数值传热学

第七章 求解椭圆型问题的涡量流函数方法

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Numerical Heat Transfer
(数值传热学)

Chapter 7 Stream Function-Vorticity Methods for Elliptic Flow and Heat Transfer

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Chapter 7 Stream-vorticity method for Solving Flow Fields

7.1 Stream-vorticity method for forced convection and discretization

7.2 Boundary conditions for stream-vorticity method

7.3 Stream-vorticity wall boundary condition

7.4 Example of stream-vorticity method for natural convection in enclosure

7.1 Stream-vorticity method for forced convection and discretization

7.1.1 Definition of stream function(流函数) and vorticity (涡量) in Cartesian coordinate

7.1.2 Stream function-vorticity governing equations for forced convection

7.1.3 Discretization of stream function-vorticity governing equations

7.1.4 Solution procedure of stream function-vorticity method

7.1 Stream function-vorticity method for forced convection and discretization

By eliminating pressure in flow governing equation the coupling between V-P is avoided; Discussion will be focused on boundary condition treatment.

7.1.1 Definition of stream function and vorticity in Cartesian coordinate

1. Stream function ψ（流函数）

For 2-D incompressible flow defining: $\frac{\partial \psi}{\partial y} = u; \frac{\partial \psi}{\partial x} = -v$

The mass conservation condition is automatically satisfied:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \cancel{\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y})} + \cancel{\frac{\partial}{\partial y}(-\frac{\partial \psi}{\partial x})} \longrightarrow 0$$

2. Vorticity ω （涡量）

For 2-D incompressible flow defining:

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

It differs from the z-component of 3-D vortex vector

$$\vec{\omega} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -\omega$$

7.1.2 Stream function-vorticity governing equations for forced convection

1. ω equation

For 2-D steady incomp. flow without grav., N-S eq.:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\eta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

In order to eliminate pressure gradient, $\frac{\partial}{\partial y}[(1)]-\frac{\partial}{\partial x}[(2)],$

According to above definition of ω, finally:

$$\rho\left(u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}\right)=\eta\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right)$$

Its conservative form is:

$$\frac{\partial(\rho u\omega)}{\partial x}+\frac{\partial(\rho v\omega)}{\partial x}=\frac{\partial}{\partial x}(\eta\frac{\partial\omega}{\partial x})+\frac{\partial}{\partial y}(\eta\frac{\partial\omega}{\partial y})$$

2. ψ equation

Substituting velocities in ω by stream function :

$$\omega=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\frac{\partial}{\partial y}(\frac{\partial\psi}{\partial y})-\frac{\partial}{\partial x}(-\frac{\partial\psi}{\partial x}) \rightarrow \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\omega=0$$

3. *T* equation

$$\frac{\partial(\rho uT)}{\partial x}+\frac{\partial(\rho vT)}{\partial x}=\frac{\partial}{\partial x}(\frac{\lambda}{c_p}\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(\frac{\lambda}{c_p}\frac{\partial T}{\partial y})+\frac{R}{c_p}$$

4. General governing equations for $\omega-\psi-T$

$$\frac{\partial(a_\phi u\phi)}{\partial x}+\frac{\partial(a_\phi v\phi)}{\partial x}=\frac{\partial}{\partial x}(\Gamma_\phi\frac{\partial\phi}{\partial x})+\frac{\partial}{\partial y}(\Gamma_\phi\frac{\partial\phi}{\partial y})+S_\phi$$

Variable	ϕ	a_ϕ	Γ_ϕ	S_ϕ
Vorticity	ω	ρ	η	0
Stream Func.	ψ	0	1	$-\omega$
Temperature	T	ρ	$\frac{\lambda}{c_p}=\frac{\eta}{Pr}$	R/c_p

In the convection term *u*,*v* are remained, rather than replacing by stream function. Such expression shows that the above equation is only a special case of the general governing equation of diffusion-convection problem.

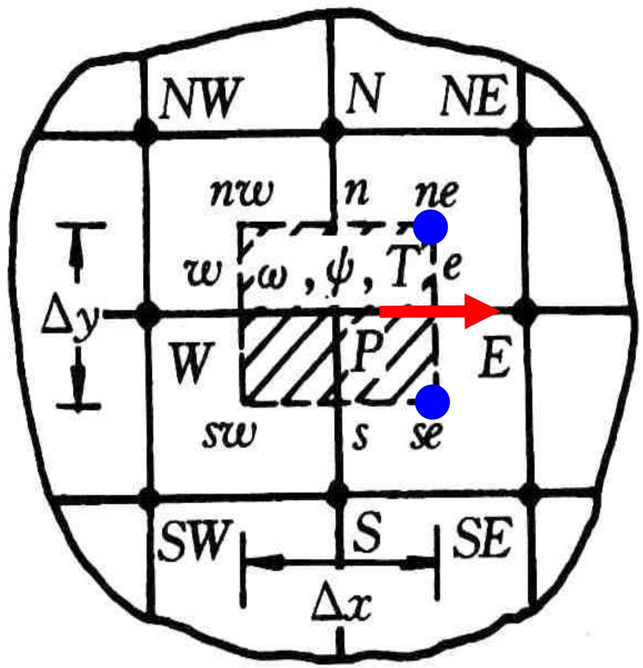
7.1.3 Discretization of stream function-vorticity governing equations

1.Adopting non-staggered grid, ω, ψ, T are stored at the same grids;

2. Discretizing by CV method, scheme and source term treatments are the same as for the primitive variable method;

3. Adopting the previous stream function to calculate the velocity required for determining coefficients of discretized equation.

$$u_e = \frac{\partial \psi}{\partial y})_e = \frac{\psi_{ne} - \psi_{se}}{\Delta y} = \frac{\psi_N + \psi_{NE} + (\cancel{\psi_P + \psi_E})}{4\Delta y} - \frac{\cancel{\psi_P + \psi_E} + (\psi_S + \psi_{SE})}{4\Delta y} = \frac{\psi_N + \psi_{NE} - \psi_S - \psi_{SE}}{4\Delta y}$$



4. B.C.: All are of 1st type except the outlet boundary.

7.1.4 Solution procedure

1. Asuming $\psi_{i,j}^{0}$, determining coefficients of ω equa. and its boundary values ;

2. Solving vorticity equation, yielding $\omega_{i,j}^{(1)}$

3. Solving ψ eq. from updated $\omega_{ij}^{(1)}$, yielding $\psi_{i,j}^{(1)}$

4. Updating boundary vorticity from $\psi_{i,j}^{(1)}$, yielding $\omega_{B}^{(1)}$

5. Repeating 2-4 from $\psi_{i,j}^{(1)}$ $\omega_{B}^{(1)}$, until converged

6. Determining *u*,*v* from converged $\psi_{i,j}$

7. Solving other scalar (标量)variables （Constant property problem）.

8. Determining inner node pressure from pressure Poisson equation:

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=2\rho[(\frac{\partial^2\psi}{\partial x^2})(\frac{\partial^2\psi}{\partial y^2})-(\frac{\partial^2\psi}{\partial x\partial y})^2]$$

The B.Cs. of pressure are all of 2nd type and can be obtained from N-S. E.:

From $$\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})=-\frac{\partial p}{\partial x}+\eta(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}) \rightarrow$$

$$\frac{\partial p}{\partial x}=\underline{\eta(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})-\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})}$$

All the right terms are known after receiving converged solution

Pressure reference needs to be selected.

7.2 Boundary conditions for stream-vorticity method

7.2.1 Inlet boundary

7.2.2 Central line

7.2.3 Outlet boundary without recirculation

7.2.4 Sharp corner

7.2 Boundary conditions for stream-vorticity method

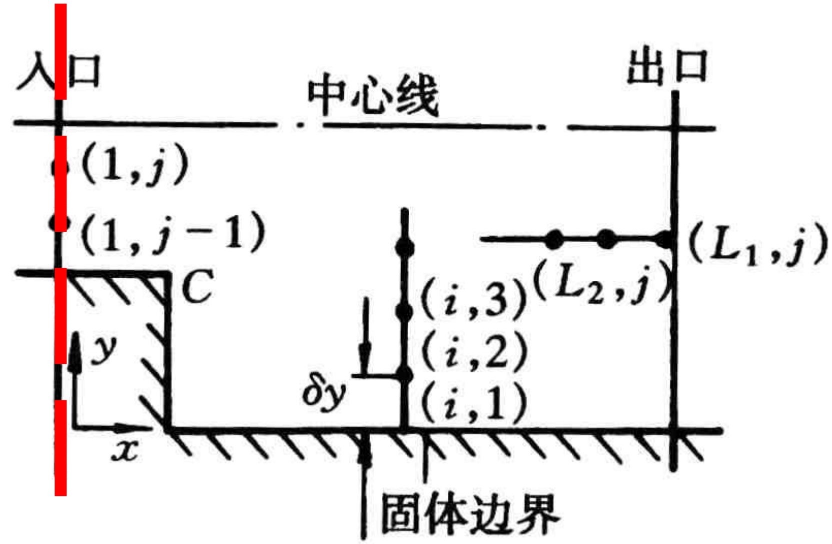
This is the focus of $\psi - \omega$ method.

7.2.1 Inlet boundary

1. Stream function ψ

Determining according to given velocity profile:

$$\psi(y) = \int u(y)dy$$



2. Vorticity ω

(1) No vortex at inlet $\omega = 0$

(2) Adopting stream function equation：

由 $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\omega=0$$

$$\omega_{1,j}=\frac{\psi_{1,j+1}-2\psi_{1,j}+\psi_{1,j-1}}{(\delta y)^2}+\frac{\psi_{1,j}-2\psi_{2,j}+\psi_{3,j}}{(\delta x)^2}$$

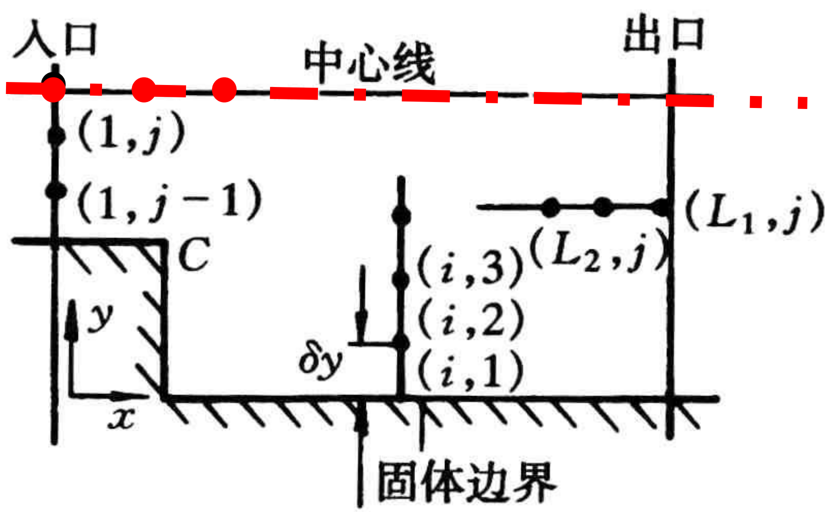
$O(\delta y^2)$ $O(\delta x)$

7.2.2 Central Line

1. Stream function ψ

At central line

v = 0 $$\frac{\partial\psi}{\partial x}=0$$



At central line flow rate keeps constant $\psi = C$

2. Vorticity

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

Symmetric condition, $\frac{\partial u}{\partial y} = 0$

At central line $v \equiv 0$, hence its derivative =0

Central line is of 1st type condition for ω, ψ .

7.2.3 Outlet boundary without recirculation

General way—Zero normal derivative at outlet boundary

(1) Fully developed assumption:

$$\frac{\partial \psi}{\partial x} = 0, \frac{\partial \omega}{\partial x} = 0$$

Two implementation methods:
- **ASTM, 2nd kind B.C.**
- **Updating boundary value method,** $\begin{cases} \psi_{L1,j} = \psi^{*}_{L2,j}; \\ \omega_{L1,j} = \omega^{*}_{L2,j} \end{cases}$

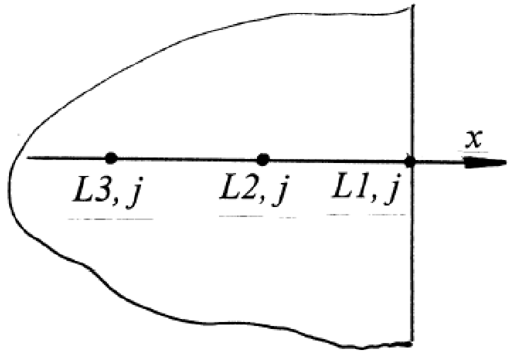
ASTM is more efficient.

(2) Less limitation assumption $\frac{\partial \omega}{\partial x} = 0 \; ; \frac{\partial^2 \psi}{\partial x^2} = 0$

Updating boundary value method

$$\begin{cases} \omega_{L1,j} = \omega^{*}_{L2,j} \\ \psi_{L1,j} = 2\psi^{*}_{L2,j} - \psi^{*}_{L3,j} \end{cases}$$

(1st order biased difference)



L1, L2, L3 are equally located

(3) Assumption with much less limitation:

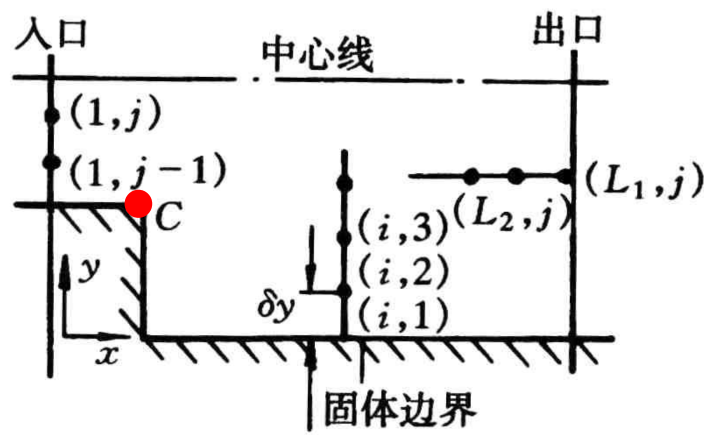
$$\frac{\partial^2 \psi}{\partial x^2} = 0; \quad \frac{\partial^2 \omega}{\partial x^2} = 0 \xrightarrow[\text{boundary values}]{\text{Updating}} \begin{cases} \omega_{L1,j} = 2\omega^*_{L2,j} - \omega^*_{L3,j} \\ \psi_{L1,j} = 2\psi^*_{L2,j} - \psi^*_{L3,j} \end{cases}$$

No effective method has been proposed when outlet boundary is located in recirculation region.

7.2.4 Sharp corner

1. Stream function ψ

Solid wall without Seepage (渗漏): $\psi \equiv const$



2. Vorticity

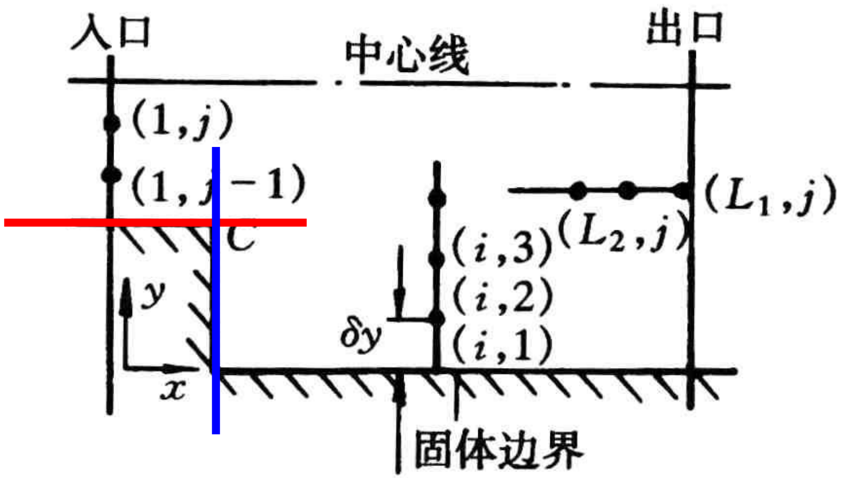
Sharp corner is a singular (奇点): derivatives are not continuous

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

At the corner C, derivatives from different direction are not equal:

$$\frac{\partial v}{\partial x})_{c^-} = 0; \quad \frac{\partial v}{\partial x})_{c^+} \neq 0$$

$$\frac{\partial u}{\partial y})_{c^-} = 0; \quad \frac{\partial u}{\partial y})_{c^+} \neq 0$$



A simple way for numerical treatment of sharp corner vorticity：

$$\omega_c = 0$$

7.3 Stream Function-Vorticity Wall Boundary Condition

7.3.1 Stream function at solid wall

7.3.2 Vorticity at solid wall

1. General principle for wall vorticity determination

2. General method for wall vorticity determination

3. Existing formulae for wall vorticity determination

4. Discussion on wall vorticity determination

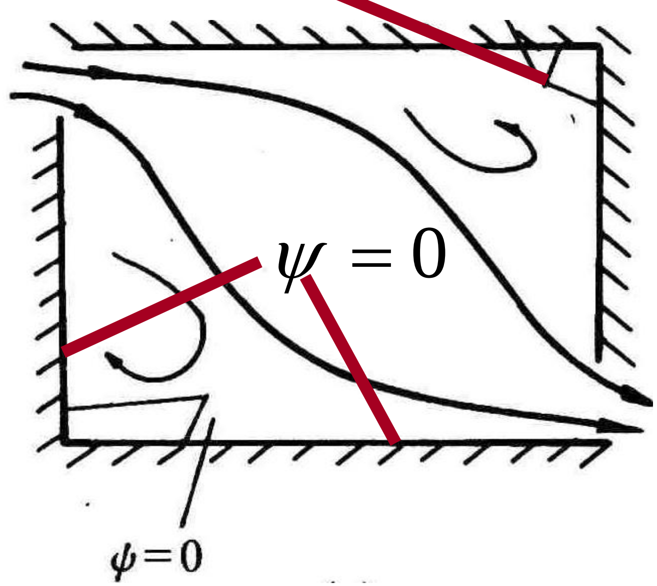
7.3 Stream function-vorticity wall boundary condition

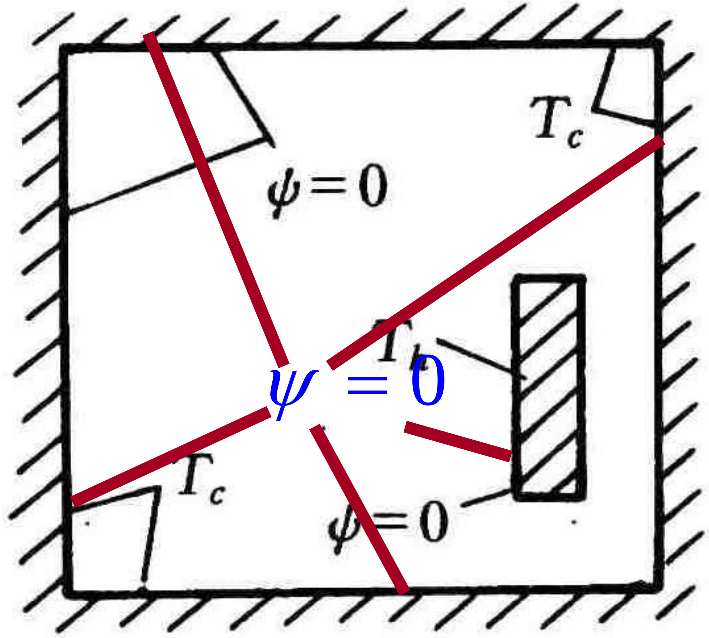
7.3.1 Stream function

1. Constant for solid wall without seepage (渗透);

2. Zero for solid wall of an enclosure.

Determined from total flow rate





7. 3.2 Vorticity

1.General principle for wall vorticity determination

Vorticity is not zero at solid wall, rather it is the solid wall where vorticity is generated.

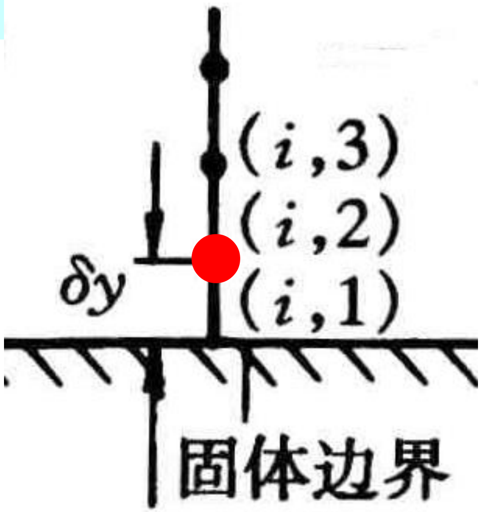
The solid wall vorticity is expressed by stream function at the wall and of the inner fluid nodes.

2. General method for wall vorticity determination

Take Taylor series expansion of stream function for the 1st inner fluid node, and find the relation between wall vorticity and stream function nearby.

3. Existing formulae for wall vorticity determination

(1) Thom equation



Conducting Taylor series expansion of $\psi(i,2)$ with respect to(i,1) :

$$\psi_{i,2}=\psi_{i,1}+\frac{\partial\psi}{\partial y})_{i,1}\delta y+\frac{\partial^2\psi}{\partial y^2})_{i,1}\frac{\delta y^2}{2}+O(\delta y^3)$$

$$\frac{\partial\psi}{\partial y})_{i,1}=u_{i,1}\xrightarrow{\text{Solidwall}}u_{i,1}=0\quad\text{Thus}\quad\frac{\partial\psi}{\partial y})_{i,1}=0$$

$$\frac{\partial^2\psi}{\partial y^2})_{i,1}=\frac{\partial}{\partial y}(\frac{\partial\psi}{\partial y})_{i,1}=(\frac{\partial u}{\partial y})_{i,1}=\frac{\partial u}{\partial y})_{i,1}-\frac{\partial v}{\partial x})_{i,1}=\omega_{i,1}$$

Zero at solid wall

$$\psi_{i,2}=\psi_{i,1}+\omega_{i,1}\frac{\delta y^2}{2}+O(\delta y^3)$$

$$\boxed{\omega_{i,1}=\omega_B=\frac{2(\psi_{i,2}-\psi_{i,1})}{\delta y^2},O(\delta y)}$$

Thom equation

(2) Woods equation

Conducting Taylor series expansion of $\psi(i,2)$ with respect to(i,1) up to 3rd-order derivatives:

$$\psi_{i,2}=\psi_{i,1}+\cancel{\frac{\partial\psi}{\partial y}}\Big)_{i,1}\delta y+\cancel{\frac{\partial^2\psi}{\partial y^2}}\Big)_{i,1}\frac{\delta y^2}{2}+\frac{\partial^3\psi}{\partial y^3}\Big)_{i,1}\frac{\delta y^3}{6}+O(\delta y^4)$$

0 $\qquad \omega_{i,1}$

$$\frac{\partial^3\psi}{\partial y^3}=\frac{\partial}{\partial y}\Big(\frac{\partial^2\psi}{\partial y^2}\Big)_{i,1}=\frac{\partial}{\partial y}(\omega)_{i,1}\longrightarrow\frac{\partial\omega}{\partial y}\Big)_{i,1}=\frac{\omega_{i,2}-\omega_{i,1}}{\delta y}+O(\delta y)$$

$$\psi_{i,2}=\psi_{i,1}+\omega_{i,1}\frac{\delta y^2}{2}+\frac{\omega_{i,2}-\omega_{i,1}}{\delta y}\frac{\delta y^3}{6}+O(\delta y^4)$$

$$\boxed{\omega_{i,1}=\frac{3(\psi_{i,2}-\psi_{i,1})}{\delta y^2}-\frac{1}{2}\omega_{i,2},O(\delta y^2)}$$

Woods eq.

(3) Jenson eq. $$\omega_{i,1}=\frac{-7\psi_{i,1}+8\psi_{i,2}-\psi_{i,3}}{2\delta y^2},O(\delta y^2)$$

4.Discussion on wall vorticity determination

(1)The determination of wall vorticity introduces nonlinearity of boundary condition---wall vorticity varies with iteration;

(2) For engineering computation usually Thom, Woods and Jenson equations are recommended.

(3) Wall vorticity needs to be underrelaxed for convergence:

Calculated from above equation

$$\omega^{(k+1)}=\omega^{(k)}+\alpha(\overline{\omega}^{(k+1)}-\omega^{(k)})$$

7.4 Examples of stream-vorticity method for convection in enclosure

7.4.1 $\psi-\omega$ governing equations in three 2-D coordinates

1. Cartesian coordinate

2. Symmetric cylindrical coordinate

3. Polar coordinate

7.4.2 Natural convection in a horizontal annular space

7.4.3 Lid-driven cavity flow

7.4 Example of stream-vorticity method for convection in enclosure

7.4.1 $\psi-\omega$ governing equations in three 2-D coordinates

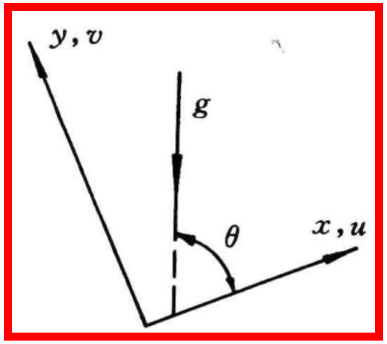
1. Cartesian coordinate

Introducing Boussinesq assumption and effective pressure:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\eta\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\eta\frac{\partial u}{\partial y}\right)+\rho g\alpha(T-T_r)\cos\theta \quad \textbf{(1)}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\eta\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\eta\frac{\partial v}{\partial y}\right)+\rho g\alpha(T-T_r)\sin\theta \quad \textbf{(2)}$$

Defining $\dfrac{\partial\psi}{\partial y}=u;\ \dfrac{\partial\psi}{\partial x}=-v\quad \omega=\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial x}$

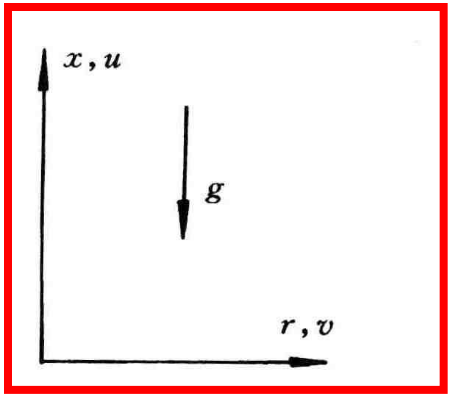


Conducting $\frac{\partial}{\partial y}[Eq.(1)]-\frac{\partial}{\partial x}[Eq.(2)]$ **and rearranging:**

$$\frac{\partial}{\partial x}(\rho\omega\frac{\partial\psi}{\partial y})-\frac{\partial}{\partial y}(\rho\omega\frac{\partial\psi}{\partial x})=\frac{\partial}{\partial x}(\eta\frac{\partial\omega}{\partial x})+\frac{\partial}{\partial y}(\eta\frac{\partial\omega}{\partial y})+\rho g\alpha(\frac{\partial T}{\partial y}\cos\theta-\frac{\partial T}{\partial x}\sin\theta)$$

This is ω **equation;** ψ **and T equations see textbook.**

2. Symmetric cylindrical coordinate



Continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}=0$$

Defining: $u=\frac{1}{r}\frac{\partial\psi}{\partial r},\ v=-\frac{1}{r}\frac{\partial\psi}{\partial x},\omega=\frac{\partial u}{\partial r}-\frac{\partial v}{\partial x}$

Taking the same operation as for Cartesian coordinate:

$$\frac{\partial}{\partial x}(\rho\frac{v}{r}\omega)+\frac{\partial}{\partial x}(\rho\frac{u}{r}\omega)=\eta[\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial r^2}+\frac{\partial(\omega/r)}{\partial r}]+\rho g\alpha\frac{\partial T}{\partial r}$$

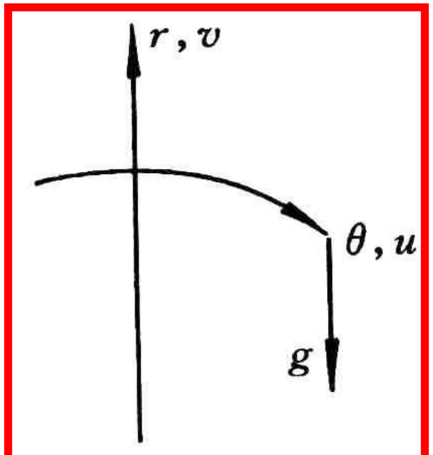
This is ω equation; ψ and T equations see textbook.

3. Polar coordinate

Continuity eq.: $$\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}+\frac{v}{r}=0$$

Defining:

$$u=-\frac{\partial \psi}{\partial r},\quad v=\frac{1}{r}\frac{\partial \psi}{\partial \theta},\quad \omega=\frac{1}{r}\left[\frac{\partial v}{\partial \theta}-\frac{\partial (ru)}{\partial r}\right]$$



Taking the same operation as for Cartesian coordinate:

$$\omega:\ \frac{\partial}{\partial x}(\rho\frac{u}{r}\omega)+\frac{\partial}{\partial x}(\rho v\omega)=\frac{1}{r}\frac{\partial}{\partial r}(\eta\frac{\partial \omega}{\partial x})+\frac{1}{r}\frac{\partial}{\partial \theta}(\frac{\eta}{r}\frac{\partial \omega}{\partial \theta})+$$
$$\rho g\alpha(\frac{1}{r}\frac{\partial T}{\partial \theta}\cos\theta+\frac{\partial T}{\partial r}\sin\theta)$$

$$\psi:\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial \theta^2}=\omega$$

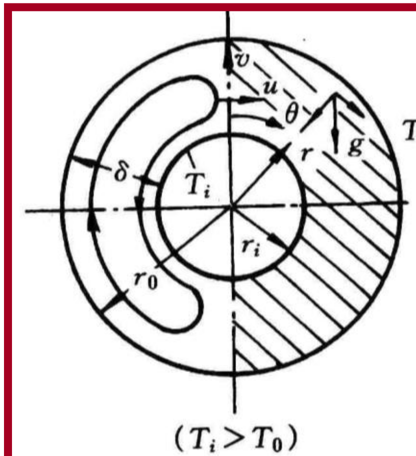
7.4.2 $\psi-\omega$ solution for natural convection in horizontal annular space

1. Presentation of simulated results

Heat conduction through cylindrical air-filled annulus(环形) with two specified temperatures

Natural convection will enhance heat transfer---- expressed by an effective thermal conductivity

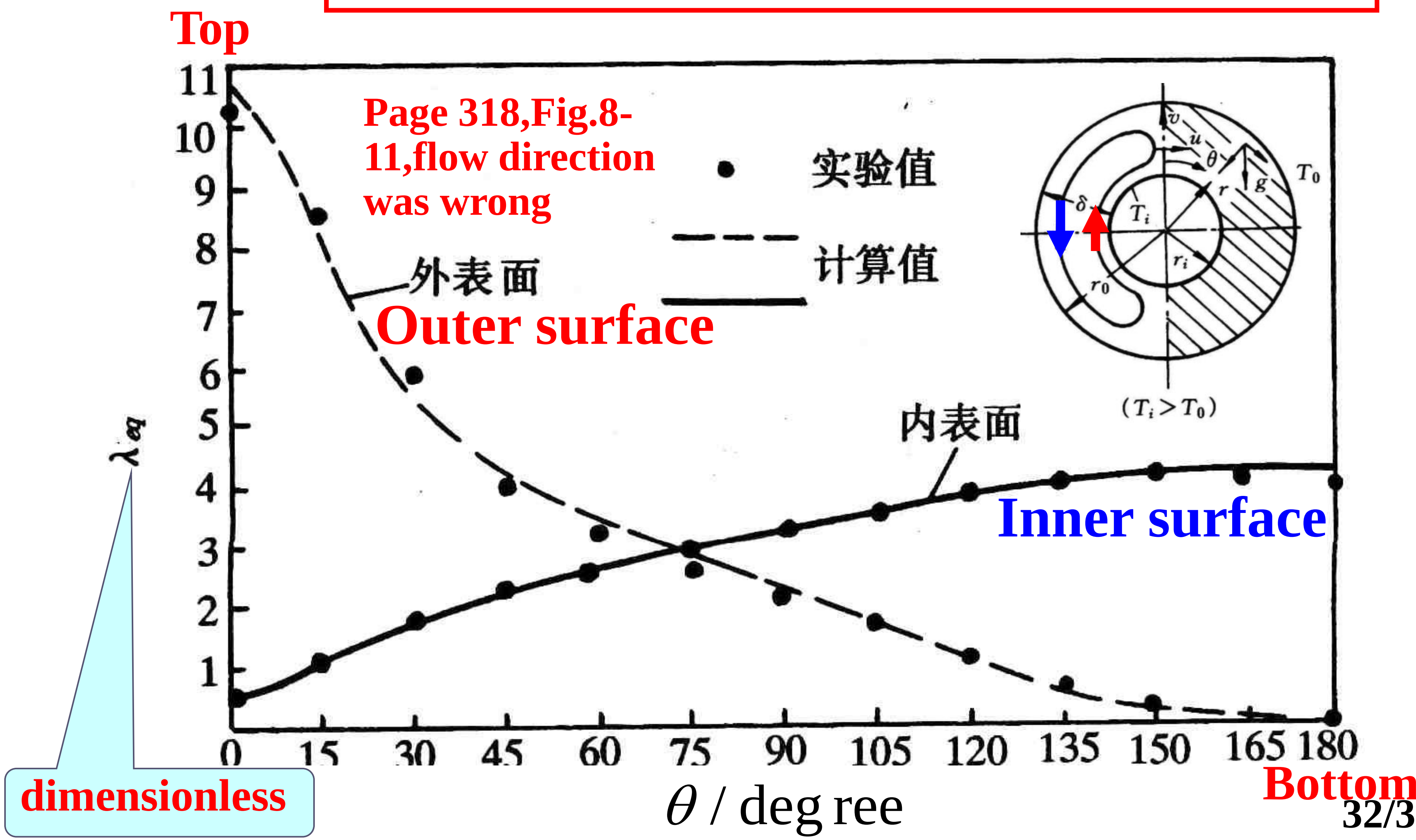
$$\phi = \frac{2\pi L \lambda_{air} \Delta T}{\ln(d_2 / d_1)}$$



$$\phi = \frac{2\pi L \lambda_{eq} \Delta T}{\ln(d_2 / d_1)}$$

$\bar{\lambda}_{\mathrm{eq}} = \dfrac{\lambda_{eq}}{\lambda_{air}}$ **presents the enhanced multiple(强化倍率):**

Kuehn T H, Goldstein R J. An experimental and theoretical study of natural convection in the annulus between horizontal concentric cylinders. J Fluid Mech, 1971, 74:605-719

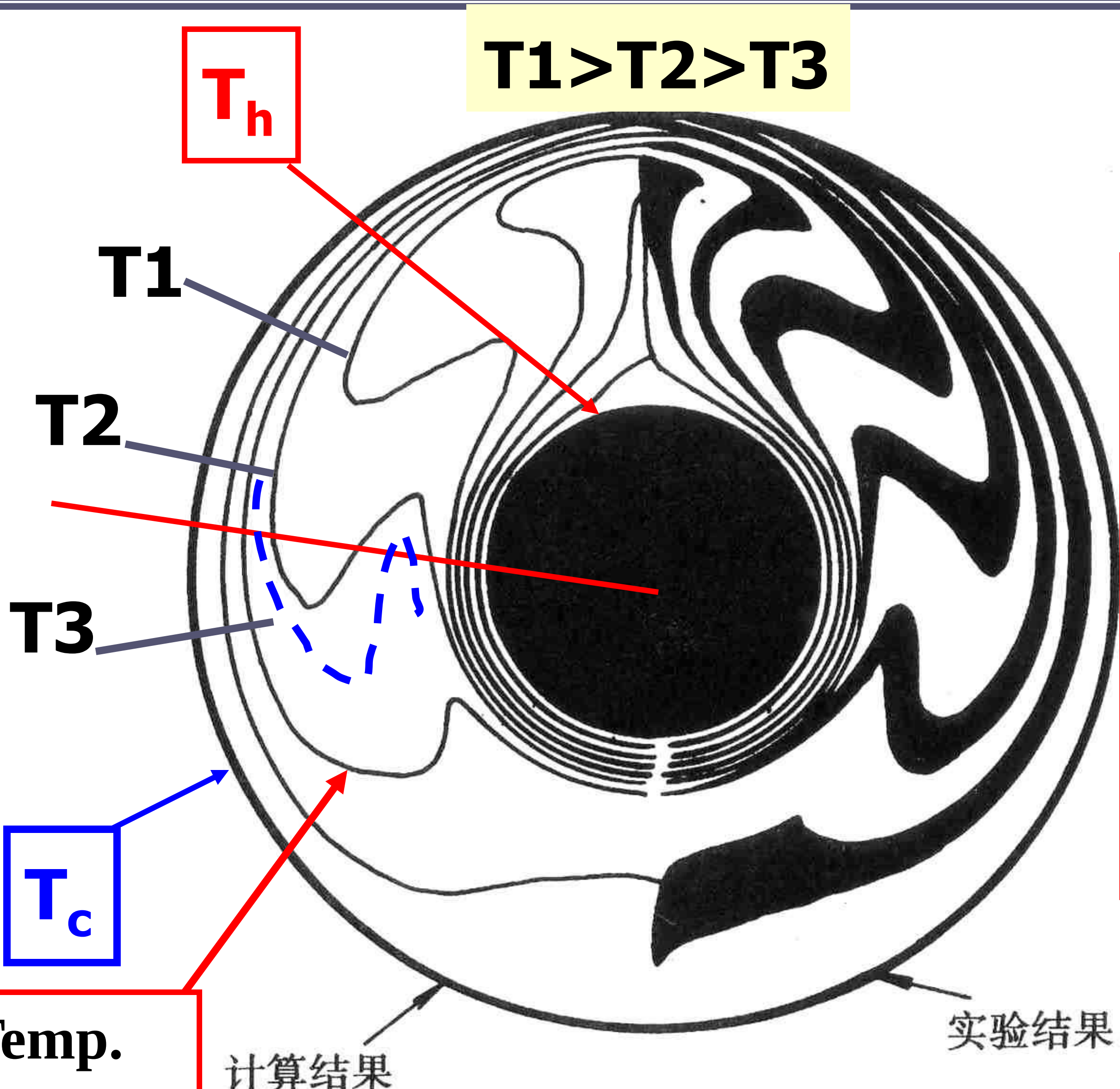


Numerical

Measured

	Test	Numer.
Ra_δ	4.7×10^4	5×10^4
Pr	0.706	0.7
δ/Di	0.8	0.8

Comparison between measured and predicted results



Temperature reverse is an indication of convection dominated flow in natural convection in enclosure

Temp. Reverse (逆转)

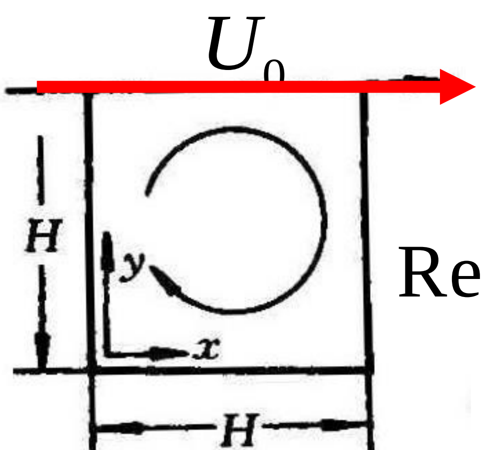
Analysis of results

2. $\psi-\omega-T$ Iteration order organization

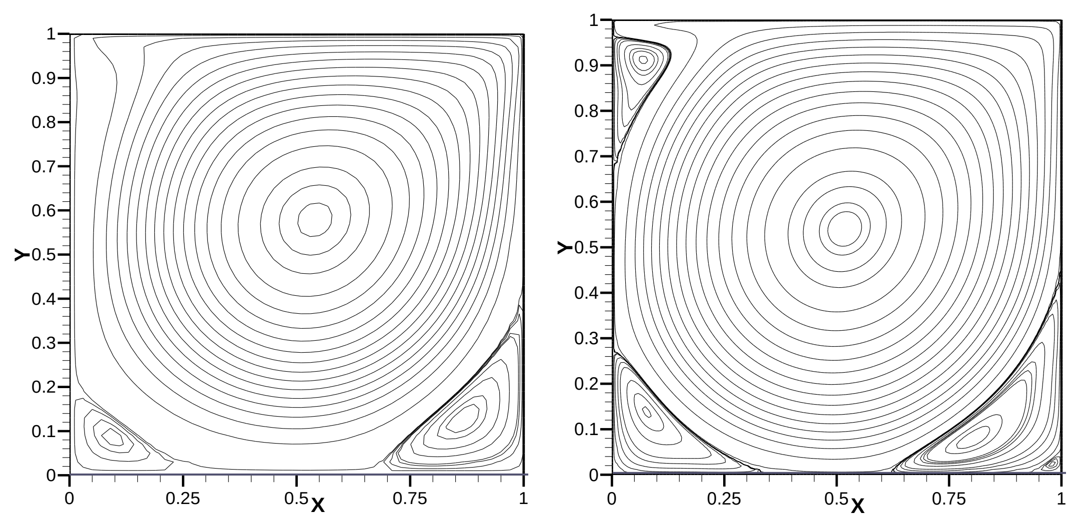
$$\color{red}{\omega-\psi-\omega-\psi-T-}\ \omega-\psi-\omega-\psi-T$$

7.4.3 $\psi-\omega$ solution for lid-driven cavity flow

Ghia et al. provided solutions for **lid-driven cavity flow (顶盖驱动流)** up to Re＝10000 with （257X257) multigrid technique 。



$$\mathrm{Re}=\frac{U_0 H}{\nu}$$



(a) *Re*=1000, 52×52; (b) *Re*=5000, 130×130;

Ghia U, Ghia N, Shin C T. J Comput Physics, 1982, 48:387-411

Most recent reference on lid-driven cavity flow

E. Ertuk, T.C. Corke and C. Gokcol. Numerical Solutions of 2-D Steady Incompressible Driven Cavity Flow at High Reynolds Numbers, *Int. J. Numer. Meth. Fluids*, 2005, vol.48,pp.747-774

Homework

8－3　8－4　8－5　8－7

Due on November 21 th

Problem # 8-3

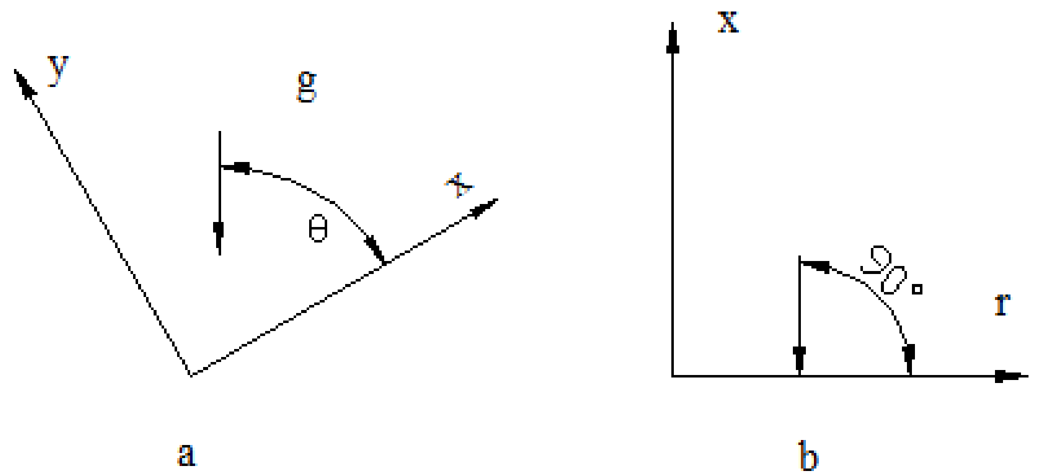
Try to drive the expression (8-23) for vorticity at boundary (wall).

Problem # 8-4

The vorticity boundary conditions in this chapter, are derived by using discrete method A. Try to use discrete methods B to derive the corresponding boundary vorticity formulate for equations (8-21), (8-22).

Problem # 8-5

Consider the three kinds of two-dimensional coordinate system as shown below; write down the expressions for effective pressure of closed cavity natural convection.



Problem # 8-7 For the lid-driven cavity flow shown in page 35 of this PPT, write down the boudnary conditions for stream-function and vorticity (Woods Eq.).

同舟共济
渡彼岸！

People in the same boat help each other to cross to the other bank, where….