数值传热学

第 11 章　湍流流动与换热的数值模拟

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Numerical Heat Transfer

(数值传热学)

Chapter 11 Numerical Simulation for Turbulent Flow and Heat Transfer

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11.8 Brief Introduction to Recent Developments

11.1 Introduction to turbulence

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11.1.3 Definitions of Reynolds time-averages and their characteristics

11.1 Introduction to turbulence

11.1.1 Present understanding of turbulence

1. Turbulence is a highly complicated unsteady flow, within which all kinds of physical quantities are randomly varying with both time and space.;

2. Navier-Stokes are valid for transient turbulent flows;

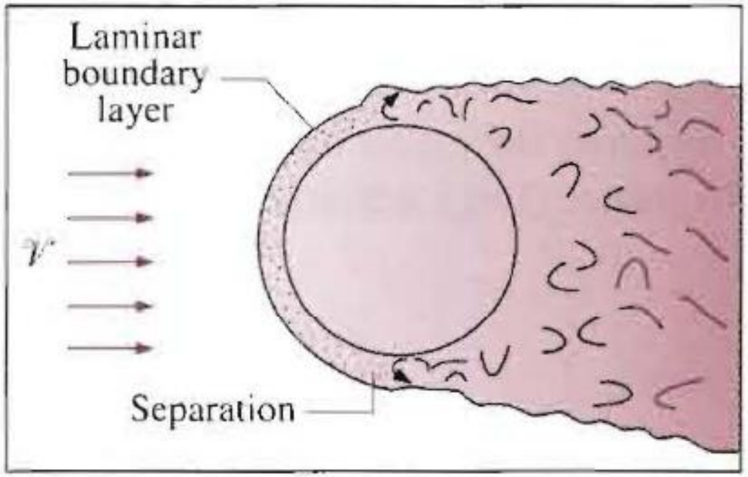
3. Turbulent flow field can be regarded as a collection of eddies(涡旋） with different geometric scales .

Remarks:

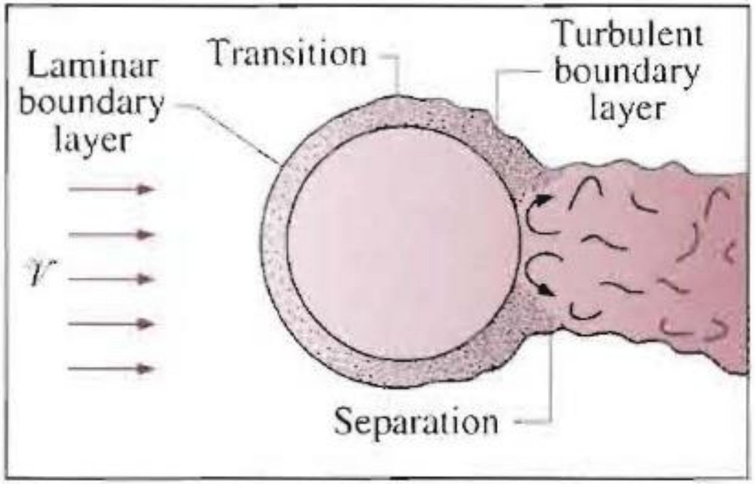
(1)Eddy vs. vortex (漩涡)： Eddy is characterized by

turbulent flow with randomness, and it covers a wide range of geometric scales;

Vortex is caused by a flow phenomenon characterized by recirculation, for example flow across a cylinder. Such vortex flow can be laminar or turbulent.



$$\mathrm{Re} < 2\times10^5 --\mathrm{Laminar}$$



$$\mathrm{Re} > 2\times10^5 --\mathrm{Turbulent}$$

Vorticity is a physical quantity defined by:

$$\vec{\omega} = \vec{\nabla} \times \vec{V}$$

Curl (旋度)of velocity vector

For a practical flow， either laminar or turbulent,

$$\omega \neq 0$$

Only for ideal fluid and potential flow $\omega = 0$

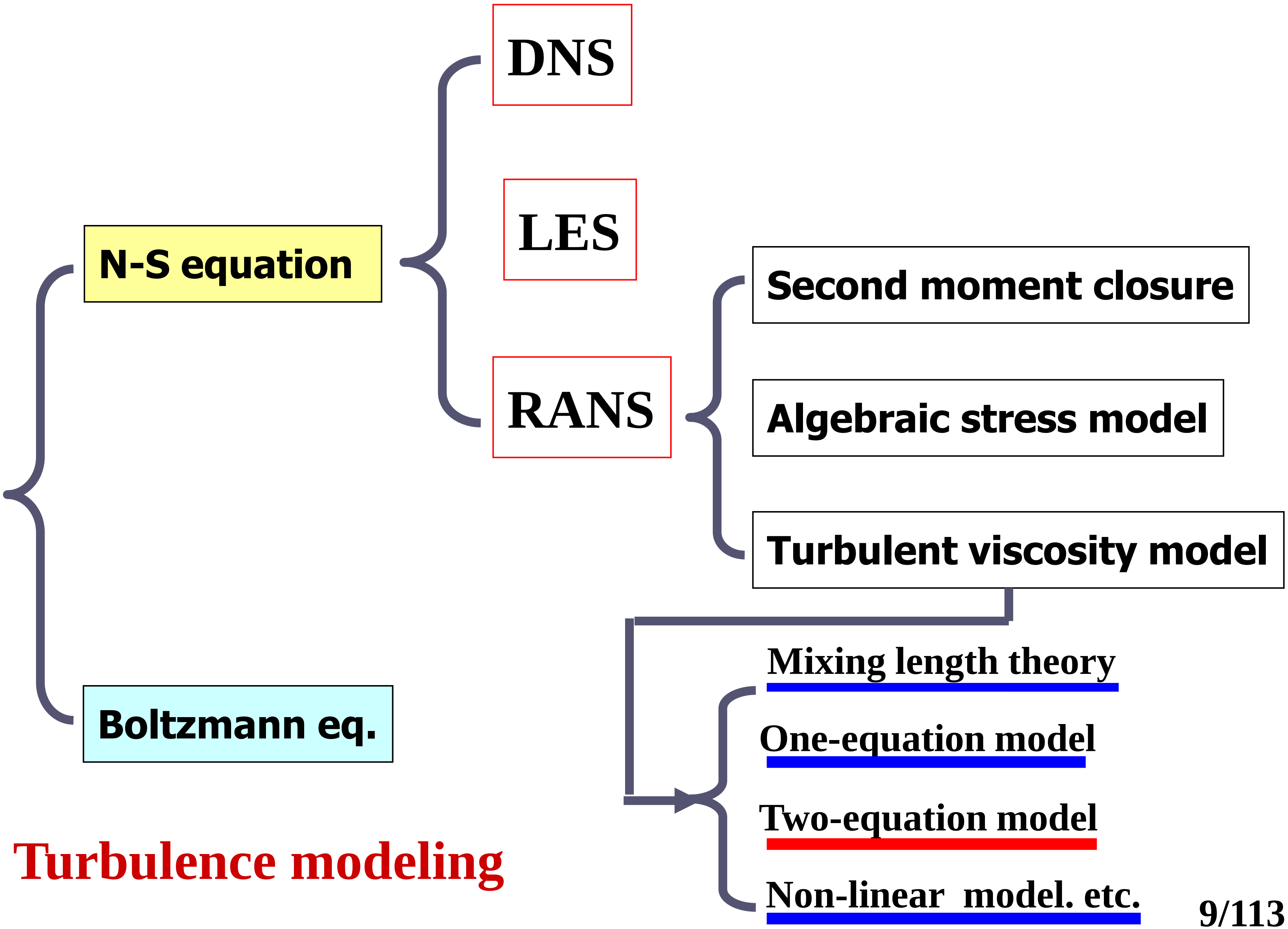
(2) A dispute (争议)happened in the later half of last century on whether N-S equations are valid for turbulent flows. The great success of direct numerical simulation gives a positive answer.

(3) Bifurcation(分岔), chaos(混沌), strange attractor (奇怪吸引子) and turbulence are regarded as the four non-linear phenomena in the 20th century.

11.1.2 Classifications of of turbulence simulation methods

Numerical methods for turbulence based on continuum assumption and Euler method can be divided into three categories: direct numerical simulation, DNS (直接模拟)，large eddy simulation, LES(大涡模拟) and Reynolds time-average N-S Eqs. method, RANS(雷诺时均) .

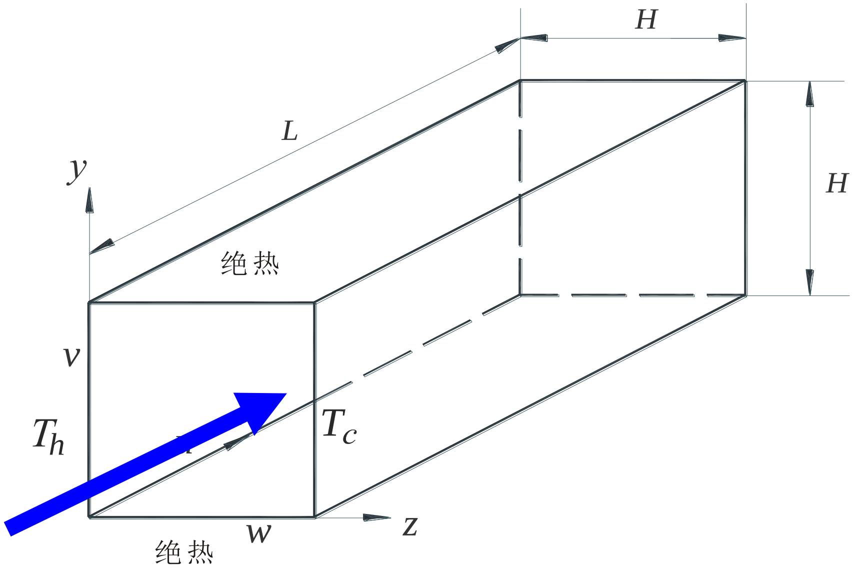
Turbulence modeling



- N-S equation
  - DNS
  - LES
  - RANS
    - Second moment closure
    - Algebraic stress model
    - Turbulent viscosity model
      - Mixing length theory
      - One-equation model
      - Two-equation model
      - Non-linear model. etc.
- Boltzmann eq.

1.DNS

In DNS very small time step and space step are needed to reveal the evolutions (演化) of eddies with different scales. Required computer resource is very High. Often high-performance computers are needed.



For a fully developed mixed convection in a square duct (L=6.4H), when Re=6400, Gr= $10^4 \sim 10^7$ DNS is conducted with 4.194×10^6 nodes(=256x128x128), and 8×10^5 time steps are needed for statistical average.

2. LES

Basic idea： Turbulent fluctuations are mainly generated by large scale eddies, which are non-isotropic(各向异性) and vary with flow situation； Small scale eddies dissipate(耗散) kinetic energy (from mechanic to thermal energy), and are almost isotropic. The N-S eqs. are used to simulate the large scale eddies, and the behavior of small scale eddies is simulated by simplified model.

LES requires less computer resource than that of DNS, even though still quite high, and has been used for engineering problems

For the above problem when simulated by LES only128x80x80=819200 grids are needed.

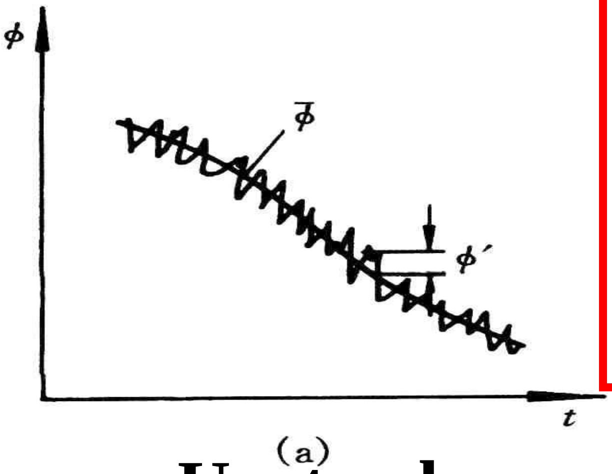
3. Reynolds time average N-S Eqs. methods

Expressing a transient term as the sum of average term and fluctuation(脉动) term. Time average is conducted for the transient N-S equations, and expressing the time average terms of the fluctuations via some function of average terms.

9.1.3 Reynolds time averages and their characteristics

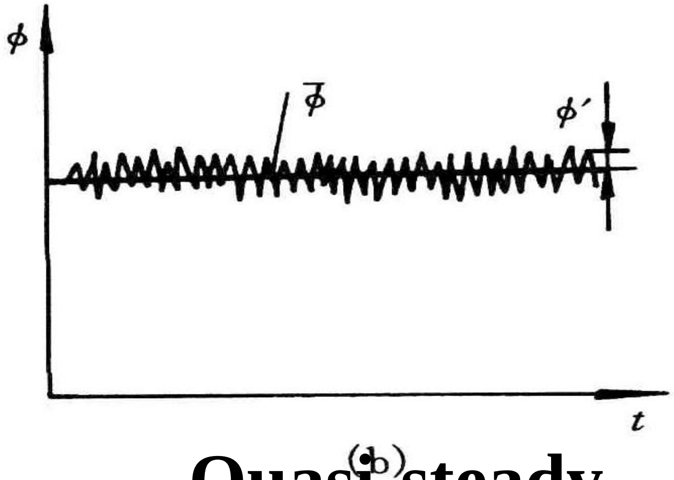
$$\phi = \overline{\phi} + \phi' \qquad \overline{\phi} = \frac{1}{\Delta t}\int_{t}^{t+\Delta t} \phi(t)dt$$

Δt is the time step, which should be large enough relative to the fluctuation but small enough with respect to the period of time average quantity.



非稳态湍流

Unsteady



准稳态湍流

Quasi-steady

Characteristics of time averages

1. $\overline{\phi'} \equiv 0;$ **2.** $\overline{\overline{\phi}} = \overline{\phi};$ **3.** $\overline{\overline{\phi}+\phi'} = \overline{\phi};$ **4.** $\overline{\overline{\phi}\phi'} = \overline{\phi}\,\overline{\phi'} = 0$

5. $\overline{\phi f} = \overline{(\overline{\phi}+\phi')(\overline{f}+f')} = \overline{\phi}\,\overline{f} + \overline{\phi' f'}$ **6.** $\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \overline{\phi}}{\partial x};$

7. $\overline{\frac{\partial \phi'}{\partial x}} = \frac{\partial \overline{\phi'}}{\partial x} = 0$ **8.** $\overline{\frac{\partial (\phi f)}{\partial x}} = \frac{\partial (\overline{\phi}\,\overline{f})}{\partial x} + \frac{\partial (\overline{\phi' f'})}{\partial x}$

11.2 Time-averaged governing equation for incompressible convective heat transfer

11.2.1 Time average governing equation

11.2.2 Ways for determining additional terms

11.2.3 Governing equations with turbulent viscosity

11.2 Time-averaged governing equation for incompressible convective heat transfer

11.2.1 Time average governing equation

1. Continuity eq.

$$\overline{\frac{\partial(\bar{u}+u')}{\partial x}}+\overline{\frac{\partial(\bar{v}+v')}{\partial y}}+\overline{\frac{\partial(\bar{w}+w')}{\partial z}}=\underbrace{\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}}_{=0}+\underbrace{\frac{\partial \overline{u'}}{\partial x}+\frac{\partial \overline{v'}}{\partial y}+\frac{\partial \overline{w'}}{\partial z}}_{=0}=0$$

Both time average velocity and time average fluctuation velocity satisfy continuity condition.

2. Momentum eq.

Taking x-direction as an example:

$$\overline{\frac{\partial(\bar{u}+u')}{\partial t}}+\overline{\frac{\partial(\bar{u}+u')^2}{\partial x}}+\overline{\frac{\partial(\bar{u}+u')(\bar{v}+v')}{\partial y}}+\overline{\frac{\partial(\bar{u}+u')(\bar{w}+w')}{\partial z}}=-\frac{1}{\rho}\overline{\frac{\partial(\bar{p}+p')}{\partial x}}+$$

$$\nu[\overline{\frac{\partial^2(\bar{u}+u')}{\partial x^2}}+\overline{\frac{\partial^2(\bar{u}+u')}{\partial y^2}}+\overline{\frac{\partial^2(\bar{u}+u')}{\partial z^2}}]$$

According to the above characteristics, yielding

$$\frac{\partial\bar{u}}{\partial t}+\frac{\partial(\bar{u}^2)}{\partial x}+\frac{\partial\bar{u}\bar{v}}{\partial y}+\frac{\partial\bar{u}\bar{w}}{\partial z}+\underline{\frac{\partial\overline{(u')^2}}{\partial x}+\frac{\partial\overline{(u'v')}}{\partial y}+\frac{\partial\overline{(u'w')}}{\partial z}}=$$

$$=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}+\nu(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial^2\bar{u}}{\partial z^2})$$

Moved to right hand side and combined with corresponding viscous term

$$\frac{\partial\bar{u}}{\partial t}+\frac{\partial(\bar{u}^2)}{\partial x}+\frac{\partial\bar{u}\bar{v}}{\partial y}+\frac{\partial\bar{u}\bar{w}}{\partial z}=$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}[\underline{\nu\frac{\partial\bar{u}}{\partial x}-\overline{(u')^2}}]+\frac{\partial}{\partial y}[\underline{\nu\frac{\partial\bar{u}}{\partial y}-\overline{(u'v')}}]+\frac{\partial}{\partial z}[\underline{\nu\frac{\partial\bar{u}}{\partial z}-\overline{(u'w')}}]$$

Rewritten in a tensor form in Cartesian coordinate：

$$\frac{\partial(\rho\bar{u})}{\partial t}+\frac{\partial(\rho\bar{u}_i\bar{u}_j)}{\partial x_j}=-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}(\eta\frac{\partial\bar{u}_i}{\partial x_j}-\rho\overline{u_i^{'}u_j^{'}})\ (i=1,2,3)$$

3. Other scalar (标量) variables

$$\frac{\partial(\rho\bar{\phi})}{\partial t}+\frac{\partial(\rho\bar{u}_j\bar{\phi})}{\partial x_j}=\frac{\partial}{\partial x_j}(\underline{\Gamma\frac{\partial\bar{\phi}}{\partial x_j}-\rho\overline{u_j^{'}\phi^{'}}})+S$$

4.Discussion on the time averaged quantity

(1) Linear term remains unchanged during time average, while product term (乘积项) generates product of fluctuations, representing the additional transport caused by fluctuation.

(2) Equations are not closed：for 3-D problem, there are five equations, with 14 unknown variables：

Five time average variables－ $\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{\phi},$

Nine products of fluctuations

$$\overline{u_i^{'} u_j^{'}} (i, j = 1, 2, 3); \qquad \overline{u_i^{'} \phi^{'}} (i = 1, 2, 3)$$

In order to close the above equations, additional relations must be added. Such additional relations are called turbulence model, or closure model (封闭模型)。

11.2.2 Ways of determining additional terms

1.Reynolds stress method

For the nine additional variables deriving their own governing equations.

However, in the derivation process new additional term of higher order (product of three variables, four variables, etc…) are introduced.; If we still go along this direction then equations for much higher order products should be derived.,,,,,. Thus we have to terminate such process at certain level. Historically some complicated models with more than 20 equations have been derived.

In the Reynolds stress models, the second moment model is quite famous and has been applied in some engineering problems. In the second moment model, for the product terms with two fluctuations their equations are derived, while for the terms with three or more fluctuations models are used to relate such terms with time average variables.

Prof. L X Zhou (周力行) in Tsinghua university contributed a lot in this regard.

2. Turbulent viscosity method

The product of fluctuations of two velocities is expressed via turbulent viscosity

(1) Definition of turbulent viscosity

In 1877 Boussinesq introduced following equation, by mimicking(比拟) the constitution equation (本构方程) of laminar fluid flow:

$$(\tau_{i,j})_t = -\rho\overline{u_i^{'}u_j^{'}} = (-p_t\delta_{i,j}) + \underline{\eta_t}(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}) - \frac{2}{3}\underline{\eta_t}\delta_{i,j}div\vec{U}$$

$$p_t = \frac{1}{3}\rho[\overline{(u^{'})^2} + \overline{(v^{'})^2} + \overline{(w^{'})^2}] = \frac{2}{3}\rho k \qquad k = \frac{1}{2}[\overline{(u^{'})^2} + \overline{(v^{'})^2} + \overline{(w^{'})^2}]$$

(2) Definition of turbulent diffusivity of other scalar variables

$$-\rho\overline{u_i^{'}\phi^{'}} = \Gamma_t \frac{\partial \overline{\phi}}{\partial x_i} \qquad \Gamma_t = \frac{\eta_t}{\Pr_t}$$

$\Pr_t$ ---turbulent Prandtl number, usually treated as a constant.

For laminar heat transfer we have

$$\Gamma_l = \frac{\lambda}{c_p} = \frac{\lambda}{c_p}\frac{\eta_l}{\eta_l} = (\frac{\lambda}{c_p\eta_l})\eta_l = \frac{\eta_l}{(\frac{c_p\eta_l}{\lambda})} = \frac{\eta_l}{\Pr_l}$$

Similarly: $\Gamma_t = \frac{\eta_t}{\Pr_t}$

Therefore for turbulent viscosity model its major task is to find $\eta_t, \Pr_t$.

The name of engineering turbulence models comes from the number of PDEqs. included in the model to determine turbulence viscosity.

11.2.3 Governing equations of viscocity models

1. Governing equations—

For simplicity of presentation, the symbol of time average "bar" is omitted hereafter:

$$\begin{cases}\dfrac{\partial u_k}{\partial x_k}=0\\[2ex] \dfrac{\partial u_i}{\partial t}+\dfrac{\partial(\rho u_k u_i)}{\partial x_k}=-\dfrac{\partial p_{eff}}{\partial x_i}+\dfrac{\partial}{\partial x_k}[\overset{\eta_{\mathrm{eff}}}{\underline{(\eta_l+\eta_t)}}\dfrac{\partial u_i}{\partial x_k}]+S_i \quad ;p_{eff}=p+p_t\\[2ex] \dfrac{\partial \phi}{\partial t}+\dfrac{\partial(\rho u_k \phi)}{\partial x_k}=\dfrac{\partial}{\partial x_k}[\underset{\Gamma_{eff}}{\underline{(\Gamma_l+\Gamma_t)}}\dfrac{\partial \phi}{\partial x_k}]+S_\phi\end{cases}$$

2. Differences from laminar governing equations:

(1) u_i, p, ϕ **-Time average; (2) Replacing** Γ **by** $\Gamma_{eff}=\Gamma+\Gamma_t$

(3) Replacing p **by** p_{eff} **(4) In source term** S_i **of** u_i **the additional terms caused by time averaging are included.**

In the Cartesian coordinates, the source terms of the three components are：

$$u:\ S=\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial x}\right)$$

$$v:\ S=\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial y}\right)$$

$$w:\ S=\frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial z}\right)$$

In laminar flow of constant properties, all source terms are zero, but for turbulent flow they are not zero.

3. Turbulent Prandtl number

Its value varies within a certain range， usually is taken as a constant

$$\Gamma_t=\frac{\eta_t}{\mathrm{Pr}_t}$$

11.3 Zero equation model and one equation model

11.3.1 Zero equation model

1. Turbulent additional stress of zero equation model
2. Equations for mixing length
3. Application range of zero eq. model

11.3.2 One equation model

1. Turbulent fluctuation kinetic energy as dependent variable
2. Prandtl-Kolmogorov equation
3. Governing equation of turbulent fluctuation kinetic energy
4. Boundary condition

11.3 Zero Equation Model and One Equation Model

11.3.1 Zero equation model

1.Turbulent additional stress of zero equation model

In zero eq. model no PDE is involved to determine turbulent viscosity. **The turbulent stress is expressed as:**

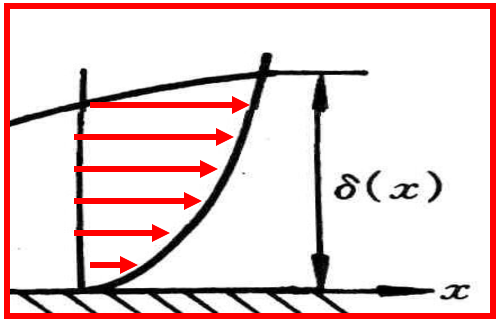
$$\tau_t = -\rho\overline{u_i^{'}u_j^{'}} = \rho\overline{u^{'}v^{'}} = \rho\nu_t\left(\frac{du}{dy}\right) = \rho\,\overbrace{l_m^2\left|\frac{du}{dy}\right|}\left(\frac{du}{dy}\right)$$

Turbulent kinetic viscosity

From dimensionality consideration

Cause of momentum exchange

From Newton shear stress eq.



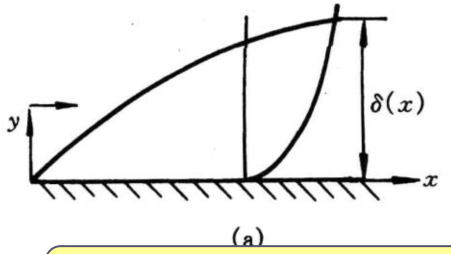
where l_m is called mixing length, whose determination is the key of zero-eq. model.

2.Equations for mixing length

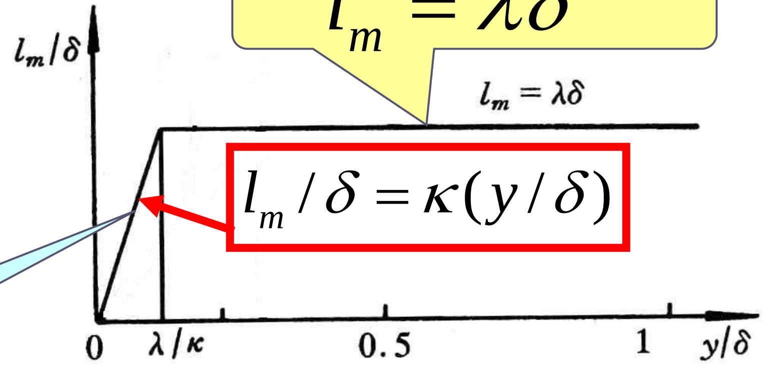
(1) Flow and HT over a plate l_m/δ vs. y/δ is a slope function (**斜坡函数**) ：

At $y/\delta=\lambda/\kappa$ $l_m=\lambda\delta$

Aut.	κ	λ
Cebeci	0.41	0.08
P-S	0.435	0.09



$l_m=\lambda\delta$



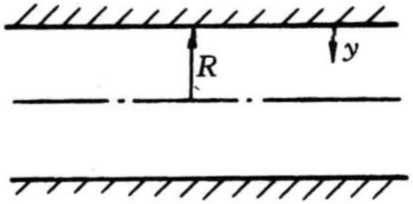
$l_m/\delta=\kappa(y/\delta)$

$l_m=\kappa y$

δ thickness of B.L.

(2) Turbulent HT in a circular tube **Nikurads eq.**

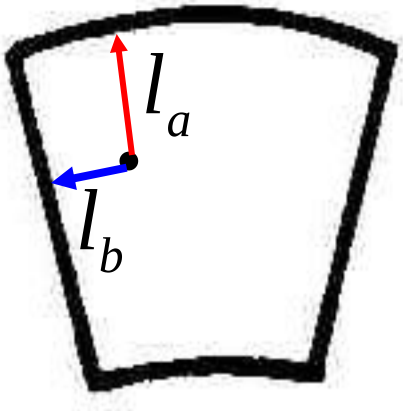
$$l_m / R = 0.14 - 0.08(1 - y/R)^2 - 0.06(1 - y/R)^4$$



Appllication range: **Re** $= 1.1\times10^5 \sim 3.2\times10^6$

(3) Fluid in a duct corner

$$\frac{1}{l_m} = \frac{1}{l_a} + \frac{1}{l_b};\ l_a, l_b$$ **from above eqs.**



(4) Modification caused by molecular viscosity — van Driest eq.

$$l_m = \kappa y[1 - \exp(-\frac{y(\tau_m/\rho)^{1/2}}{A\nu})] = \kappa y[1 - \exp(-\frac{y^+}{A})],\quad A = 26$$

Correction caused by molecular viscosity

For $\frac{y^+}{A} = 6$ **, its value = 0.997**

3. Application range of zero eq. model

(1) Boundary layer flow & HT （Flow over a wing before separation)

(2) FF & HT in straight ducts;

(3) Boundary layer type flow with weak recirculation.

Drawbacks of zero eq. model：

(1) At duct center line velocity gradient equals zero but turbulent viscosity still exists.

(2) Effects of oncoming flow turbulence is not considered

(3) Effects of turbulent flow itself is not considered

Li ZY, Hung TC, Tao WQ. Numerical simulation of fully developed turbulent flow and heat transfer in annular-sector ducts. Heat Mass Transfer,2002, 38 (4-5): 369-377

11.3.2 One-equation model

1.Turbulent fluctuation kinetic energy as dependent variable

The most important feature of turbulence is fluctuation. Fluctuation kinetic energy k is an appropriate quantity to indicate fluctuation intensity. It is taken as an dependent variable for reflecting the effects of turbulence itself.

2.Prandtl-Kolmogorov equation

Mimicking (模仿) the molecular viscosity caused by the random motion of molecules, the viscosity caused by turbulent fluctuation can be expressed by

Molecular viscosity $\eta_l \propto \rho \bar{u} \bar{\lambda}$

Turbulent viscosity $\eta_t \propto \rho k^{1/2} l$

$$\eta_t = C_\mu^{'} \rho k^{1/2} l$$

—Prandtl-Kolmogorov equation

where l **is the fluctuation scale, usually different from mixing length;**

Coefficient $C_\mu^{'}$ **is within 0.2 to 1.0;**

In order to get the distribution of *k* a related PDE is required.

3.Governing equation of turbulent kinetic energy

Starting from the definition $k=\frac{1}{2}\overline{u_i^{'}u_i^{'}}$,conducting time-average operation for N-S equations, and introducing some assumptions, following governing equation for *k* can be obtained:

$$\underbrace{\rho\frac{\partial k}{\partial t}}_{\text{transient}}+\underbrace{\rho u_j\frac{\partial k}{\partial x_j}}_{\text{convection}}=\underbrace{\frac{\partial}{\partial x_j}[(\eta_l+\frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}]}_{\text{diffusion}}+\overbrace{\underbrace{\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})}_{\text{production}}-\rho(C_D\underbrace{\frac{k^{3/2}}{l}}_{\text{dissipation}})}^{\text{source}}$$

where σ_k is called turbulent Prandtl number of *k*, and its introduction can increase the application range of the model.

4. Boundary condition treatment: wall function method

11.4 Two-Equation Model

11.4.1 Second variables related to l

11.4.2 $k-\varepsilon$ governing equations

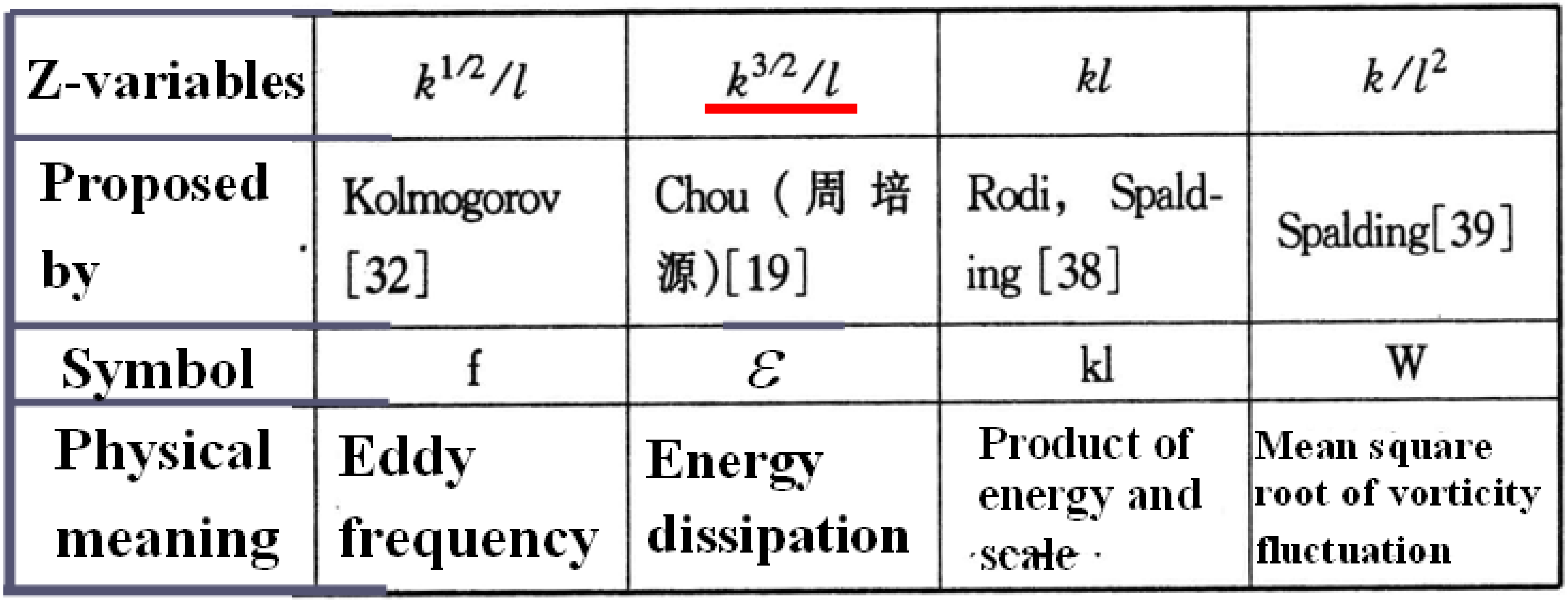
11.4.3 General governing equation for $k-\varepsilon$ model

11.4.4 Remarks

11.4 Two-Equation model

11.4.1 Second variables related to l

1. There are several physical variables related to l



Z-variables	$k^{1/2}/l$	$k^{3/2}/l$	kl	k/l^2
Proposed by	Kolmogorov [32]	Chou（周培源）[19]	Rodi，Spalding [38]	Spalding[39]
Symbol	f	ε	kl	W
Physical meaning	Eddy frequency	Energy dissipation	Product of energy and scale	Mean square root of vorticity fluctuation

$$\varepsilon = C_D \frac{k^{3/2}}{l}$$

This is the modeling definition(模拟定义). It can be regarded as the dissipation rate of fluctuation kinetic energy of unit mass; C_D is a dimensionless constant.

2. Two definitions of dissipation rate

(1) Strict definition

$$\varepsilon = \nu_l \overline{\left(\frac{\partial u_i^{'}}{\partial x_k}\right)\left(\frac{\partial u_i^{'}}{\partial x_k}\right)}$$

It represents dissipation rate of isotropic small eddies, and is used in the derivation of its governing equation.

(2) Modeling definition $\varepsilon = C_D \frac{k^{3/2}}{l}$

Understanding of its meaning： energy transit rate from larger eddies to small eddies for unit volume is proportional to ρk , and 1/*t* , where the transit time *t* is proportional to $l/k^{1/2}$, thus

$$\rho\varepsilon \sim \rho k/(\frac{l}{k^{1/2}}) \sim \rho\frac{k^{3/2}}{l} = C_D\rho\frac{k^{3/2}}{l}$$

This definition is used in the derivation process for simplifying treatment of some complicated terms.

11.4.2 $k-\varepsilon$ governing equations

(1) ε equation

Starting from strict definition, $\varepsilon = \nu_l \overline{(\frac{\partial u_i^{'}}{\partial x_k})(\frac{\partial u_i^{'}}{\partial x_k})}$

conducting time average operation for N-S equation, and adopting some assumptions (including modeling definition), yielding

$$\underbrace{\frac{\partial(\rho\varepsilon)}{\partial t}}_{\text{transient}} + \underbrace{\frac{\partial(\rho u_j\varepsilon)}{\partial x_j}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial x_j}[(\eta_l + \frac{\eta_t}{\sigma_\varepsilon})\frac{\partial\varepsilon}{\partial x_j}]}_{\text{diffusion}} + \underbrace{C_1\frac{\varepsilon}{k}\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}) - C_2\rho\frac{\varepsilon^2}{k}}_{\text{source}}$$

(2) k equation After introducing ε

k equation can be re-written as

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial(\rho u_j k)}{\partial x_j} = \frac{\partial}{\partial x_j}[(\eta_l + \frac{\eta_t}{\sigma_k})\frac{\partial k}{\partial x_j}] + \underbrace{\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}) - \overset{\substack{\boxed{-\rho(C_D\frac{k^{3/2}}{l})}\\ \big\downarrow}}{\rho\varepsilon}}_{\text{Source term}}$$

C_1, C_2 are empirical coefficients

Introducing: $G = \frac{\eta_t}{\rho}\frac{\partial u_i}{\partial x_j}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$ **called as unit mass production function**

Then source term of *k* eq. $\eta_t\frac{\partial u_j}{\partial x_i}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}) - \rho\varepsilon$

can be re-written as: $\boxed{\rho G-\rho\varepsilon}$

(3)Determination of turbulent viscosity of $k-\varepsilon$ model

$$\eta_t=C_\mu^{'}\rho k^{1/2}l=C_\mu^{'}C_D\rho k^{3/2}k^{1/2}\frac{l}{C_Dk^{3/2}}=C_\mu\rho k^2/\varepsilon$$

$$\varepsilon=C_D\frac{k^{3/2}}{l}\qquad C_\mu^{'}C_D\rightarrow C_\mu$$

11.4.3 General gov. eq. of $k-\varepsilon$ model

$$\frac{\partial(\rho\phi)}{\partial t}+div(\rho\vec{u}\phi)=div(\Gamma_\phi grad\phi)+S_\phi$$

ϕ represents: u,v,w,T,k,ε

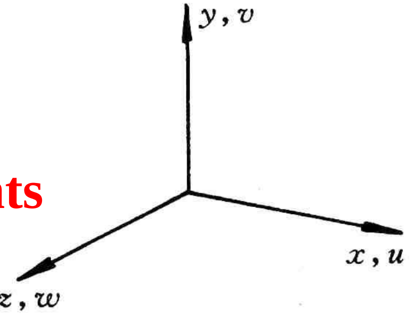
Most widely accepted values of model constants

C_1	C_2	C_μ	σ_k	σ_ε	σ_T
1.44	**1.92**	**0.09**	**1.0**	**1.3**	**0.9-1.0**

Γ_{ϕ}, S_{ϕ} depend on variable and coordinate:

$u, v, w, T, k, \varepsilon$

For Cartesian Coordinate:

控制方程	$\frac{\partial(\rho u\phi)}{\partial x}+\frac{\partial(\rho v\phi)}{\partial y}+\frac{\partial(\rho w\phi)}{\partial z}=\frac{\partial}{\partial x}(\Gamma\frac{\partial\phi}{\partial x})+\frac{\partial}{\partial y}(\Gamma\frac{\partial\phi}{\partial y})+\frac{\partial}{\partial z}(\Gamma\frac{\partial\phi}{\partial z})+S$ 对 u，v，w，k，ε，T 广义扩散系数 Γ 为： u，v，w：$\Gamma=\eta_{\text{eff}}=\eta+\eta_t$ k：$\Gamma=\eta+\frac{\eta_t}{\sigma_k}$ ε：$\Gamma=\eta+\frac{\eta_t}{\sigma_\varepsilon}$ T：$\Gamma=\frac{\eta}{Pr}+\frac{\eta_t}{\sigma_T}$ 
源项	u：$S=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}(\eta_{\text{eff}}\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(\eta_{\text{eff}}\frac{\partial v}{\partial x})+\frac{\partial}{\partial z}(\eta_{\text{eff}}\frac{\partial w}{\partial x})$ v：$S=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}(\eta_{\text{eff}}\frac{\partial u}{\partial y})+\frac{\partial}{\partial y}(\eta_{\text{eff}}\frac{\partial v}{\partial y})+\frac{\partial}{\partial z}(\eta_{\text{eff}}\frac{\partial w}{\partial y})$ w：$S=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}(\eta_{\text{eff}}\frac{\partial u}{\partial z})+\frac{\partial}{\partial y}(\eta_{\text{eff}}\frac{\partial v}{\partial z})+\frac{\partial}{\partial z}(\eta_{\text{eff}}\frac{\partial w}{\partial z})$ k：$S=\rho G_k-\rho\varepsilon$ ε：$S=\frac{\varepsilon}{k}(c_1\rho G_k-c_2\rho\varepsilon)$ $G_k=\frac{\eta_t}{\rho}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right\}$ T：S 按实际问题而定

Diffusion coefficients

Source term

11.4.4 Remarks

(1) Expansion of G term for 2D case

$$G=\frac{\eta_t}{\rho}\frac{\partial u_i}{\partial x_j}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})=\frac{\eta_t}{\rho}(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}+\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i})=$$

$$\frac{\eta_t}{\rho}(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial y})+(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial y})$$

$$G=\frac{\eta_t}{\rho}\{2[(\frac{\partial u}{\partial x})^2+(\frac{\partial v}{\partial y})^2]+(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})^2\}$$

There are 18 terms for 3D case.

(2) The above model is called standard $k-\varepsilon$ model. It can be applied to vigorously developed (旺盛发展) turbulent flow，also called as high-Re $k-\varepsilon$ model.

11.5 Wall Function Method

11.5.1 Two ways for grid settlement near wall in turbulence simulation

11.5.2 Fundamentals of wall function method

11.5.3 Boundary conditions of k, ε for standard $k-\varepsilon$ model

11.5.4 Cautions in implementing wall function method

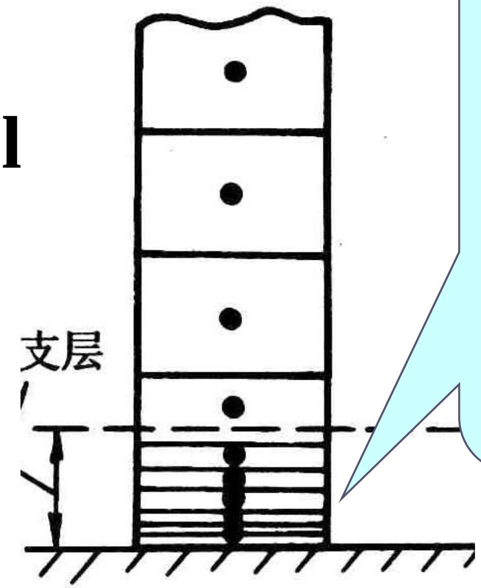
11.5 Wall Function Method

11.5.1 Two ways for grid settlement (节点设置)near wall in turbulence simulation

1.Setting enough number of grids in viscous sublayer （>10 grids）

For this treatment *k* equation can be used from vigorous turbulent flow to wall and k_w=0 for its boundary condition.

This treatment will be used in low Re $k-\varepsilon$ model.



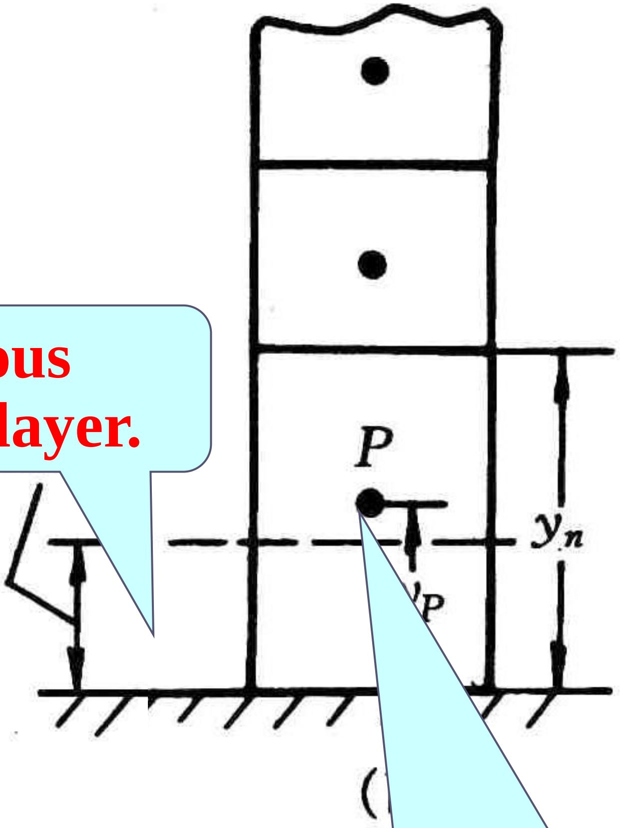
Enough number of nodes should be set in viscous sub-layer.

2. Set the 1st inner node outside the viscous sublayer

In this treatment velocity distribution near the wall should be assumed, and it is adopted in the high Re $k-\varepsilon$ model.

11.5.2 Fundamentals of WFM

1) Assuming that the dimensionless velocity and temp. distributions outside the viscous sub-layer are of logarithmic law(对数律) type.



viscous sub-layer.

The 1st inner node is set outside viscous sub-layer

(1) Logarithmic law in fluid mechanics

$$u^+=\frac{u}{v^*}=\frac{1}{\kappa}\ln(\frac{yv^*}{\nu})+B=\frac{1}{\kappa}\ln(y^+)+B=\frac{1}{\kappa}\ln(Ey^+)$$

$$v^*=\sqrt{\tau_w/\rho},\quad \kappa=0.4\sim 0.42,\quad B=5.0\sim 5.5$$

(2) Logarithmic law in turbulence model

In order that the logarithmic law can reflect some characteristics of turbulence the law is reformed as follows:

Replacing v^* **by** $C_\mu^{1/4}k^{1/2}$ **to define** y^+ **:**

$$y^+=\frac{y(C_\mu^{1/4}k^{1/2})}{\nu},$$

Introducing $C_\mu^{1/4}k^{1/2}$ **into** u^+ **definition**

$$u^+=\frac{u}{v^*}=\frac{u}{v^*}\frac{C_\mu^{1/4}k^{1/2}}{v^*}=\frac{u(C_\mu^{1/4}k^{1/2})}{\tau_w/\rho}$$

When dissipation and production of fluctuation kinetic energy are balanced, the above definitions are identical to conventional definition in fluid mechanics.

(3) Logarithmic law of temperature：mimicking definition of u⁺:

$$T^+ = \frac{(T - T_w)(C_\mu^{1/4} k^{1/2})}{(q_w / \rho c_p)}$$

Mimicking velocity

Mimicking stress

Required by dimension consistency

(4) Logarithmic laws of V & T in turbulence model

For $y^+ > 11.0$ **following distributions are adopted:**

$$u^+ = \frac{1}{\kappa} \ln(E y^+), \qquad \frac{1}{\kappa} \ln(E) = 5.0 \sim 5.5$$

$$T^+ = \frac{\sigma_t}{\kappa}\ln(Ey^+) + P\sigma_t \qquad P = 8.96(\frac{\sigma_l}{\sigma_t} - 1)(\frac{\sigma_l}{\sigma_t})^{-1/4}$$

$\sigma_l \equiv \Pr_l ; \sigma_t \equiv \Pr_t$ **If** $\sigma_l = \sigma_t$ **then** $T^+ = u^+$

Then this is Reynolds analogy(雷诺比拟) .

For $y^+ < 11.0$ **, it is regarded as laminar sublayer.**

2) Placing the 1st inner node P outside the viscous sub-layer, where logarithmic law valid ($y_P^+ > 11.0$ **)**

3) The effective turbulent viscosity and thermal conductivity between the 1st inner node and the wall should satisfy following equations:

$$\tau_w = \eta_B \frac{u_P - u_W}{y_P}, q_w = \lambda_B \frac{T_P - T_W}{y_P}$$

The equations of effective viscosity and thermal conductivity between the 1st inner node and the wall can be derived as follows:

(1) Equation for η_B : At point P, u^+ satisfy :

$$\frac{u_P(C_\mu^{1/4}k_P^{1/2})}{\tau_w/\rho} = \frac{1}{\kappa}\ln[Ey_P(\frac{C_\mu^{1/4}k_P^{1/2}}{\nu})]$$

This equation can be re-written as follows:

$$\tau_w = \frac{\rho u_P(C_\mu^{1/4}k_P^{1/2})}{\frac{1}{\kappa}\ln[Ey_P\frac{C_\mu^{1/4}k_P^{1/2}}{\nu}]} \overset{\longrightarrow}{\boxed{\text{According to Point 3}}} = \eta_B\frac{u_P - \cancelto{0}{u_W}}{y_P}$$

η_B equation can be obtained from this equation

$$\frac{\rho \cancel{u_P}(C_\mu^{1/4}k_P^{1/2})}{\frac{1}{\kappa}\ln[Ey_P\frac{C_\mu^{1/4}k_P^{1/2}}{\nu}]}=\eta_B\frac{\cancel{u_P}}{y_P}$$

$$\eta_B=\underbrace{[\frac{y_P(C_\mu^{1/4}k_P^{1/2})}{\nu}]}_{y_P^+}\underbrace{(\rho\nu)}_{\eta_l}\frac{1}{\underbrace{\frac{1}{\kappa}\ln(Ey_P^+)}_{u_P^+}}=(\frac{y_P^+}{u_P^+})\eta_l$$

In the vigorous region ， $y_P^+ >> u_P^+$ above equation shows:

turbulent viscosity is y_P^+/u_P^+ times of laminar viscosity.

For example $y_P^+=100,\ u_P^+=\frac{1}{\kappa}\ln(100)+B=\frac{1}{0.4}4.605+5.0=16.5$

Then： $\eta_B=(100/16.5)\eta_l=6.06\eta_l$

(2) Equation for λ_B : At point P, T$^+$ satisfy :

$$\frac{(T_P - T_W)(C_\mu^{1/4} k_P^{1/2})}{q_w / \rho c_p} = \frac{\sigma_t}{\kappa}\ln(E y_P^+) + \sigma_t P$$

From which:

$$q_w = \frac{\rho c_p \cancel{(T_P - T_W)}(C_\mu^{1/4} k_P^{1/2})}{\frac{\sigma_t}{\kappa}\ln(E y_P^+) + \sigma_t P} \xrightarrow{\boxed{\text{According to Point 3}}} = \lambda_B \frac{\cancel{(T_P - T_W)}}{y_P}$$

$$\lambda_B = \underbrace{\frac{(C_\mu^{1/4} k_P^{1/2}) y_P}{\nu}}_{y_P^+} \underbrace{\frac{\rho c_p \nu}{\lambda_l}}_{\frac{\eta c_p}{\lambda} = \Pr_l} \frac{\lambda_l}{\underbrace{\frac{\sigma_t}{\kappa}\ln(E y_P^+) + \sigma_t P}_{T_P^+}} = \left(\frac{y_P^+}{T_P^+}\right)\Pr_l \lambda_l$$

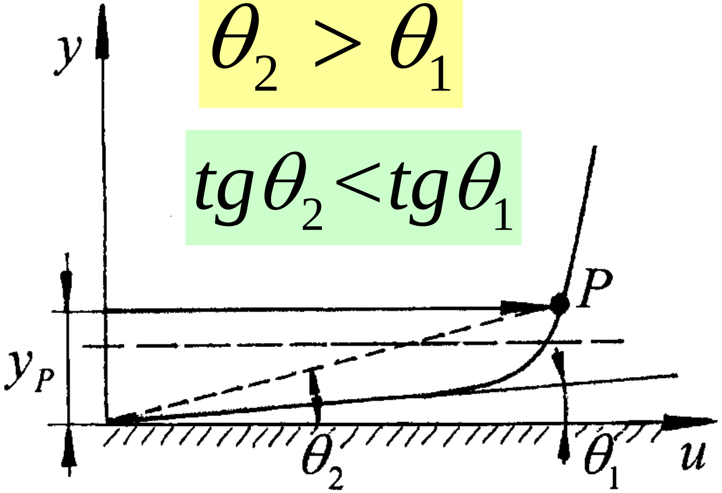
This is equivalent to magnify the molecular conductivity by $\left(\frac{y_P^+}{T_P^+}\right)\Pr_l$ times.

For $\mathrm{Pr}_l = 5.0, \mathrm{Pr}_t = 1.0, y_P^+ = 100,$

yielding $T_P^+ = 40.5, \frac{y_P^+}{T_P^+}\mathrm{Pr}_l = \frac{100}{40.5}\times 5.0 = 12.3$

The molecular conductivity is magnified by 12.3 times!

Why wall viscosity and conductivity η_B, λ_B should be magnified(放大)? This is because the 1st inner node is far from wall, leading to reduced wall gradient determined by FD method.



$\theta_2 > \theta_1$

$tg\theta_2 < tg\theta_1$

In WFM the magnified transport properties compensate (弥补) the reduced gradients so that their products will be approximately close to the true values

Wall functions refer to the expressions of η_B, λ_B

4) The boundary condition of k equation: $\frac{\partial k}{\partial n})_w = 0$

Because outside the sublayer the production of fluctuation kinetic energy is much larger than diffusion towards wall, hence diffusion to the wall is approximately taken zero.

5) The dissipation of fluctuation kinetic energy at 1st inner node is determined by the model equation:

$$\varepsilon_P = \frac{C_D k^{3/2}}{l} = \frac{C_\mu^{3/4} k_P^{\ 3/2}}{\kappa y_P}$$ **(see page 355 of text book)**

For the 1st inner node dissipation rate is specified by above equation, and computation is limited within the region surrounded by the 1st inner nodes.

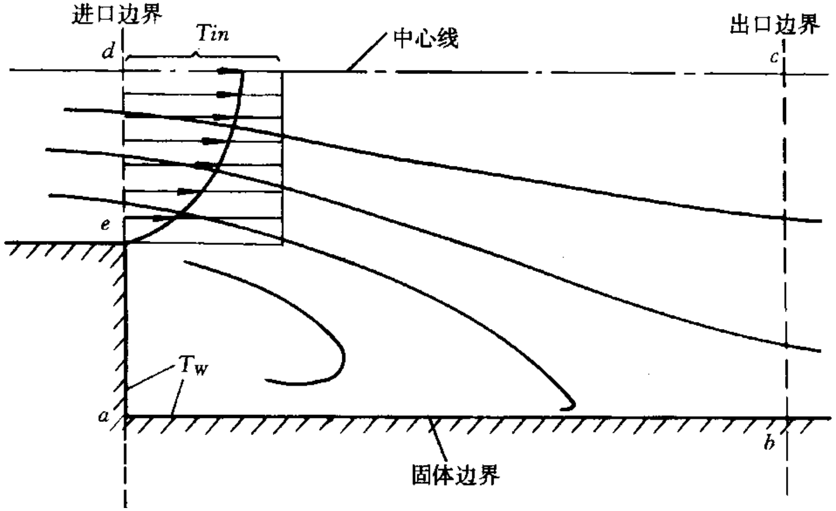
11.5.3 Boundary conditions of k,ε for standard $k-\varepsilon$ model

1. Inlet boundary

1) k: (1)Adopting test data; (2)Taking a percentage of kinetic energy of oncoming flow .For fully developed flow in ducts:0.5~1.5%;

2) ε: (1) Using model equation: $\varepsilon = \dfrac{C_{\mu}^{3/4}k^{3/2}}{\kappa y_P}$

(2)Using $\eta_t = C_{\mu}\rho k^2/\varepsilon$ assuming $\dfrac{\rho u L}{\eta_t} = 100 \sim 1000$ yielding η_t with inlet *u* and *L*.



2. At central line: $\frac{\partial k}{\partial n}=\frac{\partial \varepsilon}{\partial n}=0$

3. Outlet: Adopting local one way coordinate assumption

3. Solid wall: Adopting wall function method

(1)Velocity— Velocity normal to wall $\frac{\partial \phi}{\partial n})_w=0;$

Velocity parallel to wall $\phi_w=0,$

And wall viscosity determined by WFM.

Remarks: here velocity is the dependent variable to be solved not the one in the nonlinear part of convection term, for which wall velocity u=v=0.

(2) k— Adopting $\frac{\partial k}{\partial n}=0$ implemented via setting $\Gamma_B=0$

(3) ε— Specifying the 1st inner node by

Then cutting connection with boundary $\varepsilon_P = \dfrac{C_\mu^{3/4} k^{3/2}}{\kappa y_P}$

11.5.4 Cautions in implementing wall function method

1) Approximate range of y_P^+, x_P^+

$$11.5 \sim 30 \le (y_P^+, x_P^+) \le 200 \sim 400$$

Logarithmic law is valid in this range

2) Underrelaxation

In the iteration process η_t, k, ε must be under-relaxed. And it is organized within the solution process.

3) ε_P should be specified by large coefficient method

4) Source term treatment of k, ε

$$S_k = \rho G - \rho\varepsilon = \underbrace{\rho G}_{S_C} \underbrace{-(\rho\varepsilon / k^*)}_{S_P} k$$

$$S_\varepsilon = \frac{C_1 \rho\varepsilon G}{k} - \frac{C_2 \rho\varepsilon^2}{k} = \underbrace{\frac{C_1 \rho\varepsilon G}{k}}_{S_C} - \underbrace{\frac{C_2 \rho\varepsilon^*}{k}}_{S_P} \varepsilon$$

5) Treatment of solid located within fluid region

See pages 358－359 of textbook.

11-6 Turbulent flow and heat transfer in duct with a stepwise inlet velocity distribution

---k-epsilon turbulence model with WFM

11-6-1 Physical problem and its math formulation

Known： A stream with a central jet goes into a parallel channel; Flow is in turbulent state, AMU ＝ 10^{-6} and Pr=0.7.

Find： Adopt the standard k-Epsilon model and the wall function method to determine velocity and temperature fields in the channel.

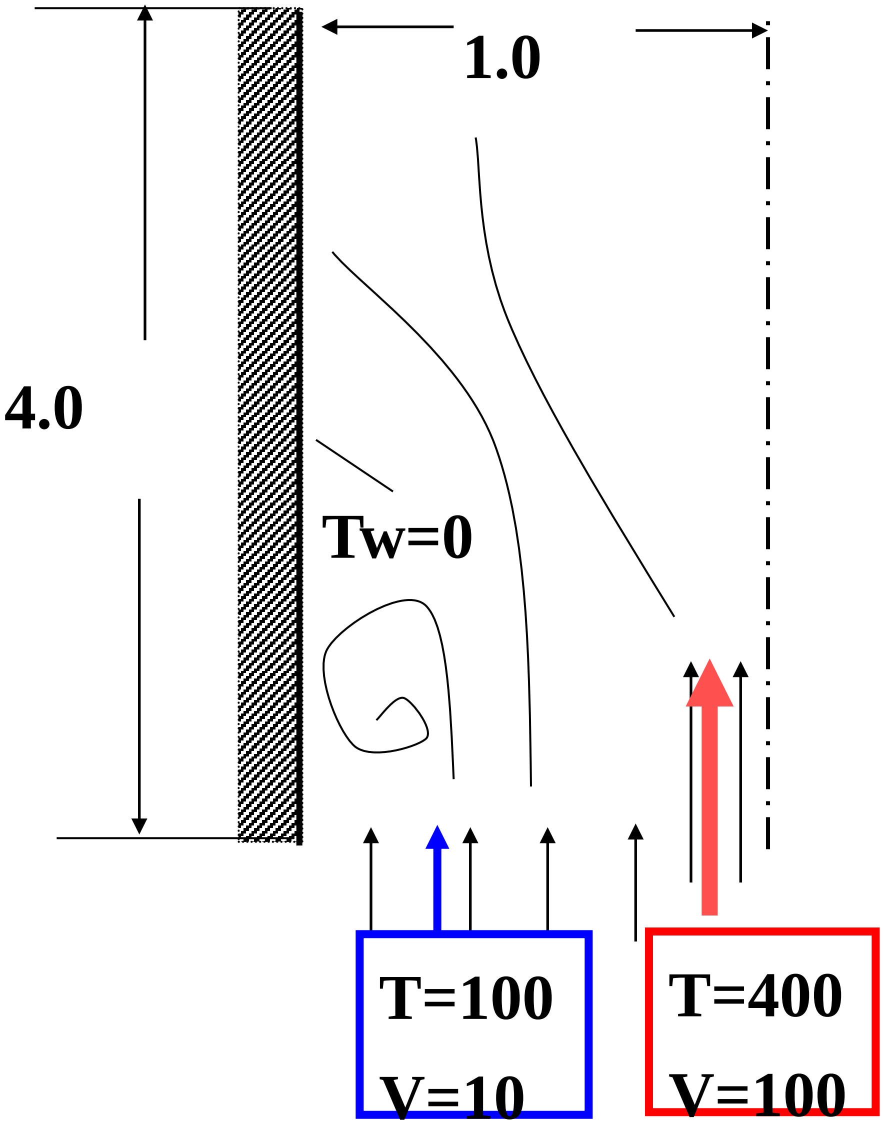


Fig. 1 of Example 8

Governing equation is：

$$div(\rho \vec{u}\phi) = div(\Gamma_{\phi} grad\phi) + S_{\phi}$$

where $\phi = u, v, T, k, \varepsilon, p, p^{'}$

The diffusion coefficients are:

NF=	1	2	3	4	5	6	7	8	11
Variable	U	V	PC	T	k	ε	AMUT ,	GEN	P
Γ_{ϕ}	η_t	η_t	/	$\frac{\eta_t c_p}{\Pr_t}$	$\frac{\eta_t}{\sigma_k}$	$\frac{\eta_t}{\sigma_{\varepsilon}}$			
α	0.8	0.8		1.0	0.6	0.6			0.6

$$S_u = \frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial x}\right)$$

$$S_v = \frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial y}\right)$$

$$S_w = \frac{\partial}{\partial x}\left(\eta_{\text{eff}}\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\eta_{\text{eff}}\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial z}\left(\eta_{\text{eff}}\frac{\partial w}{\partial z}\right)$$

k： $S = \rho G_k - \rho\varepsilon$

ε： $S = \dfrac{\varepsilon}{k}(c_1\rho G_k - c_2\rho\varepsilon)$

$$G_k = \frac{\eta_t}{\rho}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\}$$

Boundary conditions are：

（1） Inlet： k －taking 1％ of kinetic energy of oncoming flow；

Epsilon: Determining by following eq.

$$\varepsilon = \frac{c_\mu \rho k^2}{\eta_t}$$

where η_t determined by $\mathrm{Re}_{eff} = \frac{\rho V(2L_{in})}{\eta_{eff}} = 100$

（2） Wall: WFM；

（3） Outlet: local one –way;

(4) At symmetric, *u*=0, all others have their first order normal derivatives equal to zero!

11-6-2 Numerical method

(1) Source term treatment for $k-\varepsilon$

$$S_k = \eta_t G - \rho\varepsilon = \underbrace{\eta_t G}_{S_C} - \underbrace{\left(\frac{\rho\varepsilon}{k^*}\right)}_{S_P} k$$

$$S_\varepsilon = \frac{c_1 \varepsilon \eta_t G}{k} - \frac{c_2 \rho \varepsilon^2}{k} = \underbrace{\frac{c_1 \varepsilon \eta_t G}{k}}_{S_C} \underbrace{- \left(\frac{c_2 \rho \varepsilon^*}{k}\right)}_{S_P} \varepsilon$$

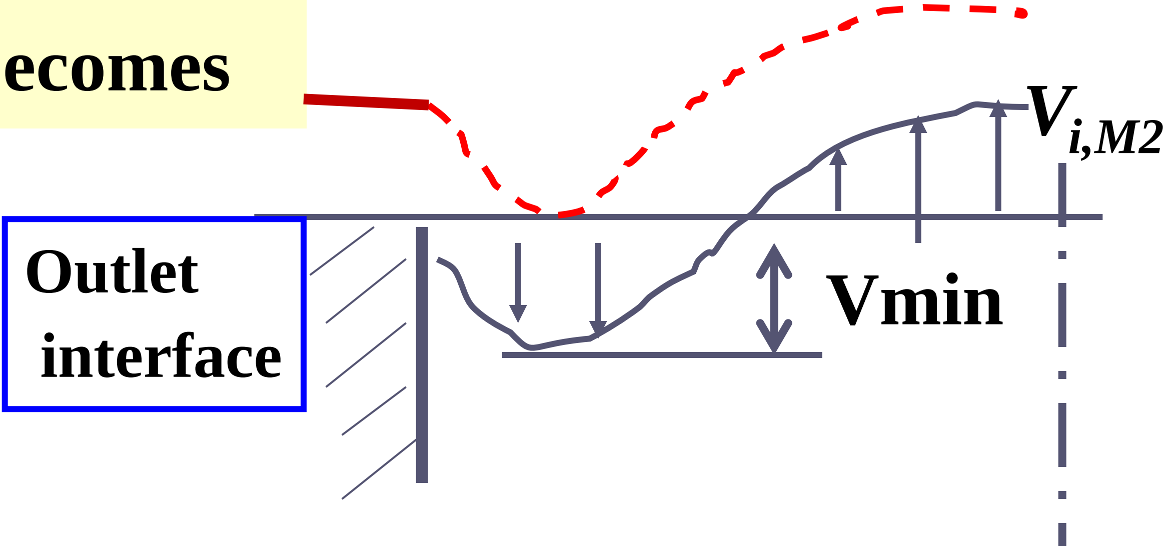
(2) Lift (提升) of outlet velocity

In order to avoid negative outlet velocity during iteration, adopt method for lifting temporary outlet velocity:

$$FACTOR = \frac{FLOWIN}{\sum_{i=2}^{L2}[(V_{i,M2} + |V_{\min}|) * RHO_{i,M1} * XCV(i)]}$$

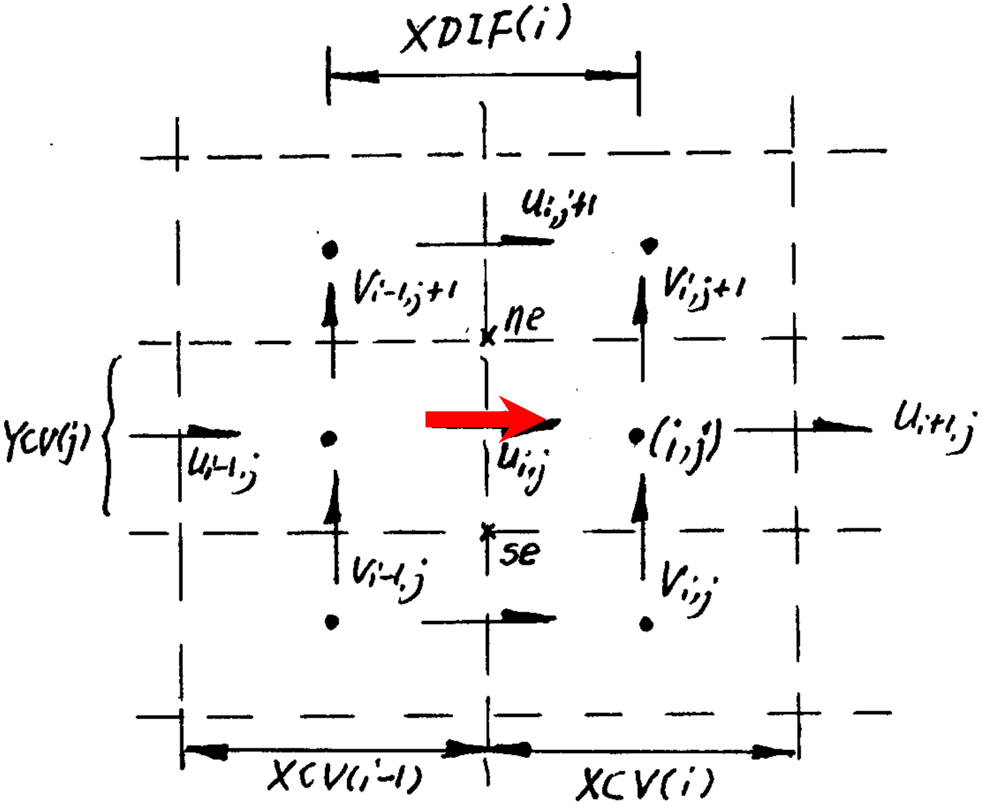
After this treatment outflow becomes

$$v_{i,M1} = FACTOR \bullet (v_{i,M2} + |v_{\min}|)$$



(3) Source term treatment of momentum equation

$$S_u = \frac{\partial}{\partial x}(\mu_t \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\mu_t \frac{\partial v}{\partial x})$$



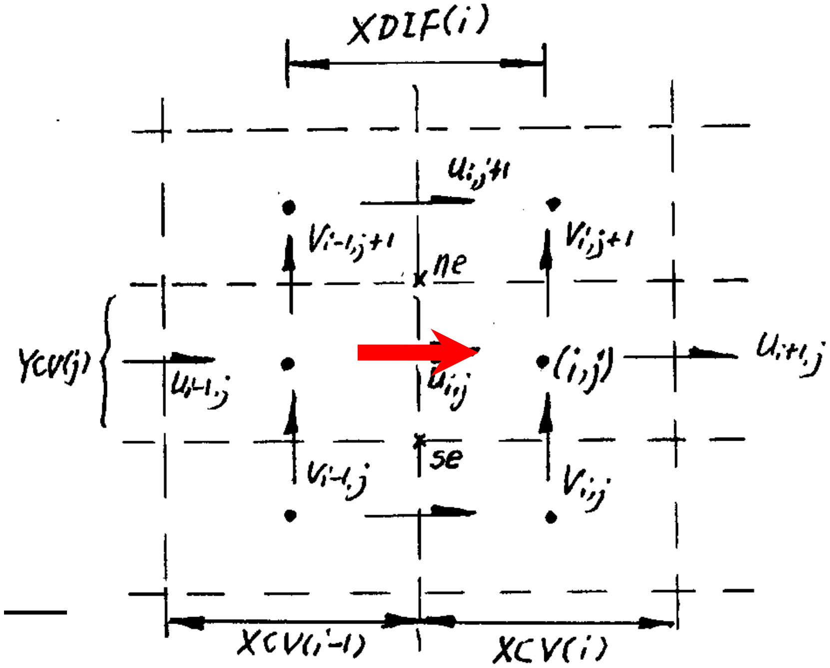
(3) Treatment of source term in u-momentum equation

$$S_u = \frac{\partial}{\partial x}(\mu_t \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\mu_t \frac{\partial v}{\partial x})$$

$$\frac{\partial}{\partial x}(\mu_t \frac{\partial u}{\partial x}) = \frac{1}{XDIF(i)}$$

$$\{GAM(i.j)\frac{u(i+1,j)-u(i,j)}{xcv(i)} -$$

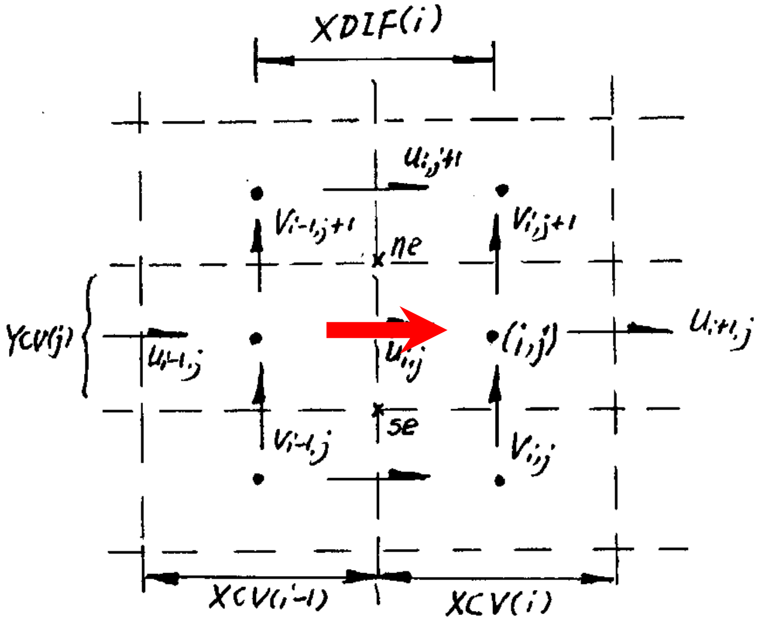
$$GAM(i-1,j)\frac{u(i,j)-u(i-1,j)}{xcv(i-1)}\}$$



$$\frac{\partial}{\partial y}(\mu_t \frac{\partial v}{\partial x}) = \frac{1}{YCV(j)}$$

$$\{\mu_{t,ne}\frac{v(i,j+1)-v(i-1,j+1)}{XDIF(i)} -$$

$$\mu_{t,se}\frac{v(i,j)-v(i-1,j)}{XDIF(i)}\}$$



(4) Flow field and temperature are solved separately

Because velocities are not coupled with temperature, the turbulent flow field can be solved first, then the fluid temperature.

11-6-3 Program reading

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      MODULE USER_L
C**********************************************************
      INTEGER*4 I,J
      REAL*8 CMU, C1, C2, PRT, PRK, PRD, PRPRT, PFN, CMU4,
     1 AFL, VMIN, REL, AMT, ALOG, GAP, GAMM, DUDX, DUDY, DVDX,
     1 DVDY, DISS, AMU, PR, FLOWIN， FL, FACTOR
       END MODULE
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
     SUBROUTINE USER
C***********************************************************
     USE START_L
     USE USER_L
     IMPLICIT NONE
C***********************************************************
C-----------------------------PROBLEM TEN-----------------------------------
C      Turbulent fluid flow and heat transfer in a parallel duct with stepwise
C                              inlet velocity distribution
C**********************************************************
```

*

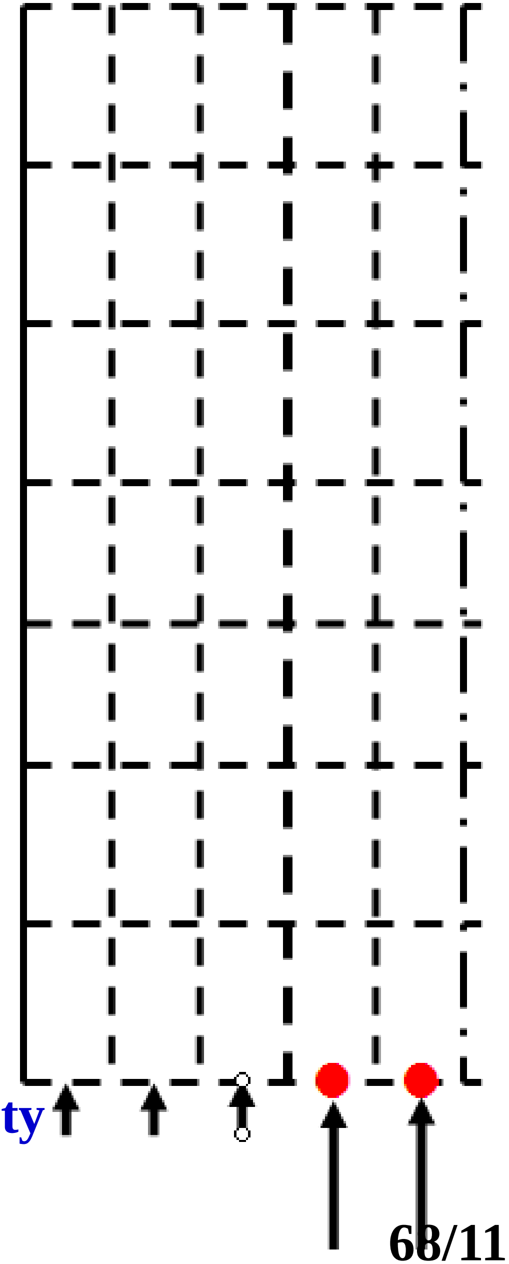
```
ENTRY GRID
TITLE(1)=‘ .VEL  U.'
TITLE(2)=‘ .VEL  V.'
TITLE(3)=‘ .STR  FN.'
TITLE(4)=‘ . TEMP .'
TITLE(5)='KIN  ENE'
TITLE(6)=‘ .DISIPA.’
TITLE(7)='TURB  VI'
TITLE(11)='PRESSURE'
TITLE(12)=‘ DENSITY'
```

!All are titles for printing

```
RELAX(1)=0.8
RELAX(2)=0.8
RELAX(5)=0.6
RELAX(6)=0.6
RELAX(13)=0.6  !  NGAM=13  for turbulent viscosity
LSOLVE(1)=.TRUE.
LSOLVE(5)=.TRUE.
LSOLVE(6)=.TRUE.
LPRINT(1)=.TRUE.
LPRINT(2)=.TRUE.
LPRINT(3)=.TRUE.
LPRINT(4)=.TRUE.
LPRINT(5)=.TRUE.
LPRINT(6)=.TRUE.
LPRINT(7)=.TRUE.
LPRINT(11)=.TRU
LAST=55
XL=1.
YL=4.
L1=7
M1=9
CPCON=1000. ! In order to resume thermal conductivity
CALL UGRID
RETURN
```

!All logical values for solving and printing

Regarding AMUT as 7th element of F(i, j, NF)



```
    ENTRY START
    DO 100 J=1,M1
    DO 101 I=1,L1
    U(I,J)=0.
    V(I,J)=10.
    V(1,J)=0.
    V(I,2)=10.
    IF(I.GT.4) V(I,2)=100.
    T(I,J)=100.
    T(1,J)=0.
    IF(I.GT.4) T(I,1)=400.
    AKE(I,J)=0.005*V(I,2)**2
    DIS(I,J)=0.1*AKE(I,J)**2
101 ENDDO
100 ENDDO
```

1% of inlet kinetic energy

η_t : determied form

$$\mathrm{Re}_{eff} = \frac{\rho V(2L_{in})}{\eta_{eff}} = 100$$

$$100 = \frac{1\times 100\times 1.0}{\eta_t}, \eta_t = 1.0$$

$$\varepsilon = C_\mu \rho k^2 / \eta_t = 0.09k^2 \approx 0.1k^2$$

```
AMU=1.E-6          !  Attention，very small values
CMU=0.09
C1=1.44
C2=1.92
PRT=0.9            !  Constants of Standard K-Epsilon
PRK=1.0
PRD=1.3
PR=0.7
PRPRT=PR/PRT                                  !  P function of WFM for T
PFN=9.*(PRPRT-1.)/PRPRT**.25
CMU4=CMU**.25
RETURN
ENTRY DENSE
RETURN
```

$$\boxed{P = 9.0(\frac{\sigma_l}{\sigma_t} - 1)(\frac{\sigma_l}{\sigma_t})^{-0.25}}$$

```
      ENTRY BOUND
      IF(ITER = = 0) THEN
      FLOWIN=0.
      DO 310 I=2,L2
      FLOWIN=FLOWIN+RHO(I,1)*V(I,2)*XCV(I)
310 ENDDO                                       ! Flow rate at inlet
      ELSE
      FL=0.
      AFL=0.
      VMIN=0.
      ENDIF
      DO 301 I=2,L2
      IF(V(I,M2)< 0.) VMIN=DMAX1(VMIN,-V(I,M2))   ! Search for Vmin
      AFL=AFL+RHO(I,M1)*XCV(I)                    ! DMAX1 () is
      FL=FL+RHO(I,M1)*V(I,M2)*XCV(I)              more accurate than
      FACTOR=FLOWIN/(FL+AFL*VMIN)                 AMAX1()
301 ENDDO
      DO 302 I=2,L2
      V(I,M1)=(V(I,M2)+VMIN)*FACTOR
302 ENDDO
      DO 303 J=2,M2
      AKE(L1,J)=AKE(L2,J)
      DIS(L1,J)=DIS(L2,J)   ! Equivalent to fully developed
303 ENDDO
      RETURN
```

$$FACTOR=\frac{FLOWIN}{\sum_{i=2}^{L2}[(V_{i,M2}+|V_{\min}|)*RHO_{i,M1}*XCV(i)]}$$

```
      ENTRY OUTPUT
      IF(ITER= =0) THEN
      PRINT 401
      WRITE(8,401)
  401 FORMAT(1X,'   ITER',6X,'SMAX',6X,'SSUM',5X,'V(6,6)',
    1 4X,'T(5,6)',4X,'KE(5,6)')
      ELSE
      PRINT 403, ITER, SMAX, SSUM, V(6,6),T(5,6), AKE(5,6)
      WRITE(8,403)  ITER,SMAX,SSUM,V(6,6),T(5,6),AKE(5,6)  403
 FORMAT(1X,I6,1P5E11.3)
      ENDIF
      IF(ITER>=50) THEN
      LSOLVE(4)=.TRUE.
      LSOLVE(1)=.FALSE.
      LSOLVE(5)=.FALSE.
      LSOLVE(6)=.FALSE.
      ENDIF
      IF  (ITER==LAST) CALL PRINT
      RETURN
```

! Flow is not coupled with temperature ! After obtaining converged flow field temperature is solved

```
     ENTRY GAMSOR          ! For pressure correction no need to visit
     IF(NF= = 3) RETURN    Following part.
     IF(NF= = 1) THEN
     REL=1.-RELAX(NGAM) ! NGAM=13 for turbulent viscosity
     DO 500 J=1,M1
     DO 501 I=1,L1
     AMT=CMU*RHO(I,J)*AKE(I,J)**2/(DIS(I,J)+1.E-30) ! Initial values
     IF(ITER= =0) AMUT(I,J)=AMT
     AMUT(I,J)=RELAX(NGAM)*AMT+REL*AMUT(I,J) ! Underrelaxation
501 ENDDO
500 ENDDO
     FACTOR=1.
     ELSE
     IF(NF== 4) FACTOR=CPCON/PRT
     IF(NF== 5) FACTOR=1./PRK
     IF(NF= = 6) FACTOR=1./PRD
     DO 520 J=1,M1
     DO 521 I=1,L1
     GAM(I,J)=AMUT(I,J)*FACTOR ! Laminar part is omitted.
     IF(NF/= 1) GAM(L1,J)=0.! symmetric(but not for u)
     GAM(I,M1)=0.! Local one way for all variables at outlet
521 ENDDO
520 ENDDO
```

$$\boxed{\Pr = \mu c_p / \lambda,\ \lambda = \eta c_p / \Pr}$$

$$\boxed{(\eta_l + \frac{\eta_t}{\sigma_k}) - \text{for k}; (\eta_l + \frac{\eta_t}{\sigma_\varepsilon}) - \text{for } \varepsilon}$$

!------- ---WFM----------------

```
    DO 530 J=2,M2
    SELECT CASE (NF)  ! For u, p',k, epsilom
    CASE (1,3,5,6)
    GAM(1,J)=0.
    CASE (2)     !   WFM for velocity v, and for temperature
    GAM(1,J)=AMU  !   First laminar viscosity is given
    XPLUS(J)=RHO(2,J)*SQRT(AKE(2,J))*CMU4*XDIF(2)/AMU
    IF(XPLUS(J)>11.5)   GAM(1,J)=AMU*XPLUS(J)/
  1 (ALOG(9.*XPLUS(J))*2.5)  ! Turbulence viscosity
    CASE (4)  !   WFM for temperature
    GAM(1,J)=AMU*CPCON/PR !   First laminar thermal conductivity
    IF(XPLUS(J)>11.5) GAM(1,J)=AMU/PRT*XPLUS(J)
  1 /(2.5*ALOG(9.*XPLUS(J))+PFN)  ! Turbulence thermal conductivity
    ENDSELECT
530 ENDDO
```

Set up zero value of boundary Gamma for u, p', k and Epsilon , thus cut the connection to their boundaries-----adiabatic condition

WFM IMPLEMENTATION

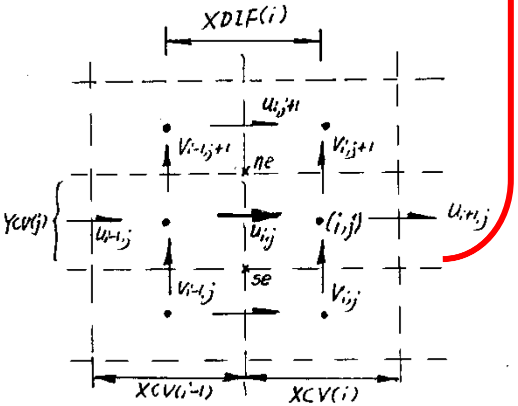
$$x^{+} = \frac{\rho x (C_{\mu}^{1/4} k^{1/2})}{\eta}$$

$$S_u = \frac{\partial}{\partial x}(\mu_t \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\mu_t \frac{\partial v}{\partial x})$$

```
 IF(NF= =1) THEN
 DO 590 J=2,M2
 DO 591 I=3,L2
 CON(I,J)=(GAM(I,J)*(U(I+1,J)-U(I,J))/XCV(I)
1 -GAM(I-1,J)*(U(I,J)-U(I-1,J))/XCV(I-1))/XDIF(I)
 GAMP=GAM(I,J+1)*GAM(I-1,J+1)/(GAM(I,J+1)+GAM(I-1,J+1)+1.E-30)
 GAMP=GAMP+GAM(I,J)*GAM(I-1,J)/(GAM(I,J)+GAM(I-1,J)+1.E-30)
 GAMM=GAM(I,J-1)*GAM(I-1,J-1)/(GAM(I,J-1)+GAM(I-1,J-1)+1.E-30)
 GAMM=GAMM+GAM(I,J)*GAM(I-1,J)/(GAM(I,J)+GAM(I-1,J)+1.E-30)
 CON(I,J)=CON(I,J)+(GAMP*(V(I,J+1)-V(I-1,J+1))
1 -GAMM*(V(I,J)-V(I-1,J)))/(YCV(J)*XDIF(I))
 AP(I,J)=0.
591 ENDDO
590 ENDDO
  RETURN
```

Source term calculation for u- eq.

$$D_{n-e} = \frac{\frac{(\delta x)_{e^-}}{(\delta y)_n}}{\Gamma_n} + \frac{\frac{(\delta x)_{e^+}}{(\delta y)_n}}{\Gamma_{ne}}$$



```
509 IF(NF= =2) THEN
      DO 594 J=3,M2
      DO 595 I=2,L2
      CON(I,J)=(GAM(I,J)*(V(I,J+1)-V(I,J))/YCV(J)-
     1  GAM(I,J-1)*(V(I,J)-V(I,J-1))/YCV(J-1))/(YDIF(J))
      GAMP=GAM(I+1,J)*GAM(I+1,J-1)/(GAM(I+1,J)+GAM(I+1,J-1)+1.E-30)
      GAMP=GAMP+GAM(I,J)*GAM(I,J-1)/(GAM(I,J)+GAM(I,J-1)+1.E-30)
      GAMM=GAM(I-1,J)*GAM(I-1,J-1)/(GAM(I-1,J)+GAM(I-1,J-1)+1.E-30)
      GAMM=GAMM+GAM(I,J)*GAM(I,J-1)/(GAM(I,J)+GAM(I,J-1)+1.E-30)
      CON(I,J)=CON(I,J)+(GAMP*(U(I+1,J)-U(I+1,J-1))
     1  -GAMM*(U(I,J)-U(I,J-1)))/(XCV(I)*YDIF(J))
      AP(I,J)=0.
595 ENDDO
594 ENDDO
      RETURN
      ENDIF
```

Source term calculation for v- eq.

```
 IF(NF= =4) THEN
     DO 596 J=2,M2
     DO 597 I=2,L2
     CON(I,J)=0.
     AP(I,J)=0.
597 ENDDO
586 ENDDO
     RETURN
```

Source term calculation for T- eq.

! Following two pages is for the source term of k- eq.:

$$S_k = \eta_t G - \rho\varepsilon = \eta_t G - (\frac{\rho\varepsilon}{k^*})k$$

! Most part is for calculation of GEN term

```
ELSE   IF(NF==5) THEN
DO 598 J=2,M2
DO 599 I=2,L2
DUDX=(U(I+1,J)-U(I,J))/XCV(I)
DVDY=(V(I,J+1)-V(I,J))/YCV(J)
IF(J= = 2) DUDY=(0.5*(U(I,J+1)-U(I,J))+0.5*(U(I+1,J+1)-
C U(I+1,J)))/YDIF(J+1)
```

```
 IF(J= =M2) DUDY=(0.5*(U(I,J)-U(I,J-1))+0.5*(U(I+1,J)-U(I+1,J-1))) /YDIF(J)
 IF(J/=2.AND.J/=M2) DUDY=(0.5*(U(I,J+1)-U(I,J-1))+0.5*(U(I+1,J+1)-
1 U(I+1,J-1)))/(YDIF(J)+YDIF(J+1))
 IF(I= =2) DVDX=(0.5*(V(I+1,J)-V(I-1,J))+0.5*(V(I+1,J+1)
1 -V(I-1,J+1)))/(XDIF(I)+XDIF(I+1))
 IF(I= =L2)  DVDX=(0.5*(V(I,J)-V(I-1,J))+0.5*(V(I,J+1)
1 -V(I-1,J+1)))/XDIF(I)
 IF(I/=2.AND.I/=L2) DVDX=(0.5*(V(I+1,J)-V(I,J))+0.5*(V(I+1,J+1)
1 -V(I,J+1)))/XDIF(I+1))
```

$$! \quad G = \frac{\eta_t}{\rho}\{2[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2] + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2\}$$

```
  GEN(I,J)=2.*(DUDX**2+DVDY**2)+(DUDY+DVDX)**2

  CONT(I,J)=GEN(I,J)*AMUT(I,J)

  AP(I,J)=-RHO(I,J)*DIS(I,J)/(AKE(I,J)+1.E-30)

598 ENDDO

599 ENDDO

  RETURN

  ENDIF
```

Sp of k-eq.

$$S_k = \eta_t G - \rho\varepsilon = \underline{\eta_t G} - (\frac{\rho\varepsilon}{k^*})k$$

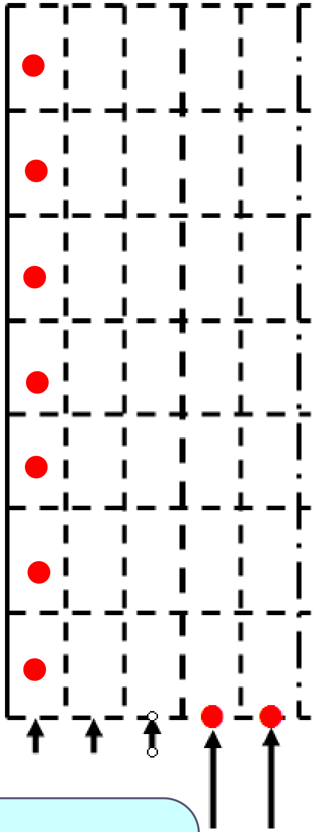
$$! \quad S_\varepsilon = \frac{c_1\varepsilon\eta_t G}{k} - \frac{c_2\rho\varepsilon^2}{k} = \frac{c_1\varepsilon\eta_t G}{k} - \left(\frac{c_2\rho\varepsilon^*}{k}\right)\varepsilon$$

●

```
    DO 600 J=2,M2
    DO 601 I=2,L2
    CON(I,J)=C1*GEN(I,J)*CMU*RHO(I,J)*AKE(I,J)
    AP(I,J)=-C2*RHO(I,J)*DIS(I,J)/(AKE(I,J)+1.E-30)
601 ENDDO
600 ENDDO
    DO 602 J=2,M2
    DISS=CMU*AKE(2,J)**1.5/(0.4*CMU4*XDIF(2))
    CON(2,J)=1.E30*DISS
    AP(2,J)=-1.E30
602 ENDDO
    RETURN
    END
```

Adopt large source term method for 1st inner node where i=2

Source term calculation for Epsilon eq.



12.8.4 Results analysis

```
          COMPUTATION IN CARTISIAN COORDINATES
          ************************************
ITER   SMAX       SSUM        V(6,6)       T(5,6)      KE(5,6)
   0  0.000E+00  0.000E+00  1.000E+01  1.000E+02  5.000E+01
   1  9.610E+00  1.192E-05  5.691E+01  3.871E+02  5.723E+01
   2  9.514E+00 -2.146E-06  5.821E+01  3.925E+02  6.918E+01
   3  1.631E+01 -2.623E-06  6.358E+01  3.929E+02  8.412E+01
   4  1.852E+01 -3.219E-06  6.502E+01  3.922E+02  1.001E+02
   5  7.633E+00  1.669E-06  6.859E+01  3.888E+02  1.158E+02
   6  3.936E+00  2.921E-06  7.270E+01  3.844E+02  1.306E+02
   7  2.899E+00  8.047E-07  7.596E+01  3.811E+02  1.450E+02
   8  2.467E+00 -1.520E-06  7.880E+01  3.781E+02  1.588E+02
   9  1.233E+00  3.767E-06  8.126E+01  3.754E+02  1.720E+02
  10  5.827E-01  8.019E-07  8.348E+01  3.730E+02  1.848E+02
  11  4.670E-01 -3.013E-07  8.535E+01  3.708E+02  1.974E+02
  12  2.132E-01 -1.404E-06  8.684E+01  3.690E+02  2.098E+02
  13  1.752E-01  2.645E-06  8.799E+01  3.675E+02  2.219E+02
  14  2.045E-01  7.786E-07  8.881E+01  3.663E+02  2.334E+02
  15  1.997E-01  1.352E-06  8.936E+01  3.654E+02  2.443E+02
```

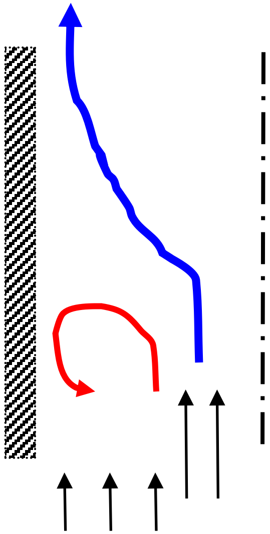
```
ITER   SMAX        SSUM         V(6,6)        T(5,6)       KE(5,6)

  16  1.952E-01 -3.356E-06  8.968E+01  3.647E+02  2.543E+02
  17  1.732E-01 -5.886E-07  8.983E+01  3.642E+02  2.633E+02
  18  1.515E-01  1.214E-06  8.985E+01  3.639E+02  2.711E+02
  19  1.275E-01  2.459E-06  8.978E+01  3.637E+02  2.778E+02
  20  1.135E-01 -1.770E-07  8.966E+01  3.636E+02  2.836E+02
  21  9.660E-02  1.088E-06  8.950E+01  3.635E+02  2.884E+02
  22  8.860E-02  1.376E-06  8.932E+01  3.635E+02  2.924E+02
  23  8.655E-02  4.222E-06  8.913E+01  3.635E+02  2.958E+02
  24  8.673E-02  1.618E-06  8.894E+01  3.635E+02  2.987E+02
  25  8.763E-02  6.519E-08  8.874E+01  3.635E+02  3.011E+02
  26  8.823E-02 -1.248E-06  8.855E+01  3.636E+02  3.032E+02
  27  8.634E-02  1.515E-06  8.837E+01  3.637E+02  3.050E+02
  28  8.221E-02 -9.155E-07  8.820E+01  3.637E+02  3.066E+02
  29  7.629E-02 -4.168E-07  8.803E+01  3.638E+02  3.079E+02
  30  6.849E-02  4.278E-06  8.789E+01  3.639E+02  3.090E+02
```

```
ITER   SMAX        SSUM         V(6,6)       T(5,6)      KE(5,6)
 31  6.000E-02 -1.577E-06  8.776E+01  3.639E+02  3.100E+02
 32  5.131E-02 -1.215E-06  8.764E+01  3.640E+02  3.109E+02
 33  4.320E-02  1.020E-06  8.753E+01  3.640E+02  3.117E+02
 34  3.870E-02 -1.668E-06  8.744E+01  3.640E+02  3.125E+02
 35  3.469E-02 -1.627E-06  8.736E+01  3.641E+02  3.132E+02
 36  3.132E-02  2.183E-06  8.728E+01  3.641E+02  3.138E+02
 37  2.813E-02 -1.673E-06  8.722E+01  3.641E+02  3.145E+02
 38  2.516E-02 -2.713E-06  8.715E+01  3.641E+02  3.151E+02
 39  2.318E-02  7.274E-07  8.710E+01  3.641E+02  3.157E+02
 40  2.092E-02  5.514E-06  8.705E+01  3.642E+02  3.163E+02
 41  1.954E-02 -5.197E-07  8.700E+01  3.642E+02  3.169E+02
 42  1.805E-02 -7.967E-07  8.695E+01  3.642E+02  3.174E+02
 43  1.683E-02  3.801E-06  8.691E+01  3.642E+02  3.179E+02
 44  1.575E-02 -3.199E-06  8.687E+01  3.642E+02  3.184E+02
 45  1.476E-02  2.365E-06  8.684E+01  3.642E+02  3.188E+02
 46  1.418E-02  2.495E-06  8.680E+01  3.642E+02  3.192E+02
 47  1.367E-02  4.471E-06  8.678E+01  3.643E+02  3.196E+02
 48  1.321E-02  5.106E-07  8.675E+01  3.643E+02  3.199E+02
 49  1.282E-02  1.093E-06  8.672E+01  3.643E+02  3.202E+02
 50  1.222E-02 -4.708E-07  8.670E+01  3.643E+02  3.205E+02
```

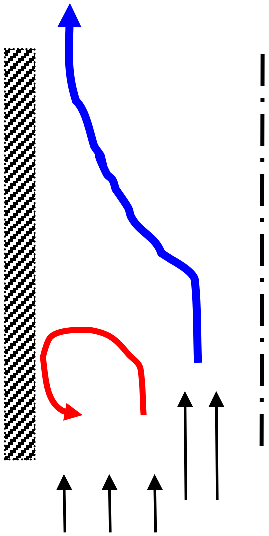
```
****************************  .VEL  U.   ****************************

 I =       2          3          4           5           6          7
J
 9   0.00E+00 0.00E+00 0.00E+00 0.00E+00  0.00E+00 0.00E+00
 8   0.00E+00 1.56E-02  3.75E-02  3.84E-02   2.04E-02 0.00E+00
 7   0.00E+00-1.65E+00-2.68E+00-2.78E+00-1.33E+00 0.00E+00
 6   0.00E+00-2.37E+00-3.56E+00-3.57E+00-1.63E+00 0.00E+00
 5   0.00E+00-2.38E+00-3.88E+00-3.98E+00-1.66E+00 0.00E+00
 4   0.00E+00-1.39E+00-3.33E+00-3.86E+00-1.45E+00 0.00E+00
 3   0.00E+00 3.74E+00-3.47E-01 -2.75E+00-8.62E-01  0.00E+00
 2   0.00E+00 4.44E+00 6.55E+0 0-2.87E+00-6.77E-01  0.00E+00
 1   0.00E+00 0.00E+00 0.00E+00  0.00E+00  0.00E+00 0.00E+00
```



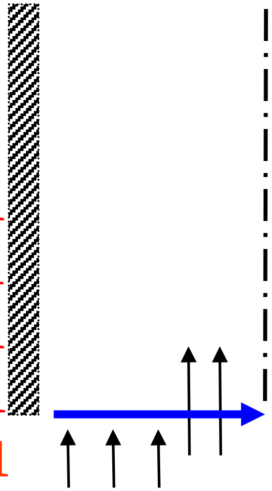
```
****************************    .VEL  V.    ****************************

 I =       1         2          3         4          5          6          7
J
 9    0.00E+00 8.87E+00 3.18E+01 4.59E+01 6.52E+01 7.82E+01 1.00E+01
 8    0.00E+00 8.87E+00 3.18E+01 4.59E+01 6.52E+01 7.82E+01 1.00E+01
 7    0.00E+00 4.16E+00 2.89E+01 4.56E+01 6.93E+01 8.20E+01 1.00E+01
 6    0.00E+00-2.61E+00 2.55E+01 4.56E+01 7.48E+01 8.67E+01 1.00E+01
 5    0.00E+00-9.41E+00 2.12E+01 4.53E+01 8.15E+01 9.14E+01 1.00E+01
 4    0.00E+00-1.34E+01 1.56E+01 4.38E+01 8.83E+01 9.56E+01 1.00E+01
 3    0.00E+00-2.70E+00 3.98E+00 3.69E+01 9.37E+01 9.81E+01 1.00E+01
 2    1.00E+01 1.00E+01 1.00E+01 1.00E+01 1.00E+02 1.00E+02 1.00E+02
```



```
****************************    .STR  FN    ****************************

 I =      2         3          4           5          6          7
J
 9   0.00E+00-1.77E+00 -8.12E+00-1.73E+01-3.03E+01-4.60E+01
 8   0.00E+00-1.77E+00 -8.14E+00-1.73E+01-3.04E+01-4.60E+01
 7   0.00E+00-8.31E-01  -6.61E+00-1.57E+01-2.96E+01-4.60E+01
 6   0.00E+00 5.21E-01  -4.58E+00-1.37E+01 -2.87E+01-4.60E+01
 5   0.00E+00 1.88E+00 -2.36E+0 0-1.14E+01-2.77E+01-4.60E+01
 4   0.00E+00 2.68E+00 -4.55E-01 -9.22E+00 -2.69E+01-4.60E+01
 3   0.00E+00 5.39E-01  -2.57E-01 -7.64E+00 -2.64E+01-4.60E+01
 2   0.00E+00-2.00E+00 -4.00E+00-6.00E+00 -2.60E+01-4.60E+01
```



Stream function increase along this direction

```
****************************     . TEMP .    ****************************

 I =      1          2          3          4          5          6          7
J
 9    0.00E+00 1.00E+02 1.00E+02 1.00E+02 1.00E+02 1.00E+02 1.00E+02
 8    0.00E+00 3.01E+02 3.26E+02 3.39E+02 3.60E+02 3.80E+02 1.00E+02
 7    0.00E+00 3.00E+02 3.21E+02 3.35E+02 3.63E+02 3.85E+02 1.00E+02
 6    0.00E+00 2.93E+02 3.10E+02 3.26E+02 3.64E+02 3.89E+02 1.00E+02
 5    0.00E+00 2.80E+02 2.93E+02 3.11E+02 3.67E+02 3.92E+02 1.00E+02
 4    0.00E+00 2.65E+02 2.69E+02 2.88E+02 3.72E+02 3.95E+02 1.00E+02
 3    0.00E+00 2.52E+02 2.36E+02 2.53E+02 3.79E+02 3.97E+02 1.00E+02
 2    0.00E+00 1.29E+02 1.16E+02 2.01E+02 3.90E+02 3.99E+02 1.00E+02
 1    0.00E+00 1.00E+02 1.00E+02 1.00E+02 4.00E+02 4.00E+02 4.00E+02
```

Given inlet temp.

- **Given wall temp**

```
****************************   KIN  ENE   ****************************

 I =      1        2         3         4         5         6          7
J
 9   5.00E-01 5.00E-01 5.00E-01  5.00E-01  5.00E+01 5.00E+01 5.00E+01
 8   5.00E-01 1.59E+02 4.93E+02 4.65E+02 3.53E+02 2.15E+02 2.15E+02
 7   5.00E-01 1.90E+02 5.34E+02 4.85E+02 3.35E+02 1.74E+02 1.74E+02
 6   5.00E-01 2.20E+02 5.83E+02 5.22E+02 3.20E+02 1.37E+02 1.37E+02
 5   5.00E-01 2.39E+02 6.06E+02 5.46E+02 2.94E+02 1.06E+02 1.06E+02
 4   5.00E-01 2.15E+02 5.40E+02 5.31E+02 2.54E+02 8.23E+01 8.23E+01
 3   5.00E-01 1.15E+02 3.30E+02 4.69E+02 2.06E+02 6.62E+01 6.62E+01
 2   5.00E-01 1.88E+01 1.03E+01 3.22E+02 1.46E+02 5.55E+01 5.55E+01
 1   5.00E-01 5.00E-01  5.00E-01  5.00E-01  5.00E+01 5.00E+01 5.00E+01
```

```
****************************   .DISIPA.   ****************************

 I =     1          2          3          4          5          6          7
J
 9   2.50E-02 2.50E-02  2.50E-02  2.50E-02  2.50E+02 2.50E+02 2.50E+02
 8   2.50E-02 8.18E+03 1.25E+04 1.13E+04 7.78E+03 3.60E+03 3.60E+03
 7   2.50E-02 1.07E+04 1.44E+04 1.28E+04 7.79E+03 2.82E+03 2.82E+03
 6   2.50E-02 1.34E+04 1.71E+04 1.53E+04 7.94E+03 2.12E+03 2.12E+03
 5   2.50E-02 1.51E+04 1.93E+04 1.80E+04 7.66E+03 1.50E+03 1.50E+03
 4   2.50E-02 1.29E+04 1.79E+04 1.98E+04 6.81E+03 1.01E+03 1.01E+03
 3   2.50E-02 5.08E+03 1.04E+04 1.99E+04 5.46E+03 6.63E+02 6.63E+02
 2   2.50E-02 3.34E+02 1.53E+02 1.52E+04 3.43E+03 4.02E+02 4.02E+02
 1   2.50E-02 2.50E-02  2.50E-02  2.50E-02  2.50E+02 2.50E+02 2.50E+02
```

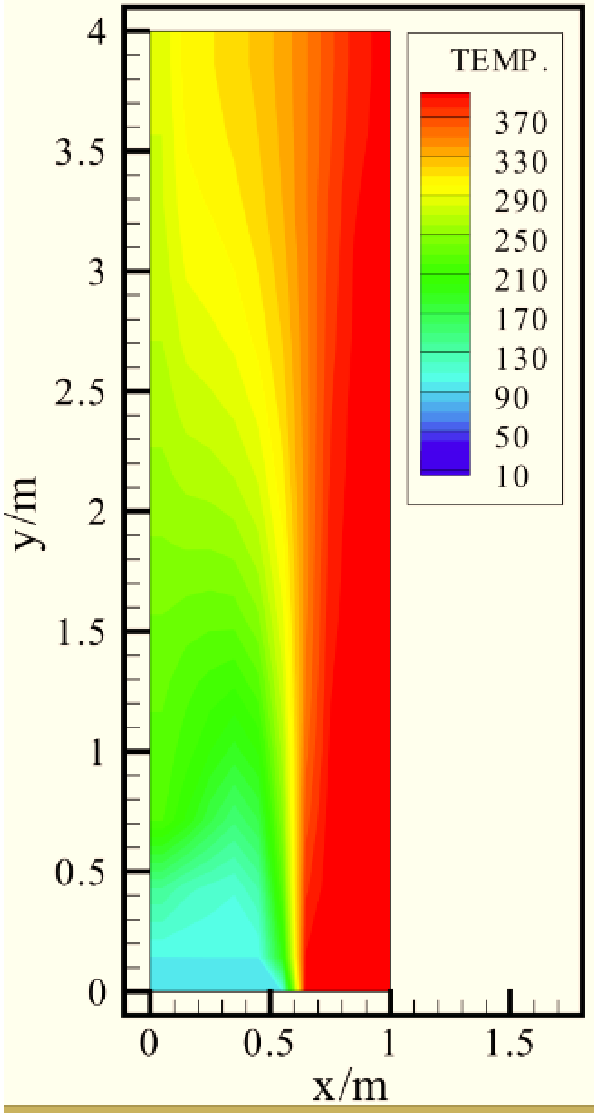
```
***************************   TURB  VI   ***************************

 I =      1          2          3          4           5           6           7
J
 9    9.00E-01 9.00E-01 9.00E-01  9.00E-01  9.00E-01  9.00E-01 9.00E-01
 8    9.00E-01 2.78E-01 1.72E+00 1.70E+00 1.42E+00 1.14E+00 1.14E+00
 7    9.00E-01 3.04E-01 1.76E+00 1.65E+00 1.29E+00 9.59E-01  9.59E-01
 6    9.00E-01 3.27E-01 1.77E+00 1.59E+00 1.16E+00 7.99E-01  7.99E-01
 5    9.00E-01 3.40E-01 1.71E+00 1.48E+00 1.01E+00 6.75E-01  6.75E-01
 4    9.00E-01 3.22E-01 1.46E+00 1.28E+00 8.54E-01  6.02E-01  6.02E-01
 3    9.00E-01 2.36E-01 9.39E-01  9.99E-01  7.00E-01  5.94E-01  5.94E-01
 2    9.00E-01 9.50E-02 6.24E-02  6.19E-01  5.58E-01  6.88E-01  6.88E-01
 1    9.00E-01 9.00E-01 9.00E-01  9.00E-01  9.00E-01  9.00E-01  9.00E-01
```

Molecular viscosity $\mu_l \approx 10^{-6}$

```
****************************    PRESSURE   ****************************

 I =     1          2          3          4          5          6          7
J
 9   1.44E+03  1.43E+03  1.41E+03 1.33E+03 1.21E+03  1.14E+03 1.12E+03
 8   1.36E+03  1.35E+03  1.33E+03 1.28E+03 1.20E+03  1.15E+03 1.13E+03
 7   1.20E+03  1.19E+03  1.17E+03 1.17E+03 1.17E+03  1.16E+03 1.16E+03
 6   9.40E+02  9.31E+02  9.11E+02 9.19E+02 9.28E+02  9.26E+02 9.25E+02
 5   6.02E+02  5.92E+02  5.72E+02 5.96E+02 6.22E+02  6.25E+02 6.27E+02
 4   2.24E+02  2.16E+02  1.99E+02 2.54E+02 3.08E+02  3.24E+02 3.32E+02
 3   4.20E+01  3.16E+01  1.09E+01 1.03E+02 1.39E+02  1.44E+02 1.46E+02
 2   1.31E+01  5.48E+00-9.74E+00-6.55E+01  2.53E+01 4.85E+01 6.02E+01
 1   0.00E+00-7.61E+00 -2.01E+01-1.50E+02-3.17E+01 1.07E+00 1.27E+01
```


TEMP.
370
330
290
250
210
170
130
90
50
10
y/m
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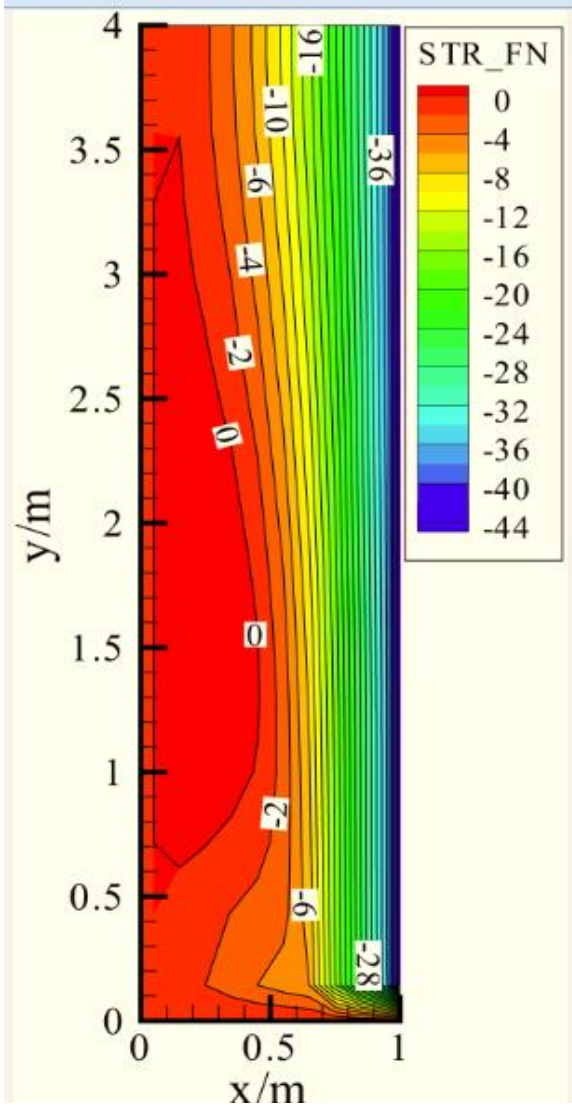
0.5
1
1.5
2
2.5
3
3.5
4
x/m
0
0.5
1
1.5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-32
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-4
-2
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0
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-6
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3
3.5
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x/m
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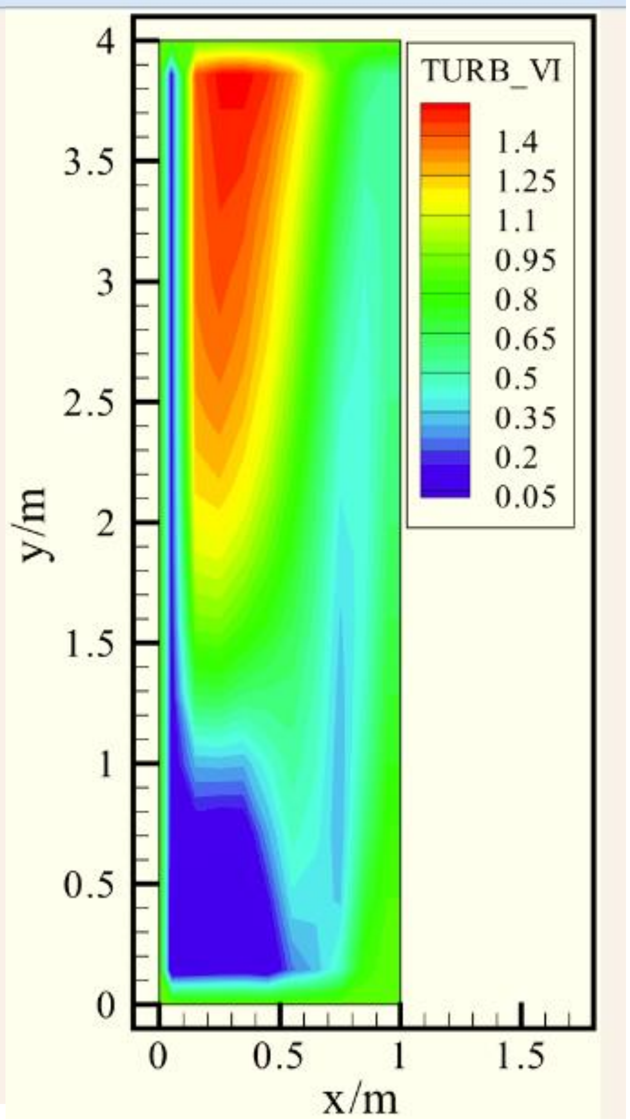
0.5
1


TURB_VI
1.4
1.25
1.1
0.95
0.8
0.65
0.5
0.35
0.2
0.05
y/m
0
0.5
1
1.5
2
2.5
3
3.5
4
x/m
0
0.5
1
1.5

11.7 Low Reynolds Number k-epsilon Model

11.7.1 Application range of standard $k-\varepsilon$ model

11.7.2 Jones—Launder low Re $k-\varepsilon$ model

11.7.3 Other low Re $k-\varepsilon$ models

11.7 Low Reynolds Number k-epsilon Model

11.7.1 Application range of standard $k-\varepsilon$ model

1. Near wall velocity distribution obeys logarithmic law
2. Shear stress is distributed uniformly from wall to 1st inner node;
3. Production and dissipation are nearly balanced for fluctuation kinetic energy.

Above assumptions are valid only when

$$\mathrm{Re}_t = \frac{\rho k^2}{\eta \varepsilon} > 150$$

When this Re less than 150， the standard $k-\varepsilon$ model can not be used. When approaching wall this Reynolds number becomes smaller and smaller. In order that simulation can be conducted from vigorous part down to the wall, model should be modified.

11.7.2 Jones－Launder low Re $k-\varepsilon$ model

1. Jones-Launder low Re model considerations(1972)

(1) Both molecular and turbulent diffusions should be considered;

(2) Effects of $\mathrm{Re}_t=\dfrac{\rho k^2}{\eta\varepsilon}$ on coefficients should be considered;

(3) Near a wall dissipation of fluctuation kinetic energy is not isotropic, and should be taken into account in k eq.

2. Jones-Launder low Reynolds $k-\varepsilon$ model

$$\frac{\partial(\rho k)}{\partial t}+\frac{\partial(\rho u_j k)}{\partial x_j}=\frac{\partial}{\partial x_j}\left[\left(\eta_l+\frac{\eta_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]+\rho G-\rho\varepsilon\underbrace{-2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2}_{D}$$

$$\frac{\partial(\rho \varepsilon)}{\partial t}+\frac{\partial(\rho u_j \varepsilon)}{\partial x_j}=\frac{\partial}{\partial x_j}\left[\left(\eta_l+\frac{\eta_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]+f_1C_1\frac{\rho G\varepsilon}{k}-f_2C_2\rho\frac{\varepsilon^2}{k}\underbrace{+2\frac{\eta_l\eta_t}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)^2}_{E}$$

$$\eta_t = C_\mu f_\mu \rho k^2 / \varepsilon$$

where $f_1 = 1.0$

$$f_2 = 1.0 - 0.3\exp(-\mathrm{Re}_t^2)$$

$$f_\mu = \exp(-2.5/(1+\mathrm{Re}_\mathrm{t}/50))$$

$$\mathrm{Re}_t = \frac{\rho k^2}{\eta\varepsilon}$$

Explanation：The vertical lines in Eqs.（9-47),(9-48) (textbook page.363) just show that the term is newly added, not the symbols of absolute value

3. Explanations for additional terms

(1) $D = -2\eta(\frac{\partial k^{1/2}}{\partial y})^2$ **(y is normal to wall), for considering that near a wall fluctuation kinetic energy is not isotropic, and with this term the condition of** $\varepsilon_w = 0$ **can be used;**

(2) The term E is for a better agreement with test data.

4. Boundary condition of J-L low Re model

$$k_w = \varepsilon_w = 0$$

11.7.3 Other low Re $k-\varepsilon$ models

Since the proposal of J-L low Re model in 1972, more than 20 variants (变体) have been proposed. The major differences between them are in four aspects:

(1) **Different values of the three modified coefficients:**

$$f_1,\ f_2,\ f_\mu$$

(2) **Different expressions of additional terms D and E ;**

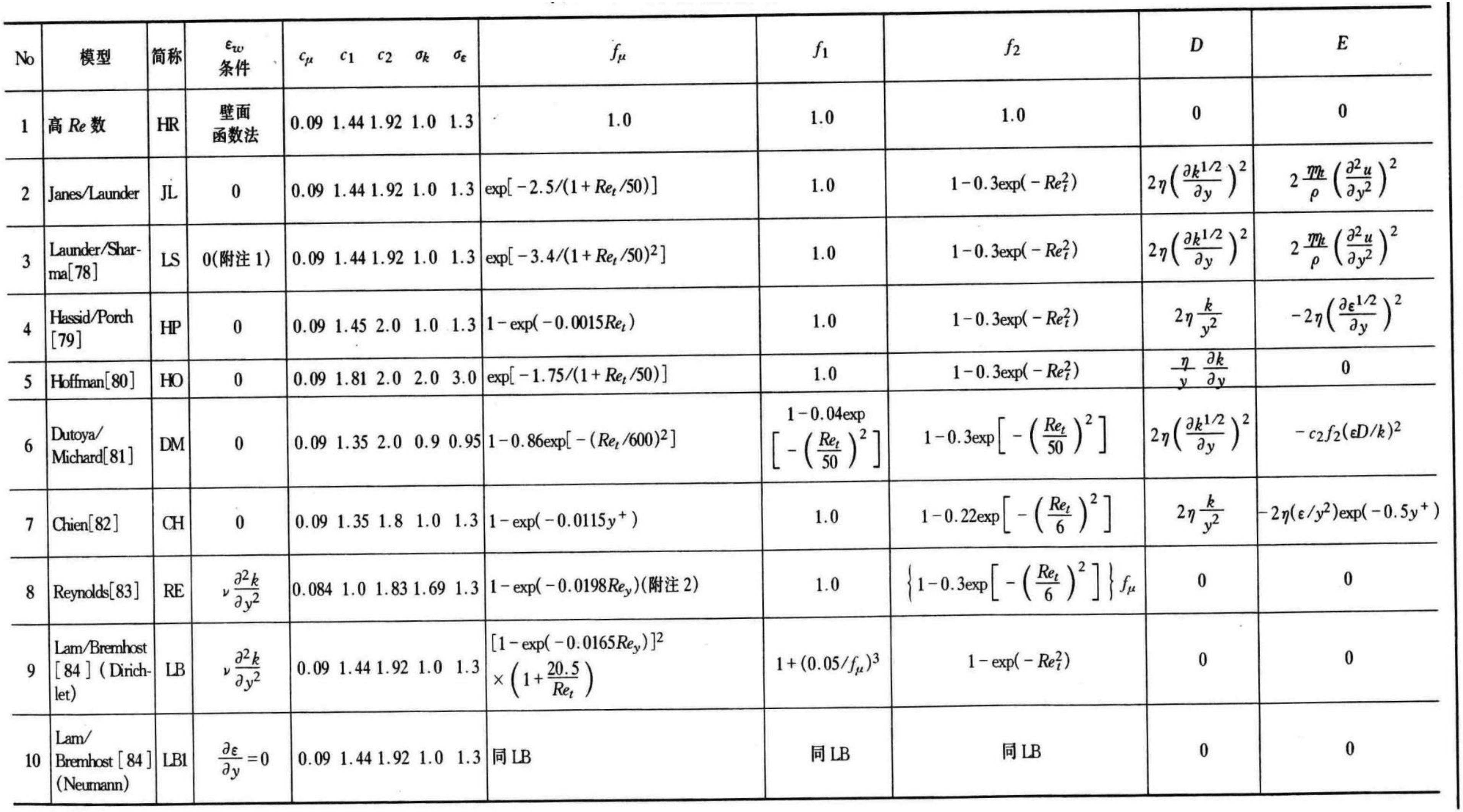
(3) **Different wall boundary condition for** ε

$$\varepsilon = 0;$$

$$\frac{\partial \varepsilon}{\partial n} = 0$$

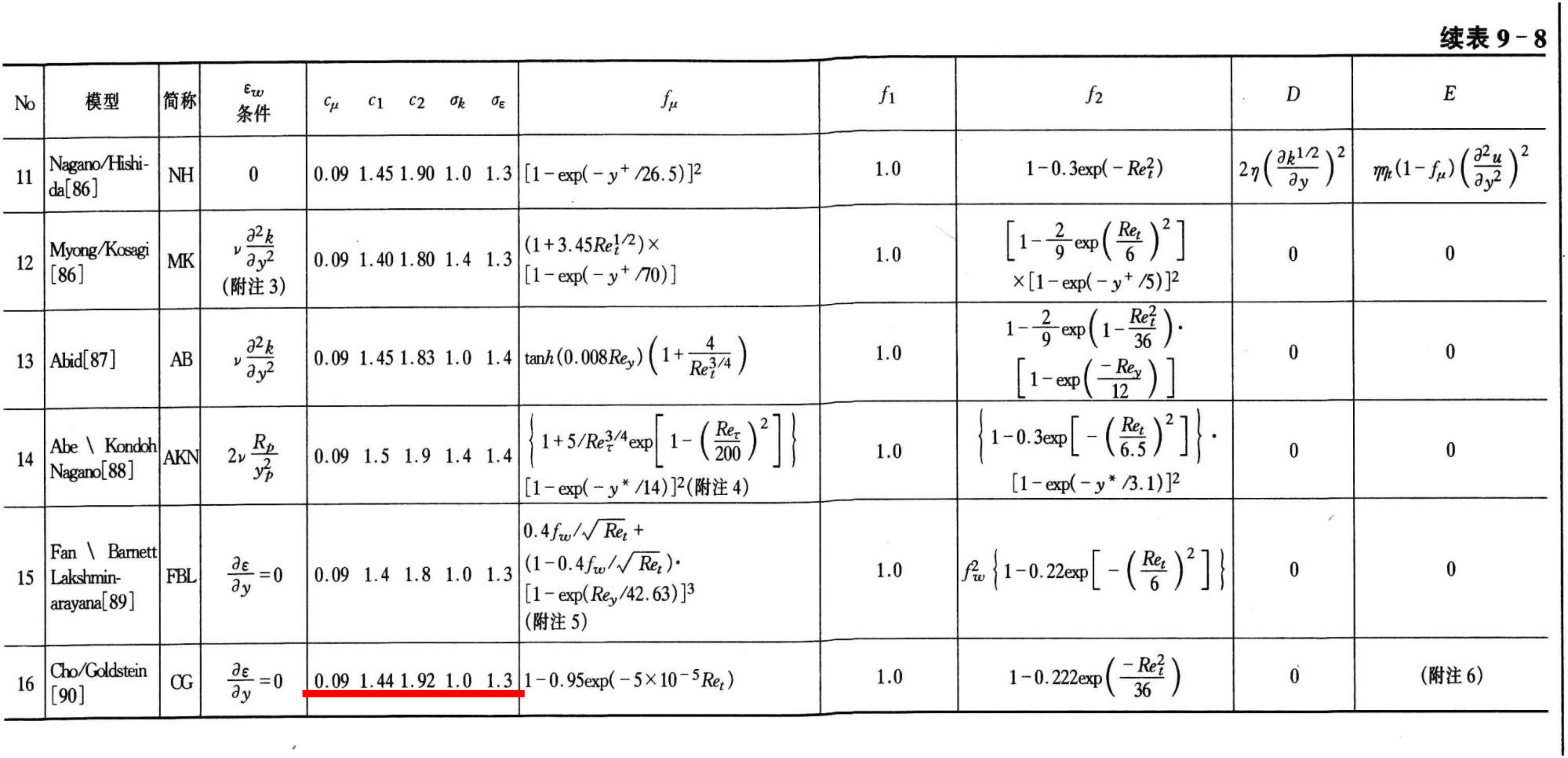
(4) **Different values of coefficients** C_1, C_2, C_μ **and constants** $\sigma_k, \sigma_\varepsilon$

Table 9-8 in Textbook



No	模型	简称	ε_w 条件	c_μ	c_1	c_2	σ_k	σ_ε	f_μ	f_1	f_2	D	E
1	高 Re 数	HR	壁面函数法	0.09	1.44	1.92	1.0	1.3	1.0	1.0	1.0	0	0
2	Janes/Launder	JL	0	0.09	1.44	1.92	1.0	1.3	$\exp[-2.5/(1+Re_t/50)]$	1.0	$1-0.3\exp(-Re_t^2)$	$2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$2\frac{\eta\mu_t}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)^2$
3	Launder/Sharma[78]	LS	0(附注 1)	0.09	1.44	1.92	1.0	1.3	$\exp[-3.4/(1+Re_t/50)^2]$	1.0	$1-0.3\exp(-Re_t^2)$	$2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$2\frac{\eta\mu_t}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)^2$
4	Hassid/Porch[79]	HP	0	0.09	1.45	2.0	1.0	1.3	$1-\exp(-0.0015Re_t)$	1.0	$1-0.3\exp(-Re_t^2)$	$2\eta\frac{k}{y^2}$	$-2\eta\left(\frac{\partial\varepsilon^{1/2}}{\partial y}\right)^2$
5	Hoffman[80]	HO	0	0.09	1.81	2.0	2.0	3.0	$\exp[-1.75/(1+Re_t/50)]$	1.0	$1-0.3\exp(-Re_t^2)$	$\frac{\eta}{y}\frac{\partial k}{\partial y}$	0
6	Dutoya/Michard[81]	DM	0	0.09	1.35	2.0	0.9	0.95	$1-0.86\exp[-(Re_t/600)^2]$	$1-0.04\exp\left[-\left(\frac{Re_t}{50}\right)^2\right]$	$1-0.3\exp\left[-\left(\frac{Re_t}{50}\right)^2\right]$	$2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$-c_2f_2(\varepsilon D/k)^2$
7	Chien[82]	CH	0	0.09	1.35	1.8	1.0	1.3	$1-\exp(-0.0115y^+)$	1.0	$1-0.22\exp\left[-\left(\frac{Re_t}{6}\right)^2\right]$	$2\eta\frac{k}{y^2}$	$-2\eta(\varepsilon/y^2)\exp(-0.5y^+)$
8	Reynolds[83]	RE	$\nu\frac{\partial^2 k}{\partial y^2}$	0.084	1.0	1.83	1.69	1.3	$1-\exp(-0.0198Re_y)$(附注 2)	1.0	$\left\{1-0.3\exp\left[-\left(\frac{Re_t}{6}\right)^2\right]\right\}f_\mu$	0	0
9	Lam/Bremhost[84](Dirichlet)	LB	$\nu\frac{\partial^2 k}{\partial y^2}$	0.09	1.44	1.92	1.0	1.3	$[1-\exp(-0.0165Re_y)]^2\times\left(1+\frac{20.5}{Re_t}\right)$	$1+(0.05/f_\mu)^3$	$1-\exp(-Re_t^2)$	0	0
10	Lam/Bremhost[84](Neumann)	LB1	$\frac{\partial\varepsilon}{\partial y}=0$	0.09	1.44	1.92	1.0	1.3	同 LB	同 LB	同 LB	0	0

Table 9-8 in Textbook（Continued）



续表 9－8

No	模型	简称	ε_w 条件	c_μ	c_1	c_2	σ_k	σ_ε	f_μ	f_1	f_2	D	E
11	Nagano/Hishida[86]	NH	0	0.09	1.45	1.90	1.0	1.3	$[1-\exp(-y^+/26.5)]^2$	1.0	$1-0.3\exp(-Re_t^2)$	$2\eta\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$\eta\eta_t(1-f_\mu)\left(\frac{\partial^2 u}{\partial y^2}\right)^2$
12	Myong/Kosagi [86]	MK	$\nu\frac{\partial^2 k}{\partial y^2}$ (附注 3)	0.09	1.40	1.80	1.4	1.3	$(1+3.45Re_t^{1/2})\times$ $[1-\exp(-y^+/70)]$	1.0	$\left[1-\frac{2}{9}\exp\left(\frac{Re_t}{6}\right)^2\right]$ $\times[1-\exp(-y^+/5)]^2$	0	0
13	Abid[87]	AB	$\nu\frac{\partial^2 k}{\partial y^2}$	0.09	1.45	1.83	1.0	1.4	$\tanh(0.008Re_y)\left(1+\frac{4}{Re_t^{3/4}}\right)$	1.0	$1-\frac{2}{9}\exp\left(1-\frac{Re_t^2}{36}\right)\cdot$ $\left[1-\exp\left(\frac{-Re_y}{12}\right)\right]$	0	0
14	Abe \ Kondoh Nagano[88]	AKN	$2\nu\frac{R_p}{y_p^2}$	0.09	1.5	1.9	1.4	1.4	$\left\{1+5/Re_\tau^{3/4}\exp\left[1-\left(\frac{Re_\tau}{200}\right)^2\right]\right\}$ $[1-\exp(-y^*/14)]^2$(附注 4)	1.0	$\left\{1-0.3\exp\left[-\left(\frac{Re_t}{6.5}\right)^2\right]\right\}\cdot$ $[1-\exp(-y^*/3.1)]^2$	0	0
15	Fan \ Barnett Lakshmin-arayana[89]	FBL	$\frac{\partial\varepsilon}{\partial y}=0$	0.09	1.4	1.8	1.0	1.3	$0.4f_w/\sqrt{Re_t}+$ $(1-0.4f_w/\sqrt{Re_t})\cdot$ $[1-\exp(Re_y/42.63)]^3$ (附注 5)	1.0	$f_w^2\left\{1-0.22\exp\left[-\left(\frac{Re_t}{6}\right)^2\right]\right\}$	0	0
16	Cho/Goldstein [90]	CG	$\frac{\partial\varepsilon}{\partial y}=0$	0.09	1.44	1.92	1.0	1.3	$1-0.95\exp(-5\times10^{-5}Re_t)$	1.0	$1-0.222\exp\left(\frac{-Re_t^2}{36}\right)$	0	(附注 6)

11.8 Brief Introduction to Recent Developments

11.8.1 Developments in $k-\varepsilon$ two-equation model

11.8.2 Brief introduction to second moment model

11.8.3 Near wall region treatment of different models

11.8.4 Chen model for indoor air movement

11.8.5 $\overline{V^2}$ - f Turbulence model for highly inhomogeneous turbulent flow

11.8 Brief Introduction to Recent Developments

11.8.1 Developments of $k-\varepsilon$ two-eq. model

1. Non-linear $k-\varepsilon$ model

In Boussinesq's constitution eq. every term is of 1st order---linear leading to $\tau_{xx}=\tau_{yy}$ for fully developed turbulent flow in parallel plate duct, which does not agree with test results.

Boussinesq's constitution eq.

$$(\tau_{i,j})_t = -\rho\overline{u_i^{'}u_j^{'}} = (-p_t\delta_{i,j}) + \eta_t\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)$$

Speziale et al. proposed a non-linear model in 1987, see reference [95] of the textbook.

2. Multi-scale $k-\varepsilon$ model

In the standard $k-\varepsilon$ model only one geometric scale is used. Actually turbulent flow fluctuations cover a wide range of time scales and geometric scales. A simple improvement is adopting two geometric scales: big eddies for carrying kinetic energy(载能涡) and small eddies for dissipating energy(耗能涡). See reference [108].

3. Renormalized group (重整化群) model

Starting from transient N-S eq. Yakhot-Orzag adopted spectral analysis (谱分析)method and derived k-epsilon equations with different coefficients and constants.

See Ref.[113] in the textbook.

3. Realizable $k-\varepsilon$ model (可实现)

In the standard k-epsilon model when fluid strain is very large the normal stress will be negative, which is not realizable; In order to establish all-cases realizable model the coefficient C_μ should be related with strain. (应变) See ref. [115] in the textbook。

9.7.2 Brief introduction to second moment model(二阶矩模型)

For the products with two fluctuations, $-\rho\overline{u_i^{'}u_j^{'}}$, their governing eqs. are derived; for products with more than two fluctuations, say $\overline{u_i^{'}u_j^{'}u_k^{'}}$, models are introduced to close the model.

1. Original form of Reynolds stress equation

$$\frac{\partial \overline{u_i'u_j'}}{\partial t} + u_k \frac{\partial \overline{u_i'u_j'}}{\partial x_k} = P_{i,j} + \pi_{i,j} + D_{i,j} - \varepsilon_{i,j}$$

where $P_{i,j} = -(\overline{u_i'u_k'}\frac{\partial u_j}{\partial x_k} + \overline{u_i'u_k'}\frac{\partial u_i}{\partial x_k}) -$ **Production term**

$$\pi_{i,j} = \overline{\frac{p'}{\rho}(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j})} -$$

Redistribution term

$$D_{i,j} = -\frac{\partial}{\partial x_k}(\overline{u_i'u_j'u_k'} - \nu \frac{\overline{\partial (u_i'u_j')}}{\partial x_k} + \delta_{i,k}\frac{\overline{u_j'p'}}{\rho}) -$$

Diffusion term

The above three terms $P_{i,j}, D_{i,j}, \pi_{i,j}$ **have to be simplified or modeled . Different treatments lead to different second moment models.**

3. Eqs. and constants in 2nd moment closure for convective heat transfer

(1) 3-D time average governing eqs.---16 eqs. are needed：

5 time average eqs. for five variables： u, v, w, p, T

6 time average fluctuation stress eqs.

3 eqs. for additional heat flux

1 eq. for k， and

1 eq. for ε

(2) Nine empirical constants.

11.8.3 Near wall region treatment of different models

All the above improvements are only for the vigorous part of turbulent flow; for near wall region the molecular viscosity must be taken into account. At present following methods are used：

1. Adopting WFM；

2. Adopting two-layer model： several choices

(1) With Re_t=150 as a deviding line(分界线）:adopting one of the above model when it is larger than 150 ; if Re_t is less than 150 low Re k-epsilon model is used.

(2) In near wall region k equation model is used, and in the vigorous part above model is adopted.

Emphasis should be paid for the near wall region.

11.8.4 Chen model for indoor air movement

Q Y Chen proposed following simple model for indoor air turbulent flow:

$$\eta_t = 0.03874 \rho v l$$

$\rho -$ **air density**

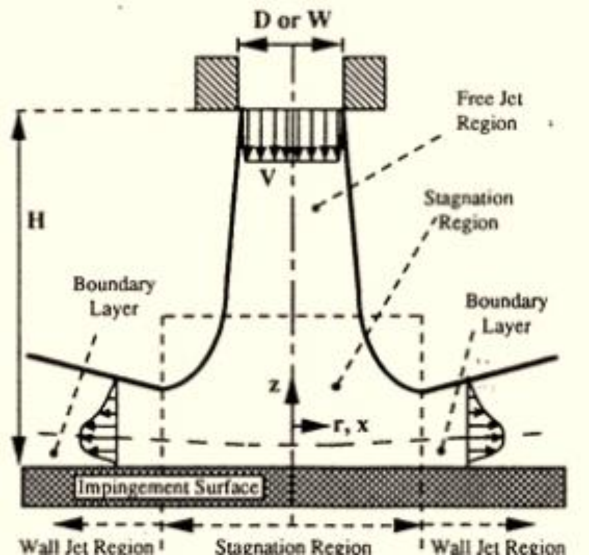
$v -$ **local time average velocity**

$l -$ **the shortest distance to the wall**

Qingyang Chen, Weiran Xu. A zero-equation model for indoor airflow simulation. Energy and Building, 1998, 28, 137-144

11.8.5 $\overline{v^2}-f$ Turbulence model for highly inhomogeneous turbulent flow

For highly inhomogeneous flow and heat transfer, such as jet impingement flow, this $\overline{v^2}-f$ turbulence model may obtain reasonable simulation results.



[1] Durbin PA. Near wall turbulence closure modeling without damping functions. Theoretical and Computational Fluid Dynamics, 1991, 3:1-13

[2] Laurence D, Popovac M, and Uribe JC., and Utsyuzhinikov SV. A robust formulation of $\overline{v^2}-f$ model, Flow, Turbulence and Combustion, 2004, 73, 169-185

[3] Hanjalic K, Laurence D, Popovac M, and Uribe JC. $\overline{v^2}/k-f$ turbulence model and its applications to forced and natural convections, Engineering Turbulence Modeling and Experiments, 2005, 6: 67-86

Home work

9-1 9-2 9-4 9-5

Due on Dec. 14th

Home work

Problem # 9-1

Take the following data to estimate the difference between the fluid thermodynamic pressure and turbulent effective pressure; for the air flow through the wind tunnel, the pressure of the air is 1 bar, the average velocity is $u = 50$ m/s, the temperature of air is $20\,^{\circ}C$, and turbulence intensity $\sqrt{u'^2}/u = 5\%$, (which is a quite large value). Assumed that the turbulence is isotropic, i.e. various statistical values regardless of the direction of turbulence, here is $\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$

Problem # 9-2

Try to write k equation in three dimension Cartesian coordinates (see 9-21)

Problem # 9-4

In a two-dimensional boundary layer flow, if the generation of turbulent kinetic energy and dissipation balanced each other, try this

$$\sqrt{\tau_w / \rho} = C_\mu^{1/4} k^{1/2} .$$

Problem # 9-5

The definition of turbulent kinetic energy dissipation rate is $\varepsilon = \nu[\overline{(\frac{\partial u'_i}{\partial x})^2}$. Try to write the expression of ε in three dimension Cartesian coordinates, and identify its dimension and unit (SI). Then to analyze c_μ, c_1, c_2 are dimensionless numbers or not.

同舟共济
渡彼岸！

People in the same boat help each other to cross to the other bank, where….