^n}}{2\Delta x} \longrightarrow \phi_{i+1}^{n+1} = \left(\frac{u\Delta t}{2\Delta x}\right)\varepsilon$$

Disturbance is transferred downstream!

Physically reasonable!

For Point (i-1) at (n+1) instant:

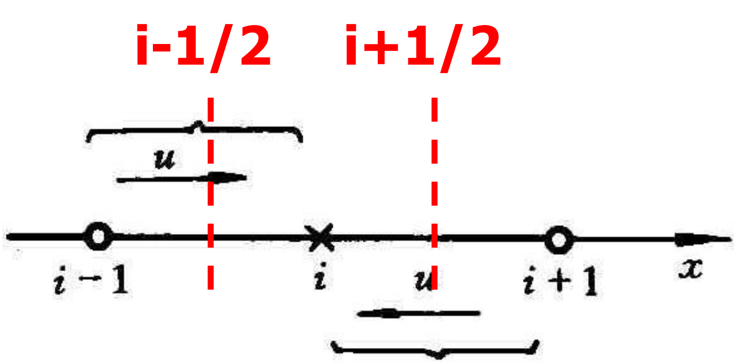
$$\frac{\phi_{i-1}^{n+1} - \cancelto{0}{\phi_{i-1}^n}}{\Delta t} = -u\frac{\overset{=\varepsilon}{\phi_i^n} - \cancelto{0}{\phi_{i-2}^n}}{2\Delta x} \longrightarrow \phi_{i-1}^{n+1} = -\left(\frac{u\Delta t}{2\Delta x}\right)\varepsilon$$

Disturbance is transferred upstream， and its sign is the opposite to the original one!

CD of convective term does not possess T.C.

3.4.4 Upwind scheme (迎风格式) of convective term possesses transport character

1. **Basic idea of US－Towards the oncoming flow(来流) taking some information for construction (构造)of the discretized form**

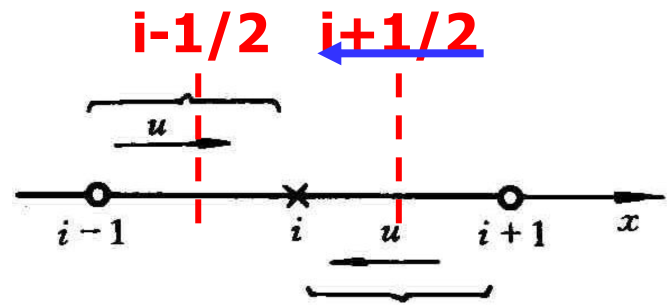


Upwind and/or downwind are related (相对于) to velocity direction

2. Definitions in FVM and FDM

(1) FDM — Bias finite difference form of 1st derivative at a point (某点一阶导数的偏差分格式)

$$\left(\frac{\partial\phi}{\partial x}\right)_i = \begin{cases} \dfrac{\phi_i - \phi_{i-1}}{\delta x}, & u > 0 \\[2ex] \dfrac{\phi_{i+1} - \phi_i}{\delta x}, & u < 0 \end{cases}$$



(2) FVM — Interpolation of interface value of ϕ

$$\phi_{i+1/2} = \begin{cases} \phi_i, & u > 0 \\ \phi_{i+1}, & u < 0 \end{cases}$$

Above two expressions have 1st-order accuracy，first-order upwind difference，FUD .

Two definitions have the same order of accuracy.

$$\frac{1}{\Delta x}\int_{i-1/2}^{i+1/2}\frac{\partial\phi}{\partial x}dx=\frac{1}{\Delta x}(\phi_{i+1/2}-\phi_{i-1/2})\ \overset{u>0}{=}\ \frac{\phi_i-\phi_{i-1}}{\Delta x}$$

FVM: integral average; FDM: discretized form at i.

3. FUD possesses transport character

$$\frac{\partial\phi}{\partial t}=-u\frac{\partial\phi}{\partial x}\ \xrightarrow{u>0}\ \frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=-u\frac{\phi_i^n-\phi_{i-1}^n}{\Delta x}$$

For Point (i+1) at (n+1) instant

$= \varepsilon$

$$\frac{\phi_{i+1}^{n+1}-\cancelto{0}{\phi_{i+1}^n}}{\Delta t}=-u\frac{\cancelto{0}{\phi_{i+1}^n}-\phi_i^n}{\Delta x}$$

Thus: $\phi_{i+1}^{n+1}=\varepsilon(\dfrac{u\Delta t}{\Delta x})$

For point (i-1) - (n+1):

$$\frac{\phi_{i-1}^{n+1}-\cancelto{0}{\phi_{i-1}^{n}}}{\Delta t}=-u\frac{\cancelto{0}{\phi_{i-1}^{n}}-\cancelto{0}{\phi_{i-2}^{n}}}{\Delta x}$$

Thus $\phi_{i-1}^{n+1}=0$

Disturbance is not transferred upstream; FUD possesses transport character.

3.4.5 Discussion on transportive character of discretized convective term

1. **Transportiv character (T.C.) is an important property of discretized convective term; Those who possess T.C. are absolutely stable;**

2. **Within the stable range, CD is superior to (优于) FUD; Strong convection may lead solution by CD oscillating while solution by FUD not.**

3. For those schemes who do not possess T.C. in order to get an absolutely stable solution the coefficients of the scheme should satisfy certain conditions. （替代教材73页4－5行的“凡是不具有迁移特性的对流项…因而只是条件地稳定）；

4. Numerical solution with FUD often has large numerical error; FUD is not recommended for the final solution; while in the debugging (调试) stage it may be used for its absolutely stability. Upwind idea once was widely used to construct higher-order schemes.

Home work: 3－1，3－3， 3－4， 3－6， 3－9

Due in：2017－10－09

Problem # 3-1

The Dufort-Frankel scheme for 1-D transient conduction equation:

$$\frac{T_i^{n+1}-T_i^{n-1}}{2\Delta t}=\frac{a}{\Delta x^2}\left(T_{i+1}^n-T_i^{n+1}-T_i^{n-1}+T_{i-1}^n\right)$$

This equation contains three time-levels i.e. $(\mathrm{n}-1,\mathrm{n},\mathrm{n}+1)$ so it is called three-level scheme. Write down the expression for truncation error and get the consistency condition.

Problem # 3-3

In the 2-D diffusion-convection equation:

$$\rho\frac{\partial\phi}{\partial t}+\rho(u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y})=\Gamma(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2})$$

u,v,ρ,Γ all are known constants.

Its one discretized scheme is as follows:

$$\rho\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n}}{\Delta t}+\rho u\frac{\phi_{i,j}^{n}-\phi_{i-1,j}^{n}}{\Delta x}+\rho v\frac{\phi_{i,j}^{n}-\phi_{i,j-1}^{n}}{\Delta y}=$$

$$\Gamma\frac{\phi_{i+1,j}^{n}-2\phi_{i,j}^{n}+\phi_{i-1,j}^{n}}{\Delta x^{2}}+\Gamma\frac{\phi_{i,j+1}^{n}-2\phi_{i,j}^{n}+\phi_{i,j-1}^{n}}{\Delta y^{2}}$$

By applying von Neumann analysis method showing that the stability condition is :

$$\Delta t\le\frac{1}{\frac{2a}{\Delta x^{2}}+\frac{2a}{\Delta y^{2}}+\frac{u}{\Delta x}+\frac{v}{\Delta y}},\ a=\frac{\Gamma}{\rho}$$

Problem # 3-4

There is a heat exchanger pipe with fully developed velocity field, where temperature field is described by the following equation

$$u\frac{\partial T}{\partial x}=\frac{\nu}{P_r}\frac{\partial^2 T}{\partial y^2}$$

Using explicit scheme for discretization of the given equation and find out the stability condition.

Problem # 3-7

Show the abosulute stable character of the C-N scheme for 1-D transient heat condcution equation in Cartesion coordinate.

Problem # 3-9

Show the conservation character of the central scheme for the convection term.

Announcement

No class on September 27!

Meet you in the evening of October 9 at 7:10 PM

Have a long National-Day and Mid-Autumn vocation!

同舟共济
渡彼岸！

People in the same boat help each other to cross to the other bank, where….