数值传热学

第四章 扩散方程的数值解及其应用(2)

主讲 陶文铨

西安交通大学能源与动力工程学院
热流科学与工程教育部重点实验室

2016年10月10日, 西安

Numerical Heat Transfer
(数值传热学)
Chapter 4 Numerical Solution of Diffusion Equation and its Applications(2)

Instructor Tao, Wen-Quan

CFD-NHT-EHT Center
Key Laboratory of Thermo-Fluid Science & Engineering
Xi'an Jiaotong University
Xi'an, 2016-Oct-10

Contents

4.4 TDMA & ADI Methods for Solving ABEs

4.4.1 TDMA algorithm (算法) for 1-D conduction problem

1. General form of algebraic equations of 1-D conduction problems

2. Thomas algorithm

3. Treatment of 1st kind boundary condition

4.4.2 ADI method for solving multi-dimensional problem

1. Introduction

2. ADI iteration of Peaceman-Rachford

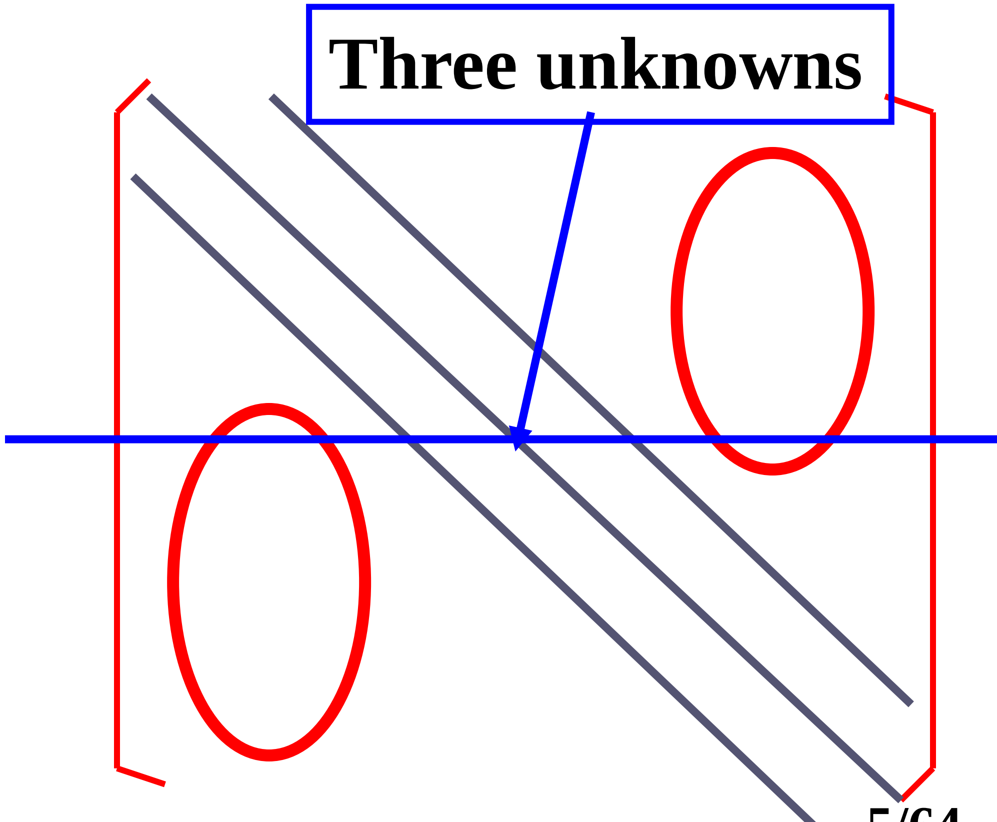
4.4 TDMA & ADI Methods for Solving ABEqs

4.4.1 TDMA algorithm for 1-D conduction problem

1.General form of algebraic equations. of 1-D conduction problems

The ABEqs for steady and unsteady （f>0) problems take the form

$$a_P T_P = a_E T_E + a_W T_W + b$$



The matrix of the coefficients is a tri-diagonal (三对角) one .

2. Thomas algorithm(算法)

Rewrite above equation：

$$A_i T_i = B_i T_{i+1} + C_i T_{i-1} + D_i, \; i = 1,2,.....M1 \qquad \textbf{(a)}$$

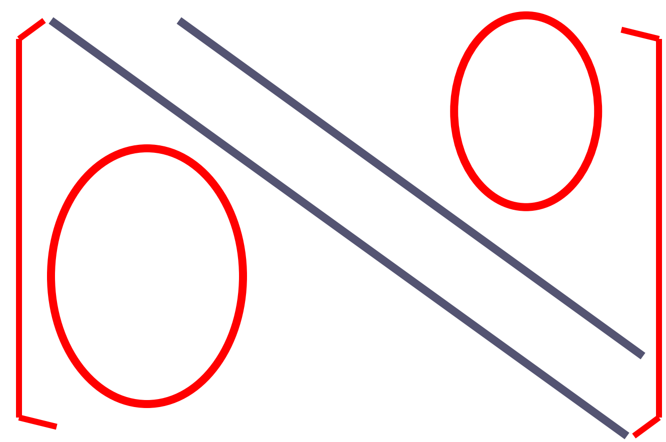
End conditions：i=1, C_i=0; i＝M1, B_i＝0

(1) Elimination (消元)－Reducing the unknowns at each line from 3 to 2

Assuming the eq. after elimination as

$$T_{i-1} = P_{i-1} T_i + Q_{i-1} \qquad \textbf{(b)}$$

Coefficient has been treated to one.



The purpose of the elimination procedure is to find the relationship between P_i, Q_i with A_i, B_i, C_i, D_i:

Multiplying Eq.(b) by C_i, and adding to Eq.(a):

$$A_i T_i = B_i T_{i+1} + C_i T_{i-1} + D_i \quad \text{(a)}$$

$$C_i T_{i-1} = C_i P_{i-1} T_i + C_i Q_{i-1} \quad \text{(b)}$$

$$A_i T_i - C_i P_{i-1} T_i = B_i T_{i+1} + D_i + C_i Q_{i-1}$$

Yielding

$$T_i = \left(\frac{B_i}{A_i - C_i P_{i-1}}\right) T_{i+1} + \frac{D_i + C_i Q_{i-1}}{A_i - C_i P_{i-1}}$$

Comparing with

$$T_{i-1} = P_{i-1} T_i + Q_{i-1}$$

$$P_i = \frac{B_i}{A_i - C_i P_{i-1}}; \qquad Q_i = \frac{D_i + C_i Q_{i-1}}{A_i - C_i P_{i-1}};$$

The above equations are recurrent(递归的)—i.e.,

In order to get P_i, Q_i, P_1 and Q_1 must be known.

In order to get P_1, Q_1, use Eq.(a)

$$A_i T_i = B_i T_{i+1} + C_i T_{i-1} + D_i,\ i = 1, 2, \ldots. M1 \quad \textbf{(a)}$$

End condition： $i=1$, $C_i=0$; $i=M1$, $B_i=0$

Applying Eq.(a) to *i=1*, and comparing it with Eq. (b) , the expressions of P_1, Q_1 can be

obtained: $i=1, C_1=0, \quad A_1T_1 = B_1T_2 + D_1$

$$T_1 = \frac{B_1}{A_1}T_2 + \frac{D_1}{A_1} \longrightarrow P_1 = \frac{B_1}{A_1}; \quad Q_1 = \frac{D_1}{A_1}$$

(2) Back substitution(回代) —Starting from M1 via Eq.(b) to get T_i subsequently（顺序地）

$$T_{M1} = P_{M1}T_{M1+1} + Q_{M1}, \qquad P_i = \frac{B_i}{A_i - C_iP_{i-1}};$$

End condition:
i＝M1, B_i＝0

$$\longrightarrow P_{M1} = 0$$

$$T_{M1} = Q_{M1} \quad \boxed{T_{i-1} = P_{i-1}T_i + Q_{i-1}}$$ **to get:** $T_{M1-1}, \ldots\ldots T_2, T_1.$

3. Implementation of Thomas algorithm for 1st kind B.C.

For 1st kind B.C., the solution region is from i=2,….to M1-1=M2.

Applying Eq.(b) to i=1 with given $T_{1,given}$:

$$T_1 = P_1 T_2 + Q_1 \longrightarrow P_1 = 0; \quad Q_1 = T_{1,given}$$

Because T_{M1} is known, back substitution should be started from M_2: $T_{M2} = P_{M2} T_{M1} + Q_2$

When the ASTM is adopted to deal with B.C. of 2nd, and 3rd kind, the numerical B.C. for all cases is regarded as 1st kind, and the above treatment should be adopted.

4.4 TDMA & ADI Methods for Solving ABEs

4.4.1 TDMA algorithm for 1-D conduction problem

1.General form of algebraic equations. of 1-D conduction problems

2.Thomas algorithm

3.Treatment of 1st kind boundary condition

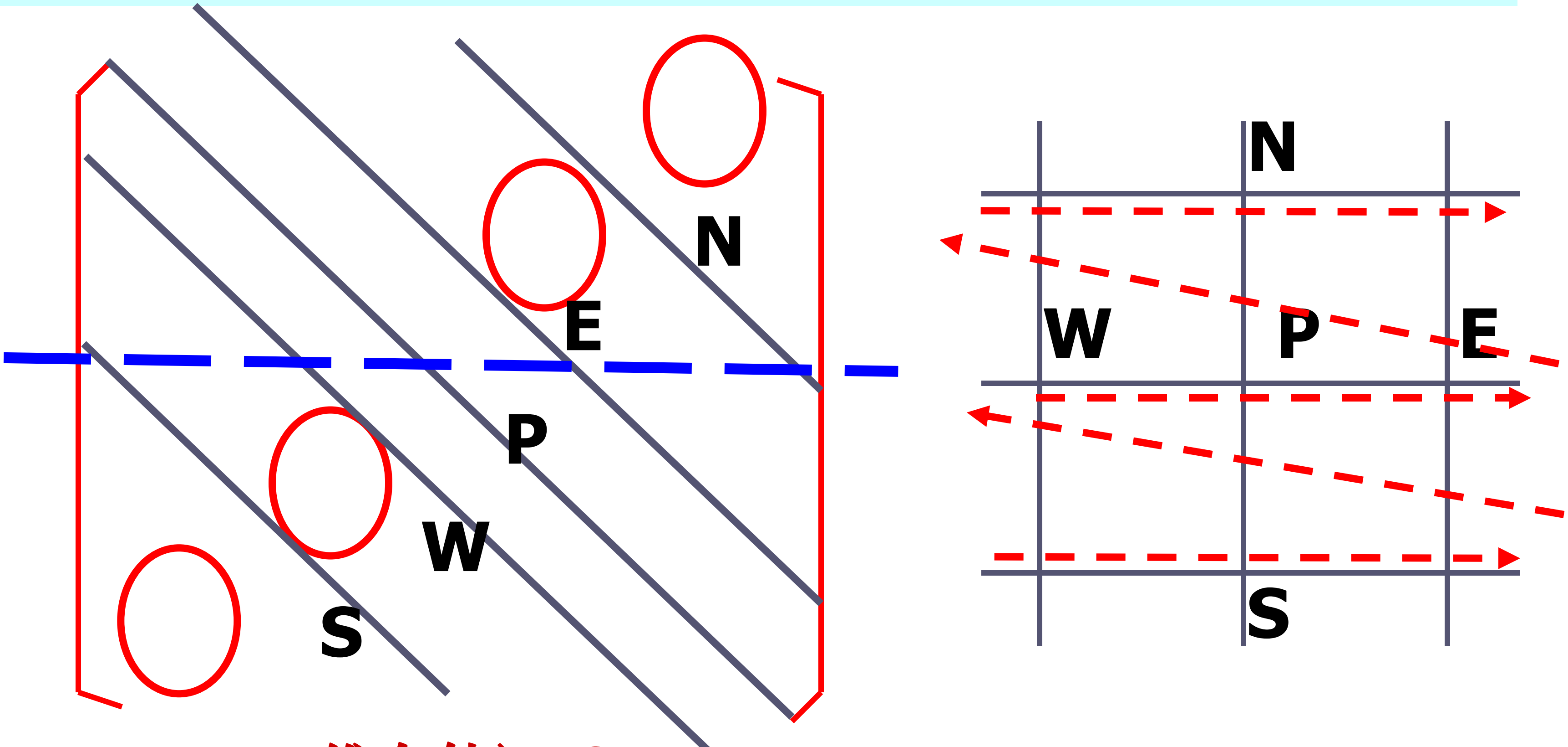
4.4.2 ADI method for solving multi-dimensional problem

1. Introduction

2. ADI iteration of Peaceman-Rachford

4.4.2 ADI method for solving multi-dimensional problem

1. Introduction



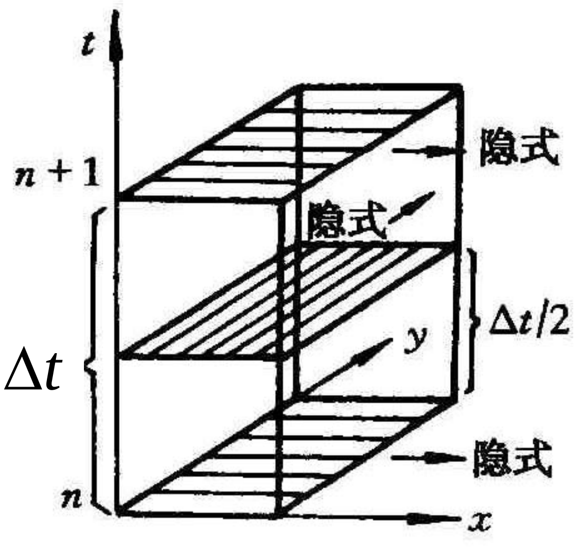
1-D storage (一维存储） of variables and its relation to matrix coefficients

Numerical methods for solving ABEqs. of 2-D problems.

(1) Penta-diagonal algorithm(PDMA,五对角阵算法)

(2) Alternative-direction Implicit (ADI,交替方向隐式方法)

2. 3-D Peaceman-Rachford ADI method



2－D ADI

Dividing Δt into three uniform parts

In the 1st $\Delta t\ /\ 3$ implicit in x direction, and explicit in y, z directions;

In the 2nd and 3rd $\Delta t\,/\,3$ implicit in y, z direction, respectively.

Set $u_{i,j,k}$, $v_{i,j,k}$ the temporary(临时的) solutions at two sub-time levels

$\delta_x^2 T_{i,j,k}^n$ **-CD for 2nd derivative at n time level in x direction**

1st sub-time level

$$\frac{u_{i,j,k} - T_{i,j,k}^n}{\Delta t/3} = a(\delta_x^2 u_{i,j,k} + \delta_y^2 T_{i,j,k}^n + \delta_z^2 T_{i,j,k}^n)$$

2nd sub-time level:

$$\frac{v_{i,j,k} - u_{i,j,k}^n}{\Delta t/3} = a(\delta_x^2 u_{i,j,k} + \delta_y^2 v_{i,j,k} + \delta_z^2 u_{i,j,k})$$

3rd sub-time level

$$\frac{T_{i,j,k}^{n+1} - v_{i,j,k}^n}{\Delta t/3} = a(\delta_x^2 v_{i,j,k} + \delta_y^2 v_{i,j,k}^n + \delta_z^2 T_{i,j,k}^{n+1})$$

Stability condition by von Neumann method:

$$a\Delta t\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}\right)\le 1.5$$

Is the allowed maximum time step three times of 1-D case?

Actually, No!

For 2-D case P-R method is absolutely stable.

3. Two "ADI" methods: ADI-implicit(交替方向隐式) for transient problems and ADI-iteration(交替方向迭代) multi-dimensional problems. They are very similar.

4.5 Introduction to Fully Developed HT in Tubes and Ducts

4.5.1 Definition of FDHT in tubes and ducts

1. Simple fully developed heat transfer

2. Complicated fully developed heat transfer

4.5.2 Boundary conditions for existence of FDHT

4.5.3 Collection of partial examples

4.5 Introduction to Fully Developed HT in Tubes and Ducts

4.5.1 Definition of FDHT in tubes and ducts

1. Simple fully developed heat transfer

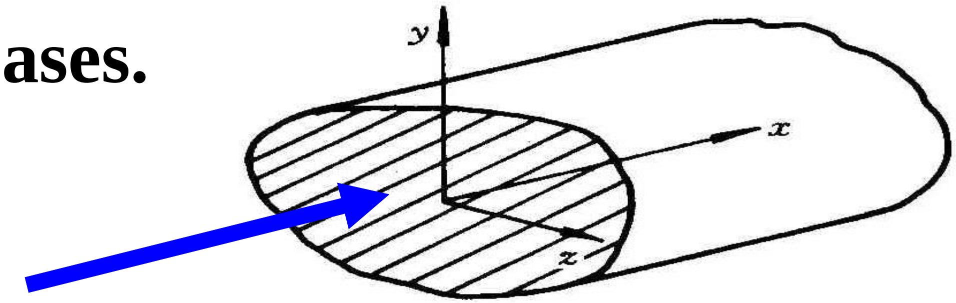
Physically: Velocity components normal to flow direction equal zero; Fluid dimensionless temperature distribution is independent on(无关) the position in the flow direction

Mathematically: Both dimensionless momentum and energy equations are of **diffusion type.**

Present chapter is limited to simple cases.

FDHT in straight duct is an example of simple cases.

$$\frac{\partial}{\partial x}\left(\frac{T_{w,m}-T}{T_{w,m}-T_b}\right)=0$$

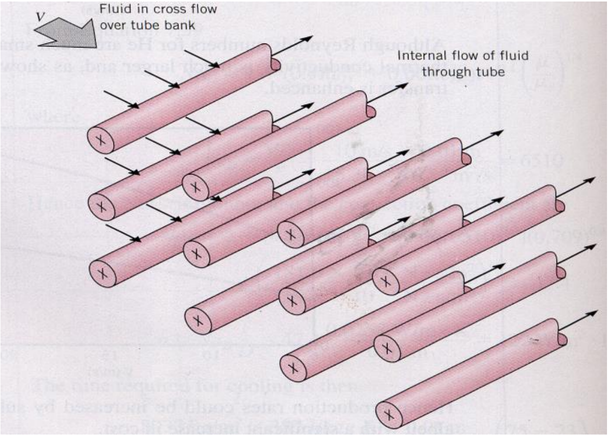


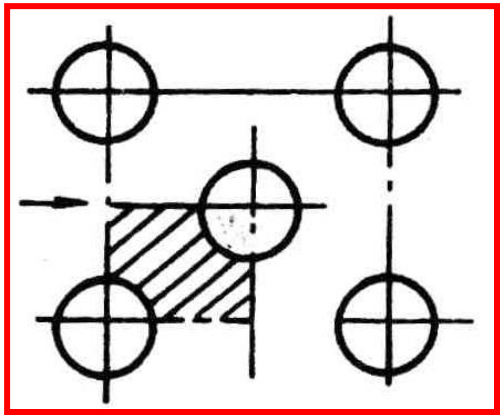
2. Complicated FDHT

In the cross section normal to flow direction there exist velocity components , and the dimensionless temperature depends on the axial position, often exhibits periodic (周期的） character. The full Navier-Stokes equations must be solved。

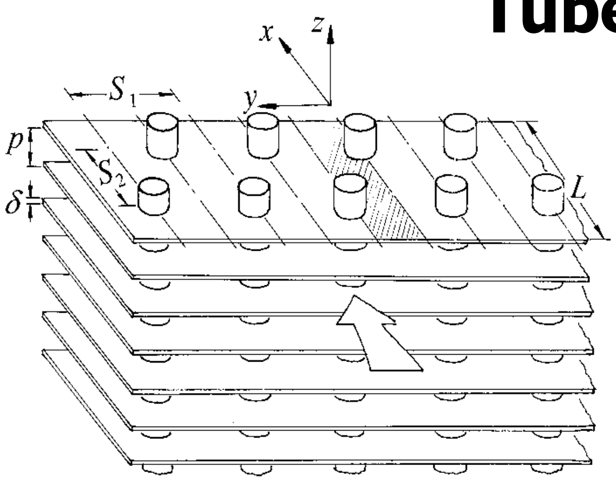
This subject is discussed in Chapter 11 of the textbook.

Examples of complicated FDHT

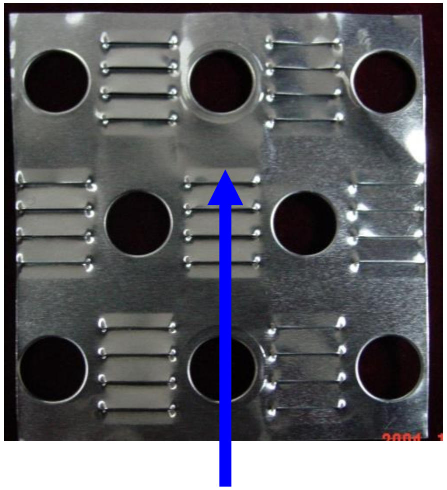


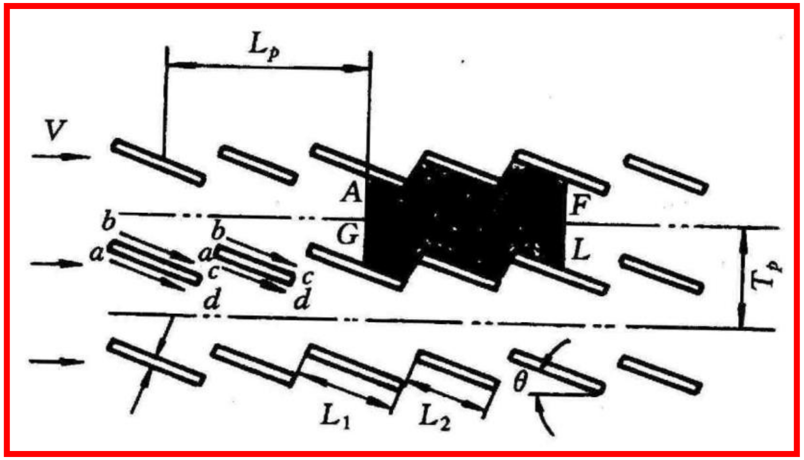


Tube bundle (bank)（管束）



Fin-and-tube heat exchanger





Louver fin (百叶窗翅片)

4.5.2 Boundary conditions for existence of FDHT

1. Axial and circumferential (周向) uniform wall temperature: $T_w = Constant$

2. Axial uniform wall heat flux and circumferential uniform wall temperature

$$q_x = Const, \quad T_w = f(x)$$

3. Axial and circumferential uniform wall heat flux

$$q = Const$$

4. Exponential (指数的)variation of axial heat flux:

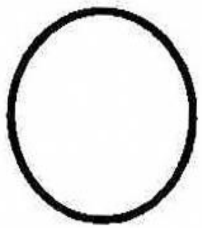
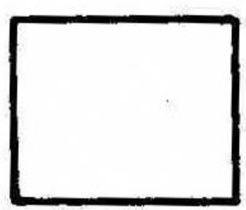
$$q_x = C_1 e^{C_2 x}$$

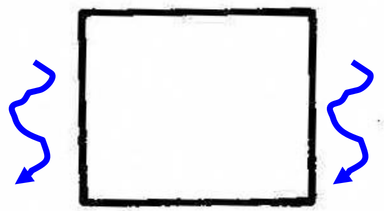
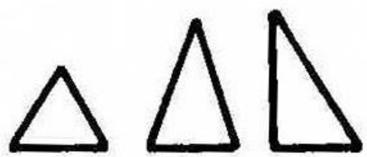
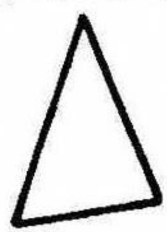
Monograph (专著) of R K Shah and A L London:

Laminar flow forced convection in ducts. Advances in heat transfer. Supplement 1, New York: Academic Press, 1978

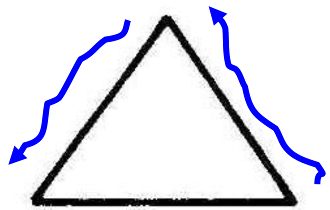
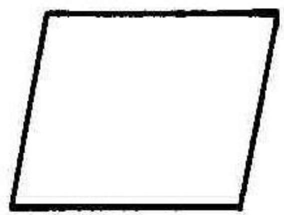
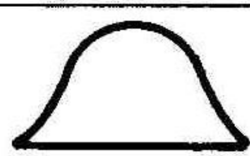
4.5.3 Collection of partial examples

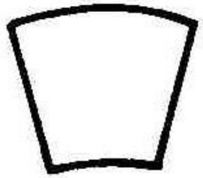
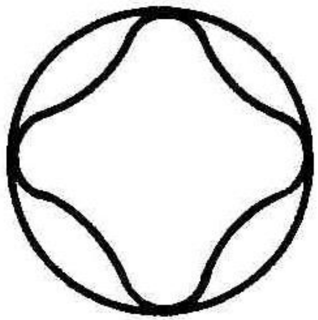
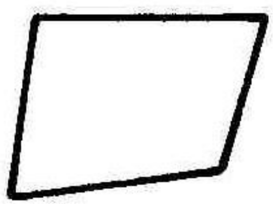
Table 4-5 Examples of simple FDHT

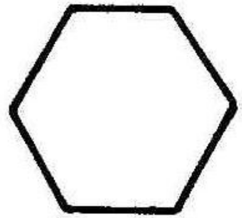
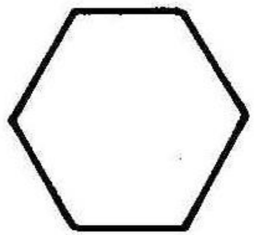
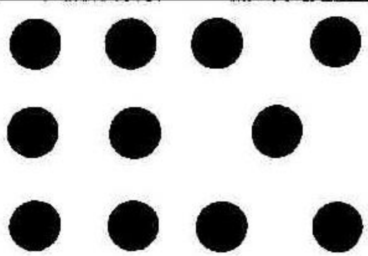
No	Cross section	B. Condition	Refs
1.		均匀壁温;给定周向热流分布;轴向热流呈指数变化;外部对流换热	[23,24,25,26,27]
2		均匀壁温;均匀热流及其组合	转引自[23]
3		均匀壁温;周向任意分布热流;轴向均匀热流;一组对边均匀壁温,另一线绝热	[28,29,30]

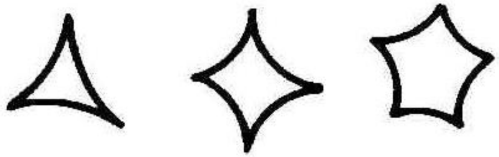
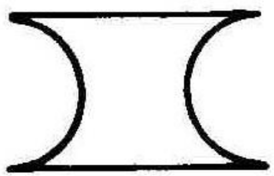
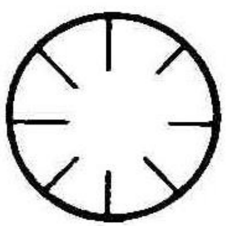
No	Cross section	B. Condition	Refs
No	流　道　截　面	热边界条件	文　献
4		外部对流换热条件	[31]
5		周向均匀壁温、轴向均匀热流；一个边均匀加热而其余两边绝热	[30,31－34]
6		均匀壁温	[34]

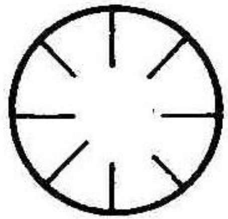
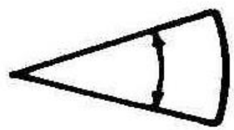
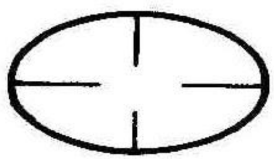
续表4－5

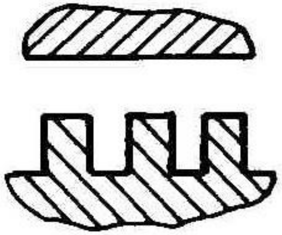
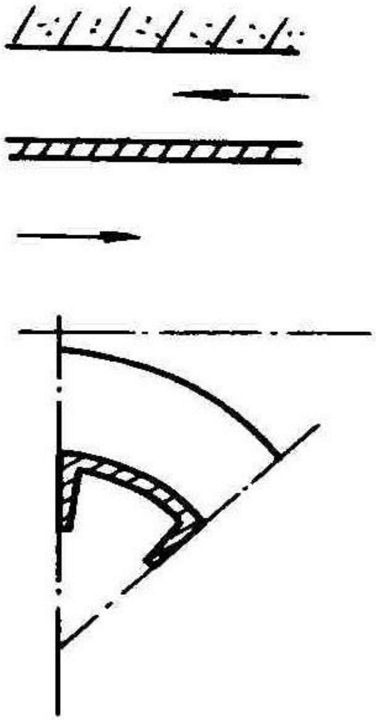
No	Cross section	B. Condition	Refs
No	流 道 截 面	热边界条件	文 献
7		外部对流换热条件	[31]
8		周向、轴向都是均匀热流	[35,36]
9	 正弦曲线	均匀壁温；周向均匀壁温、轴向均匀热流	[37]

No	Cross section	B. Condition	Refs
No	流 道 截 面	热边界条件	文 献
10	 环扇形	均匀壁温;轴向均匀热流,周向均匀壁温	[38]
11		均匀壁温	[39]
12		均匀壁温;周向均匀壁温、轴向均匀热流	[40]

No	流 道 截 面	热边界条件	文 献
13		外部对流换热条件	[31]
14		均匀热流:各组对边具有相同的均匀热流,不同组之间其值不同;均匀壁温;周向均匀壁温,轴向均匀热流	[41,42]
15	 纵掠管束	周向均匀壁温、轴向均匀热流;轴向、周向都是均匀热流	[43-47]

No	流 道 截 面	热边界条件	文 献
16	 等圆相切	均匀壁温 (流场及阻力分析)	[23,48,49]
17		(流场及阻力分析)	[49]
18		均匀壁温	[50]
19		均匀壁温;均匀热流;轴向均匀热流、周向均匀壁温	[51,52]

No	流 道 截 面	热边界条件	文 献
20	 (垂直向上流动)	轴向均匀热流、周向均匀壁温，并考虑自然对流	[53]
21		均匀壁温	[54]
22		均匀壁温	[55]
23		均匀壁温；圆弧均匀壁温、底面绝热	[56]
24		换热元件上周向、轴向均匀热流；流道壁面绝热	[57]

No	流　道　截　面	热边界条件	文　献
25	 流动垂直低面	上、下表面各为均匀热流	[58]
26		套管外表面绝热；内管表面为耦合条件	[59]

4.6 FDHT in Circular Tubes

4.6.1. Physical and Mathematical Models

4.6.2. Governing equations and their non-dimensional forms

4.6.3. Conditions for unique solution

4.6.4. Numerical solution method

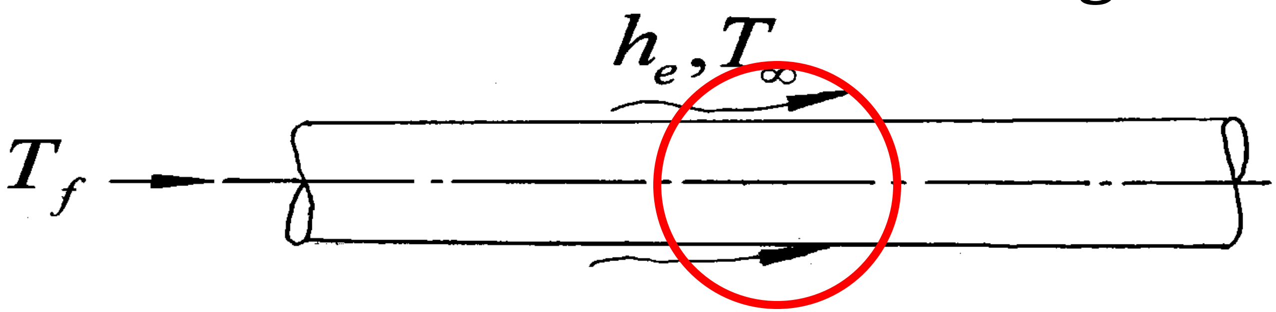
4.6.5. Treatment of numerical results

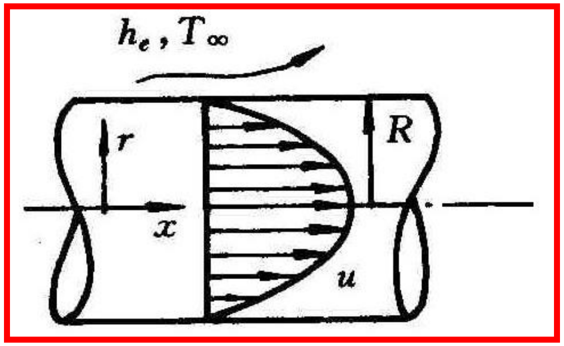
4.6.6. Discussion on numerical results

4.6 Fully Developed HT in Circular Tubes

4.6.1. Physical and mathematical models

A laminar flow in a long tube is cooled (heated) by an external fluid with temp. T_∞ and heat transfer coefficient h_e . Determine the heat transfer coefficient and Nusselt number in the FDHT region.





1.Simplification (简化) assumptions

(1) Thermo-physical properties are constant ;

(2) Axial heat conduction in the fluid is neglected

(3) Viscous dissipation (耗散)is neglected;

(4) Natural convection is neglected;

(5) Wall thermal resistance is neglected;

(6) The flow is fully developed:

$$\frac{u}{u_m} = 2[1-(\frac{r}{R})^2]; \qquad v = 0$$

2. Mathematical formulation (描述)

(1) Energy equation

Cylindrical coordinate, symmetric temp. distribution, and no natural convection (A4):

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+S_T$$

FD flow (A6)

No axial cond. (A2)

No dissipation (A3)

$$\rho c_p u\frac{\partial T}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r\frac{\partial T}{\partial r}\right)$$

Type of eq.?

2-D parabolic eq.!

(2) Boundary condition

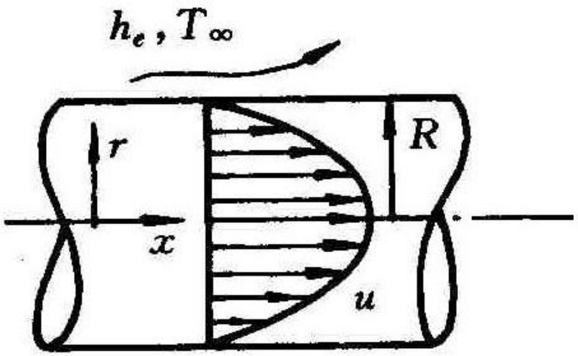
$$r=0,\ \frac{\partial T}{\partial r}=0$$ (Symmetric condition);

$$r=R,\ -\lambda\frac{\partial T}{\partial r}=h_e(T-T_\infty)$$ (External convective condition!)

λ: Internal fluid thermal conductivity

h_e: External (外部)convective heat transfer

No wall thermal resistance(A5), tube outer radius =R; 。



4.6.2. Governing eqs. and dimensionless forms

From FD condition a dimensionless temp. can be introduced, transforming the PDE to ordinary eq..

Defining $$\Theta = \frac{T - T_\infty}{T_b - T_\infty} \leftarrow \frac{T - T}{T_b - T} \leftarrow \frac{T - T}{T - T}$$

Then: $$T = \Theta(T_b - T_\infty) + T_\infty; \quad \frac{\partial T}{\partial x} = \Theta \frac{\partial T_b}{\partial x} = \Theta \frac{dT_b}{dx}$$

Defining dimensionless space and "time" coordinates:

$$\eta = \frac{r}{R}; \qquad X = \frac{x}{R \bullet Pe}$$

$$Pe = \frac{2R\rho c_p u_m}{\lambda} = \frac{2Ru_m}{a}$$

Constant properties（A 1）

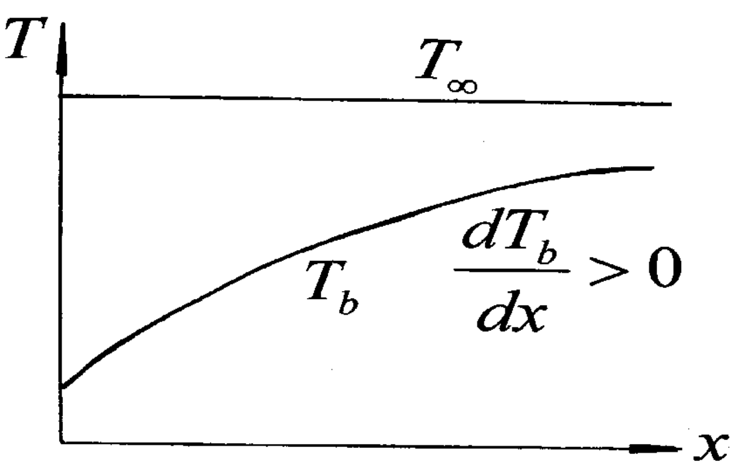
Energy eq. can be rewritten as：

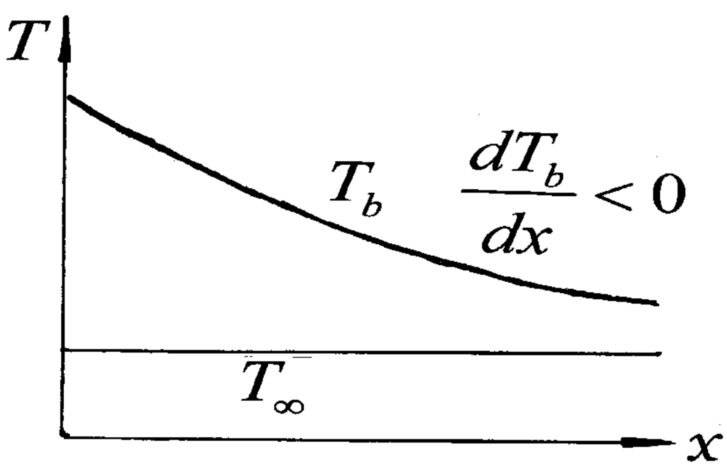
$$\frac{dT_b / dX}{T_b - T_\infty} = \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\Theta}{d\eta}\right) / \left(\frac{1}{2}\Theta\frac{u}{u_m}\right) = -\Lambda$$

$\Lambda > 0$

Dependent on X only

Dependent on η only





Λ is called **eigenvalue (特征值)**

Following ordinary dimensionless temp. eq. can be obtained

$$\frac{1}{\eta}\frac{d}{d\eta}(\eta\frac{d\Theta}{d\eta})/(\frac{1}{2}\Theta\frac{u}{u_m})=-\Lambda \qquad \textbf{(a)}$$

The original two B.Cs. are transformed (转换成) into:

$$\eta=0,\ \frac{d\Theta}{d\eta}=0; \qquad \textbf{(b)}$$

$$\eta=1,\ -\frac{d(\frac{T-T_\infty}{T_b-T_\infty})}{d(\frac{r}{R})}=(\frac{h_e R}{\lambda})\frac{T-T_\infty}{T_b-T_\infty} \rightarrow \frac{d\Theta}{d\eta})_{\eta=1}=-Bi\Theta_w \qquad \textbf{(c)}$$

Question：whether from Eqs.(a)-(c) a unique (唯一的) solution can be obtained?

4.6.3. Analysis of condition for unique solution

Because of the **homogeneous (齐次性)** character ：

Every term in the differential equation contains a linear part of dependent variable or its 1st/2nd derivative.

$$\frac{1}{\eta}\frac{d}{d\eta}(\eta\frac{d\Theta}{d\eta})/(\frac{1}{2}\Theta\frac{u}{u_m})=-\Lambda \longrightarrow \frac{1}{\eta}\frac{d}{d\eta}(\eta\frac{d\Theta}{d\eta})=-\Lambda(\frac{1}{2}\Theta\frac{u}{u_m})$$

In addition, the given B.Cs. are also **homogeneous:**

$$\eta=0,\ \frac{d\Theta}{d\eta}=0; \qquad \frac{d\Theta}{d\eta})_{\eta=1}=-Bi\Theta_w$$

For the above mathematical formulation there exists an uncertainty （不确定性）of being able to be multiplied by a constant.

While in order to solve the problem, the value of Λ in the formulation has to be determined.

In order to get a unique solution and to specify the eigenvalue, we need to supply one more condition!

We examine the definition of dimenionless temperature:

$$\Theta_b = \left(\frac{T - T_\infty}{T_b - T_\infty}\right)_b = \frac{T_b - T_\infty}{T_b - T_\infty} \equiv 1.0$$

Physically, the averaged temp. is defined by

$$\Theta_b = \frac{\int_0^R 2\pi r u \Theta dr}{\pi R^2 u_m} = 2\int_0^1 \frac{r}{R}\frac{u}{u_m}\Theta d\left(\frac{r}{R}\right) = 1$$

Thus the complete formulation is:

$$\frac{1}{\eta}\frac{d}{d\eta}(\eta\frac{d\Theta}{d\eta})+\Lambda(\frac{1}{2}\Theta\frac{u}{u_m})=0 \quad \text{(a)}$$

$$\eta=0,\ \frac{d\Theta}{d\eta}=0; \quad \text{(b)}$$

$$\frac{d\Theta}{d\eta})_{\eta=1}=-Bi\Theta_w \quad \text{(c)}$$

$$\int_0^1\eta\frac{u}{u_m}\Theta d\eta=1/2 \quad \text{(d)}$$

Non-homogeneous term!

4.6.4. Numerical solution method

$$\frac{1}{\eta}\frac{d}{d\eta}(\eta\frac{d\Theta}{d\eta})+\Lambda(\frac{1}{2}\Theta\frac{u}{u_m})=0$$

This is a 1-D conduction equation with a source term!

$\frac{\Lambda}{2}\Theta\frac{u}{u_m}$ **,whose value should be determined during the solution process iteratively.**

Patankar－Sparrow proposed following numerical solution method:

(1) Let $\Theta=\Lambda\phi$

Because of the homogeneous character, the form of the equation is not changed only replacing Θ by ϕ .

$$\frac{1}{\eta}\frac{d}{d\eta}(\eta\frac{d\phi}{d\eta})+\Lambda(\frac{1}{2}\phi\frac{u}{u_m})=0 \qquad \text{(a)}$$

$$\eta=0,\ \frac{d\phi}{d\eta}=0; \qquad \text{(b)}$$

$$\frac{d\phi}{d\eta})_{\eta=1}=-Bi\phi_w \qquad \text{(c)}$$

$$\int_0^1 \eta\frac{u}{u_m}\Lambda\phi d\eta=1/2 \qquad \text{(d)}$$

Non-homogeneous term

$$\Lambda=1/(2\int_0^1 \eta\frac{u}{u_m}\phi d\eta)$$

It can be used to iteratively determine the eigenvalue.

(2) Assuming an initial field ϕ^* , get Λ^*

(3) Solving an ordinary differential eq. with a source term to get an improved ϕ

(4) Repeating the above procedure until:

$$\left|\frac{\phi^* - \phi}{\phi}\right| \le \varepsilon, \qquad \varepsilon = 10^{-3} \sim 10^{-6}$$

This iterative procedure is easy to approach convergence:

$$S = \Lambda \frac{1}{2}\frac{u}{u_m}\phi = \frac{(u/u_m)\phi}{4\int_0^1 \eta (u/u_m)\phi d\eta} = \frac{(1-\eta^2)\phi}{4\int_0^1 \eta (1-\eta^2)\phi d\eta}$$

$$\Lambda = 1/(2\int_0^1 \eta \frac{u}{u_m}\phi d\eta)$$

ϕ exists in both numerator and denominator, thus only the distribution, rather than absolute value will affect the source term.

4.6.5. Treatment of numerical results

Two ways for obtaining heat transfer coefficient:

1. From solved temp. distribution using Fourier's law of heat conduction and Newton's law of cooling:

$$r = R, -\lambda \frac{\partial T}{\partial r} = h(T_w - T_b)$$

For inner fluid

$$h = -\lambda \frac{\partial T}{\partial r})_{r=R} \frac{1}{T_w - T_b}$$

Different from Boundary condition

$$r = R, -\lambda \frac{\partial T}{\partial r} = h_e(T - T_\infty)$$

2. From the eigenvalue（特征值）：

From heat balance between inner and external heat transfer

$$h(T_b - T_w) = h_e(T_w - T_\infty)$$

Inner — External

Get:

$$h = h_e \frac{T_w - T_\infty}{T_b - T_w} \rightarrow h = h_e \frac{1}{\dfrac{T_b - T_w}{T_w - T_\infty}} \rightarrow \frac{h_e}{\dfrac{T_b - T_\infty + T_\infty - T_w}{T_w - T_\infty}}$$

$$\rightarrow \frac{h_e}{\dfrac{T_b - T_\infty}{T_w - T_\infty} - 1} \rightarrow h = \frac{h_e}{\dfrac{1}{\dfrac{T_w - T_\infty}{T_b - T_\infty}} - 1} = \frac{h_e}{\dfrac{1}{\Theta_w} - 1} \rightarrow$$

$$h = \frac{h_e}{\frac{1}{\Theta_w} - 1} = \frac{h_e \Theta_w}{1 - \Theta_w} = \frac{h_e \Lambda \phi_w}{1 - \Lambda \phi_w}$$

$$Nu = \frac{2Rh}{\lambda} = \frac{2R}{\lambda} \frac{h_e \Lambda \phi_w}{1 - \Lambda \phi_w} = \frac{2Bi \Lambda \phi_w}{1 - \Lambda \phi_w}$$

From the specified values Bi, the corresponding eigenvalues, Λ, can be obtained. Thus it is not necessary to find the 1st derivative at the wall of function ϕ for determining Nusselt number.

4.6.6. Discussion on numerical results

Table 4-6 Numerical results of FDHT in tubes

Bi	Λ	Nu
0	0	4.364 $(Nu)_q$
0.1	0.381 8	4.330
0.25	0.894 3	4.284
0.5	1.615	4.221
1	2.690	4.122
2	3.995	3.997
5	5.547	3.840
10	6.326	3.758
100	7.195	3.663
∞	7.314	3.657 $(Nu)_T$

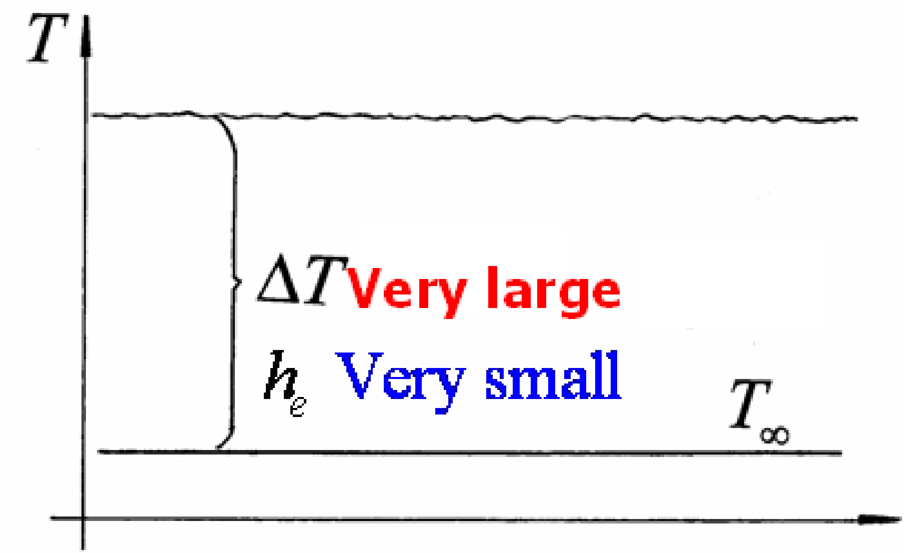
1. Bi effect:

From definition $Bi = \frac{Rh_e}{\lambda}$

$Bi \to \infty, \quad h_e \to \infty$ External heat transfer is very strong, the wall temp. approaches fluid temp.

This is corresponding to constant wall temp condition, Thus **Nu=3.66**

$Bi \to 0, \; h_e \to 0$ **Is this adiabatic? No!**



Product of very small HT coefficient and very large temp. difference makes heat flux almost constant.

$$q = h_e \Delta T \approx const$$

2. Computer implementation of $\mathrm{Bi} \to \infty$ and $\mathrm{Bi} = 0$

Bi $\to \infty$ by progressively (逐渐地) increasing Bi:

Bi$= 10^5,\ 10^6\ ,10^7$.....

Bi$=$0 by progressively decreasing Bi:

Bi= 0.1, 0.01, 0.001, 0.0001, 0.00001,

Double decision (双精度)must be used for Computation：

$$Nu = \frac{2Bi\Lambda\phi_w}{1-\Lambda\phi_w},\ Bi \to 0,\ \Lambda \to 0,\ \Lambda\phi_w \to 1 \longrightarrow \frac{0}{0}$$

4.7 Fully Developed HT in Rectangle Ducts

4.7.1 Physical and mathematical models

4.7.2 Governing eqs. and their dimensionless forms

4.7.3 Condition for unique solution

4.7.4 Treatment of numerical results

4.7.5 Other cases

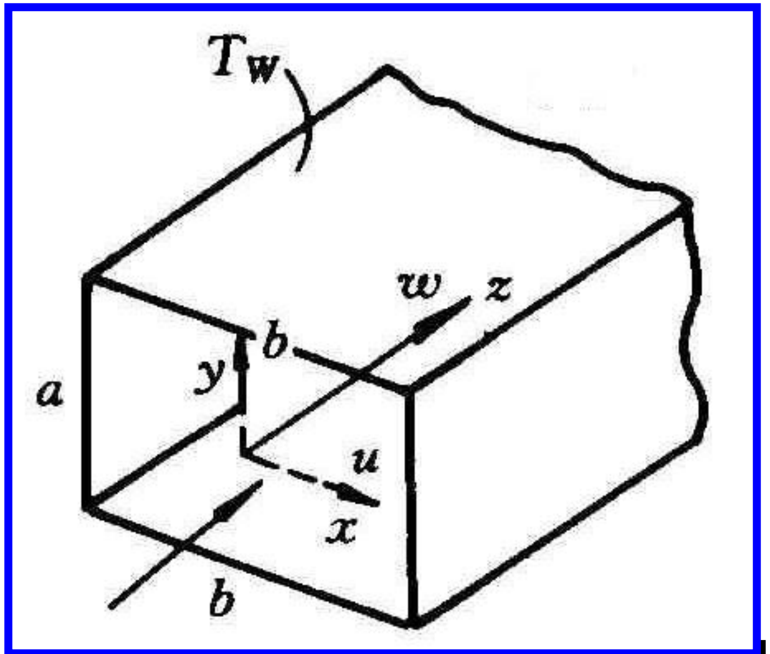
4.7 Fully Developed HT in Rectangle Ducts

4.7.1 Physical and mathematical models

Fluid with constant properties flows in a long rectangle duct with a constant wall temp. Determine the friction factor and HT coefficient in the fully developed region for laminar flow.

1. Momentum eq.

For the fully developed flow $u=v=0$, only the component in z-direction is not zero. Its governing equation:



$$\rho(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z})=-\frac{\partial p}{\partial z}+\eta(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})$$

$$\eta(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})-\frac{\partial p}{\partial z}=0$$

Neglecting cross section variation

$$\eta(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})-\frac{dp}{dz}=0$$

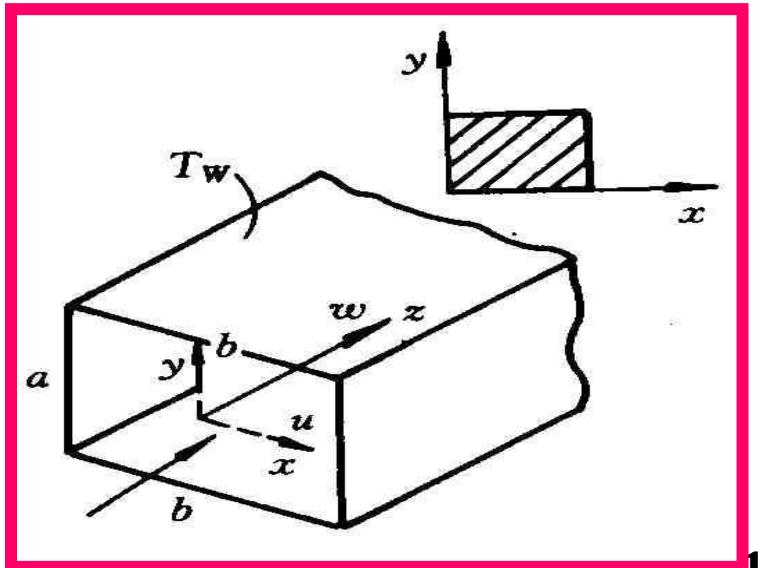
Taking ¼ region as the computational domain because of symmetry. Boundary conditions are:

At the wall， *w*=0;

At center line，

First order normal derivative equals zero：

$$\frac{\partial w}{\partial n}=0$$



Defining a dimensionless velocity as：

$$W=\frac{\eta w}{-D^2\dfrac{dp}{dz}}$$

where D is the referenced length, say: *D*＝*a*, or *D*=*b*.

Defining dimensionless coordinates： *X*＝*x/D*, Y=*y/D*, yields：

$$\eta\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\frac{dp}{dz}=0 \rightarrow \begin{cases}\dfrac{\partial^2 W}{\partial X^2}+\dfrac{\partial^2 W}{\partial Y^2}+1=0\\ \text{At wall, } W=0;\\ \text{At center lines, } \dfrac{\partial W}{\partial n}=0\end{cases}$$

It is a heat conduction problem with a source term!

2. Energy equation

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)$$

(0, 0, 0 — the $u\frac{\partial T}{\partial x}$, $v\frac{\partial T}{\partial y}$ and $\frac{\partial}{\partial z}(\lambda\frac{\partial T}{\partial z})$ terms are crossed out)

Neglecting axial heat conduction

Thus: $$\rho c_p w\frac{\partial T}{\partial z}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)$$

Type of equation? **Parabolic! Z is a one-way coordinate**

Boundary conditions:

At the wall, $T=T_w$;

At the center line, $\frac{\partial T}{\partial n}=0$

4.7.2 Dimensionless governing equation

We should define an appropriate dimensionless temperature such that the dimension of the problem can be reduced from 3 to 2: Separating the one-way coordinate z from the two-way coordinates x,y。

$$\Theta = \frac{T_w - T}{T_w - T_b} \leftarrow \frac{T - T}{T - T_b} \leftarrow \frac{T - T}{T - T}$$

Then $T = \Theta(T_b - T_w) + T_w$

$$\frac{\partial T}{\partial z} = \Theta \frac{\partial (T_b - T_w)}{\partial z}$$

$$Pe = \frac{\rho c_p w_m D}{\lambda}$$

Defining: $X = x / D, Y = y / D, Z = z / (DPe)$

One-way coordinate!

Dimensionless governing eq.

$$\frac{\partial (T_b - T_w)}{\partial Z}\frac{1}{T_b - T_w} = \frac{\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2}}{\frac{W}{W_m}\Theta} = -\Lambda$$

$\Lambda > 0$

Dependent on Z only

Dependent on X, Y only

Thus:

$$\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} + \Lambda \frac{W}{W_m}\Theta = 0;$$

$$\frac{d(T_b - T_w)}{dZ}\frac{1}{T_b - T_w} = -\Lambda$$

At the wall $\Theta = 0$

At center line, $\frac{\partial \Theta}{\partial n} = 0$

Heat conduction with an inner source!

4.7.3 Analysis on the unique solution condition

Because of the homogeneous character, these also exists an uncertainty of being magnifying by any times!

Introducing average temperature (difference):

$$T_w - T_b = \frac{\int_A (T_w - T) w dA}{\int_A w dA} \rightarrow \frac{T_w - T_b}{T_w - T_b} = \frac{\int_A \frac{T_w - T}{T_w - T_b} w dA}{w_m A}$$

$$1 = \frac{1}{A}\int_A \frac{T_w - T}{T_w - T_b}\frac{w}{w_m} dA \rightarrow 1 = \frac{1}{A}\int_A \Theta\left(\frac{W}{W_m}\right) dA$$

It is the additional condition for the unique solution.

Numerical solution method is the same as that for a circular tube.

4.7.4 Treatment of numerical results

After receiving converged velocity and temperature fields, friction factor and Nusselt number can be obtained as follows:

1.fRe—**for laminar problems fRe =constant:**

$$f\,\mathrm{Re} = [-\frac{D_e \dfrac{dp}{dz}}{\dfrac{1}{2}\rho w_m^2}](\frac{w_m D_e}{\nu})$$

Definition of W

$$W = \frac{\eta w}{-D^2 \dfrac{dp}{dz}}$$

$$f\,\mathrm{Re} = \frac{2}{W_m}(\frac{D}{D_e})^2$$

2. Nu— **Making an energy balance：**

$$\rho c_p w_m A \frac{dT_b}{dz} = qP$$ **,*P* is the duct circumference length**

$$\frac{d(T_b - T_w)}{dZ}\frac{1}{T_b - T_w} = -\Lambda$$ **i.e.,** $$\frac{dT_b}{dZ} = \frac{dT_b}{dz} DPe = (T_w - T_b)\Lambda$$

$$\frac{dT_b}{dz} = \frac{1}{DPe}(T_w - T_b)\Lambda$$ **Substituting in**

$$\rho c_p w_m A \frac{dT_b}{dz} = qP$$

yields $$q = \frac{A\rho c_p w_m}{P}\frac{dT_b}{dz} = \frac{A\rho c_p w_m}{P}\frac{1}{DPe}\Lambda(T_w - T_b)$$

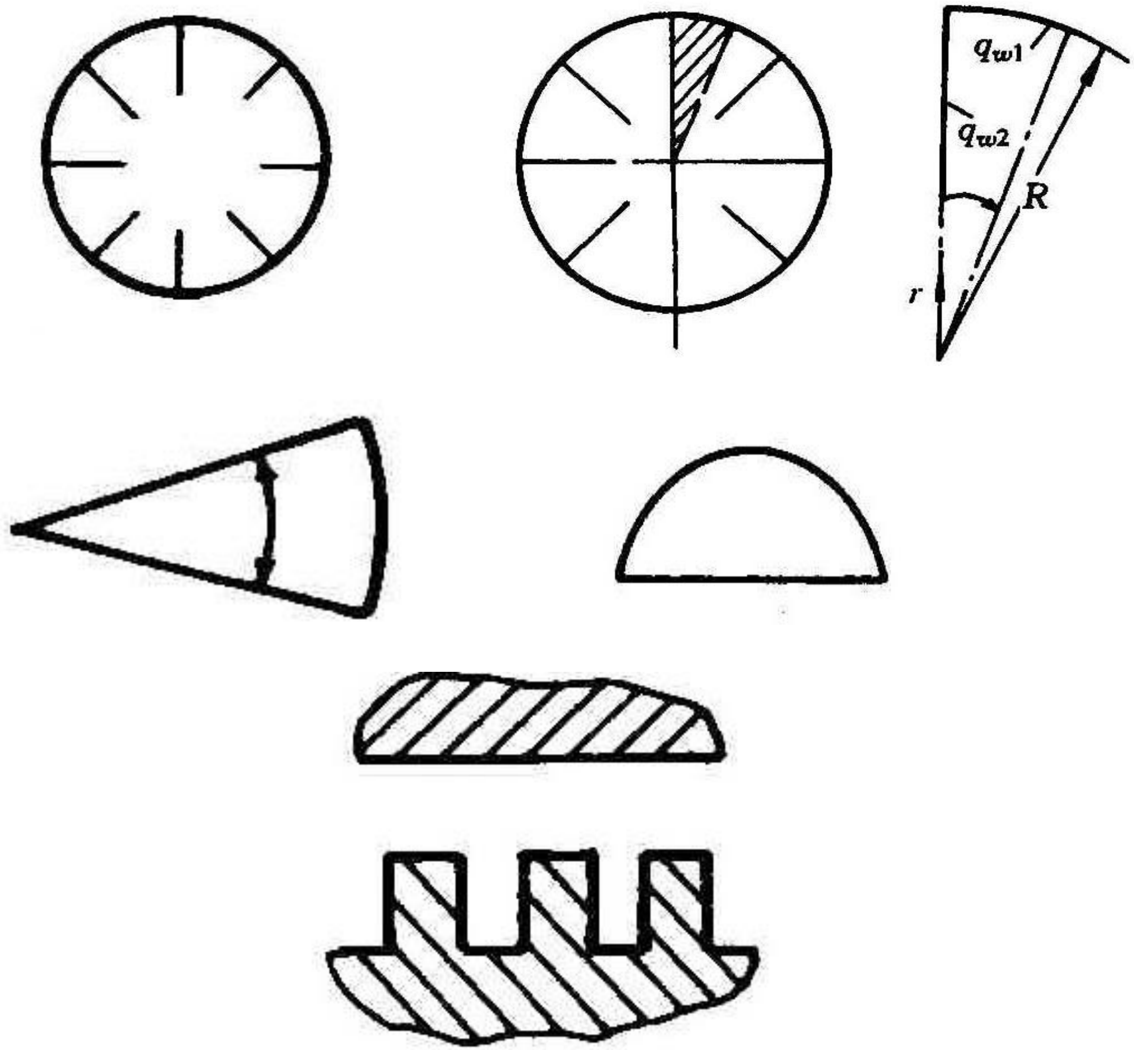
$$Pe = \frac{\rho c_p w_m D}{\lambda}$$

yields: $$q = \frac{A}{P}\frac{\lambda}{D^2}\Lambda(T_w - T_b)$$

$$Nu = \frac{hD_e}{\lambda} = \frac{q}{T_w - T_b}\frac{D_e}{\lambda} = \frac{1}{T_w - T_b}\frac{D_e}{\lambda}\frac{A}{P}\frac{\lambda}{D^2}\Lambda(T_w - T_b) = \frac{1}{4}\left(\frac{D_e}{D}\right)^2\Lambda$$

$$D_e = \frac{4A}{P}$$

4.7.5 Other cases



Home Work 3

4－2,

4－3,

4－12,

4－14,

4－19

Due in October 19

Problem # 4-2

Figure as shown below, is one-dimensional steady-state heat conduction problem, where $T_1 = 100$, $\lambda = 5$, $S = 150$, $T_f = 20$, $h = 15$ are known, the unit of every item is in System International (SI) . Try to determine the value of T_2, T_3 by numerical calculation, according to your results, to prove that the entire computational domain meets the requirements of the overall conservation even if only three nodes were took.

Problem#4-3: A large plate with thickness of 0.1 m, uniform source $S=50\times10^3\ \mathrm{W/m^3}$, $\lambda = 10\ \mathrm{W/(m \bullet {}^\circ C)}$ **; One of its wall is kept at 75 ,while the other wall is cooled by a fluid with** $T_f = 25^\circ\mathrm{C}$ **and heat transfer coefficient** $h = 50\ \mathrm{W/m^2 \bullet {}^\circ C}$.

Divide the plate thickness into three uniform elements, and by using Practice A, determine the inner node temperature. Take 2nd order accuracy for the inner node, and 1st order / 2nd order accuracy (two cases) for the right boundary node.

Problem # 4-12

Write a program using TDMA algorithm, and use the following method to check its accuracy: set arbitrary values of the coefficients A_i, B_i and C_i (i = 1,10). But B_1 and C_{10} should not be zero. Then setting the reasonable values of temperature $T_1, ..., T_{10}$, calculate the corresponding constants D_i. Apply your program for solving T_i by using the values of A_i, B_i, C_i and D_i, and compare the results with the given value.

Problem # 4-14

According to the problem discussed in section 4.6(The fully developed heat convection in a circular tube), try to analyze the following three dimensionless temperature definitions of $\Theta = \frac{T - T_w}{T_b - T_w}$, $\Theta = \frac{T - T_\infty}{T_w - T_\infty}$ and $\Theta = \frac{T - T_w}{T_\infty - T_w}$, which one is acceptable for separation of variables.

Problem #4-19: Shown in Fig.4-25 is a channel for cooling an electronic device :laminar fluid flow and heat transfer in the direction normal to the page are fully developed, and the top and bottom wall have uniform heat fluxes q_1 and q_2, respectively.

Try:

(1)Determine the domain for numerical simulation and give mathematical formulation;

(2) Propose a numerical method which can be sued to simulate the dimensionless temperature distributions of the channel at different ratio of q_1/q_2

Important Announcement:

No lecture on Oct. 17.
We will meet on Oct 19!

同舟共济
渡彼岸！

People in the same boat help each other to cross to the other bank, where….