数值传热学

第七章 代数方程组的求解方法

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Numerical Heat Transfer

(数值传热学)

Chapter 7 Solution Methods for Algebraic Equations

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Chapter 7 Solution Methods for Algebraic Equations

7.1 Introduction to Solution Methods of ABEqs

7.2 Construction of Iteration Methods of Linear Algebraic Equations

7.3 Convergence Conditions and Acceleration Methods for Solving Linear ABEqs.

7.4 Block Correction Method –Promoting Conservation Satisfaction

7.5 Multigrid Techniques –Promoting Simultaneous Attenuation of Different Wave-length Components

7.1 Introduction to Solution Methods of ABEqs

7.1.1 Matrix feature of multi-dimensional discretized equation

7.1.2 Direct method and iteration method for solving ABEqs.

7.1.3 Major idea and key issues of iteration methods

7.1.4 Criteria for terminating iteration

7.1 Introduction to Solution Methods of ABEqs

7.1.1 Matrix feature of multi-dimensional discretized equation

For 2-D, 3-D flow and heat transfer problems, the discretized equations with 2nd order accuracy:

2-D $$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b$$

3-D $$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + a_F\phi_F + a_B\phi_B + b$$

For a 2D case with L1×M1unknown variables, the general algebraic equation of kth variable is:

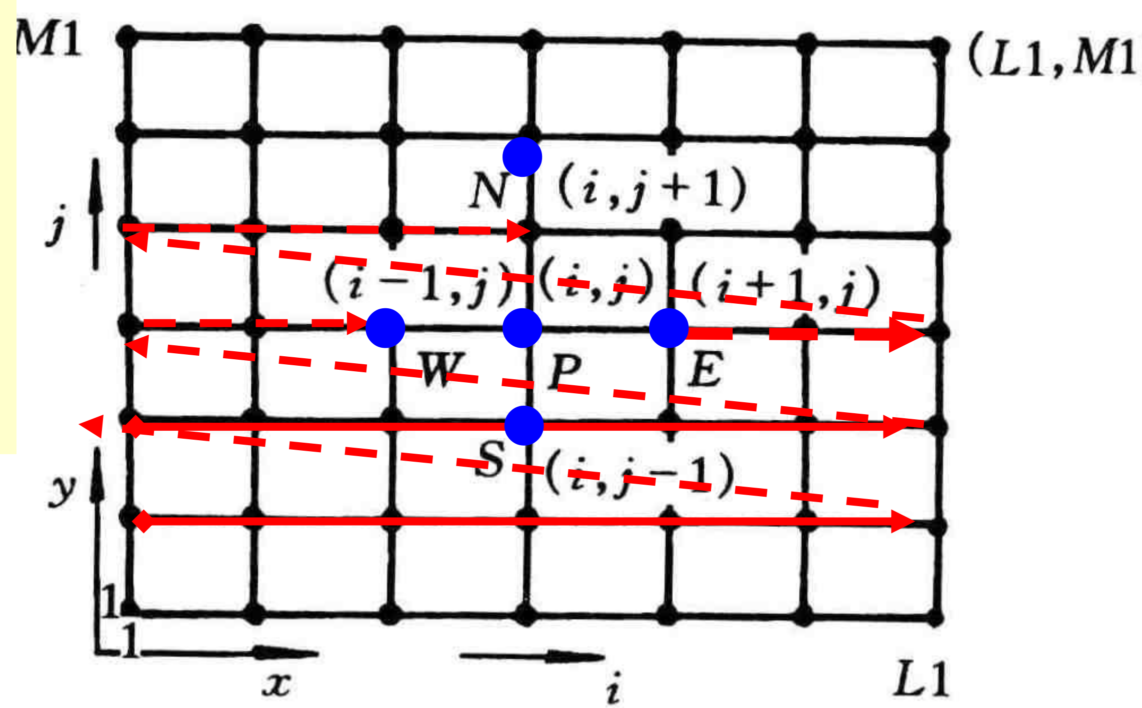
$$a_{k,1}\phi_{,1} + a_{k,2}\phi_2 + \ldots. + a_{k,k-L1}\phi_{k-L1} + a_{k,k-L1+1}\phi_{k-L1+1} + \ldots + a_{k,k-1}\phi_{k-1}$$

$$+a_{k,k}\phi_k + a_{k,k+1}\phi_{k+1} + \ldots + a_{k,k+L1}\phi_{k+L1} + \ldots + a_{k,L1\times M1}\phi_{L1\times M1} = b_k$$

For 2-D problem with 2nd order accuracy there are only five coefficients at the left hand side are not equal to zero, and the matrix is of quasi (准)five-diagonal, a large scale sparse matrix (大型稀疏矩阵).

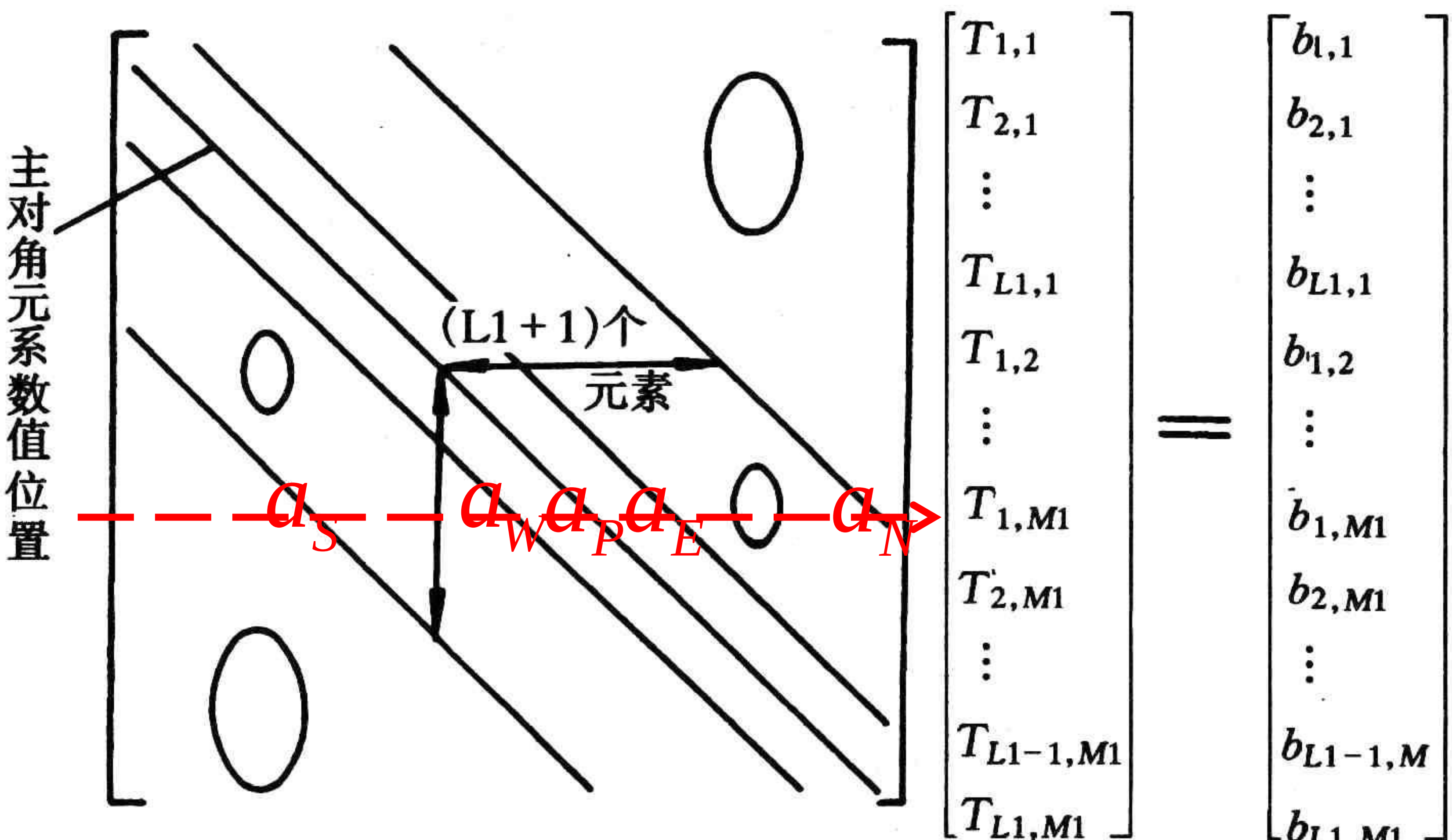
If the 1-D storage of the coefficients is conducted as shown right, then the order of coefficients in one line are:

$$a_S, a_W, a_P, a_E, a_N$$



$$\cancelto{0}{a_{k,1}\phi_{,1}} + \cancelto{0}{a_{k,2}\phi_2} + \ldots + \underline{a_{k,k-L1}\phi_{k-L1}} + \cancelto{0}{a_{k,k-L1+1}\phi_{k-L1+1}} + \ldots + \underline{a_{k,k-1}\phi_{k-1}}$$

$$\underline{+a_{k,k}\phi_k} + \underline{a_{k,k+1}\phi_{k+1}} + \ldots + \underline{a_{k,k+L1}\phi_{k+L1}} + \ldots + \cancelto{0}{a_{k,L1\times M1}\phi_{L1\times M1}} = b_k$$



Features of ABEqs. of discretized multi-dimensional flow and heat transfer problems:

1) Conduction of constant properties in uniform grid—matrix is symmetric and positive definite(正定、对称）;

2) Other cases: matrix is neither symmetric nor positive definite.

ABEqs. of large scale sparse matrix are usually solved by iteration methods (迭代法).

7.1.2 Direct method and iterative method for solving ABEqs.

1.Direct method(直接法)

Accurate solution can be obtained via a finite times of operations if there is no round-off error, such as TDMA，PDMA.

2. Iterative method（迭代法）

From an initial field the solution is progressively improved via the ABEqs. and terminated （终止） when a pre-specified criterion is satisfied.

The ABEqs. of fluid flow and heat transfer problems usually are solved by iteration methods:

✓ Non-lineairity of the problems. The coefficients need to be updated. There is no need to get the true solution for temporary （临时的） coefficients;

✓ Huge time of direct method. The operation times of direct method is proportional to $N^{2.5\sim3}$, where N is the number of unknown variables. When N is very large the operation times becomes very large, often unmanageable!

7.1.3 Major Idea and Key Issues of Iteration Methods

1. Major idea

In matrix form the ABEqs. is ： $\vec{A}\vec{\phi}=\vec{b}$ Its solution is $\vec{\phi}=(\vec{A})^{-1}\vec{b}$. Iteration method is to construct a series of $\overrightarrow{\phi^{k}}$ in multi-dimensional space R (the number of dimensions equals the number of unknowns) such that

when $k\to\infty$ $\vec{\phi}^{(k)}\to(\overleftarrow{A})^{-1}\bar{b}$

For the kth iteration $\vec{\phi}^{(k)}=f(\vec{A},\vec{b},\vec{\phi}^{(k-1)})$

2. Key issues of iteration methods

1) How to construct the iteration series?

2) Is the series converged?

3) How to accelerate the convergence speed?

7.1.4 Criteria for terminating iteration

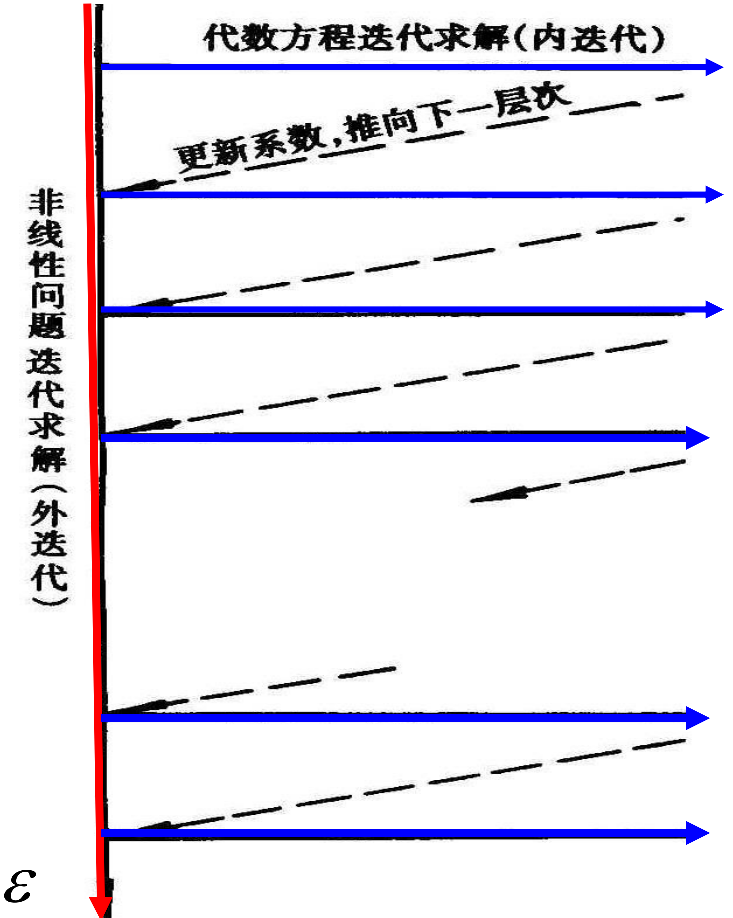
(1) Specifying iteration number;

(2) Specifying the norm of p'eq. residual less than a certain small value;

(3) Specifying the relative norm of p'eq. residual less than a certain small value;

(4) Specifying relative change of variable less than a small value;

$$\left|\frac{\phi^{(k+1)}-\phi^{(k)}}{\phi_{\max}^{(k+1)}}\right|_{\max} \le \varepsilon; \qquad \left|\frac{\phi^{(k+1)}-\phi^{(k)}}{\phi^{(k+1)}+\varepsilon_0}\right|_{\max} \le \varepsilon$$



7.2 Construction of Iteration Methods of Linear Algebraic Equations

7.2.1 Point (explicit) iteration

7.2.2 Block (implicit) iteration

7.2.3 Alternative direction iteration－ADI

7.2 Construction of Iteration Methods of Linear Algebraic Equations.

7.2.1 Point (explicit) iteration

The updating is conducted from node to node; After every node has been visited a cycle of iteration is finished; The updated value at each node is explicitly related to the others (values of previous iteration).

1. Jakob iteration

In the updating of every node the previous cycle values of neighboring nodes are used; The convergence speed is independent of iteration direction

2. Gauss－Seidel iteration

Present values are used for updating.

3. SOR/SUR iteration

$$\phi^{(k+1)} = \phi^{(k)} + \alpha(\phi^{(k+1)} - \phi^{(k)}) \begin{cases} \alpha < 1 & \textbf{Under-} \\ (0 \le \alpha \le 2) & \\ \alpha > 1 & \textbf{Over-} \end{cases}$$

Remarks： This relaxation is for the linear ABEqs., not for the non-linearity.

8.2.2 Block (implicit) iteration （块(隐式)）

1. Basic idea

Dividing the solution domain into several regions, within each region direct solution method is used, while from block to block iteration is used， also called implicit iteration

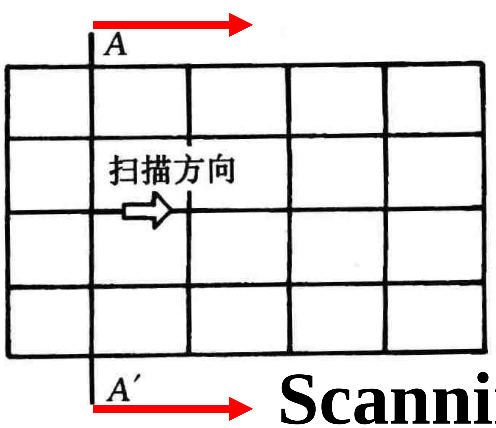
2. Line iteration（线迭代)-the most fundamental of block iteration

The smallest block is a line: At the same line TDMA is used for direct solution, from line to line iterative method is used.

Solving in N-S direction and scanning (扫描) in E-W D.:

Jakob: $$a_P\phi_P^{(k+1)} = a_N\phi_N^{(k+1)} + a_S\phi_S^{(k+1)} + [\underline{a_E\phi_E^{(k)} + a_W\phi_W^{(k)} + b}]$$

G-S: $$a_P\phi_P^{(k+1)} = a_N\phi_N^{(k+1)} + a_S\phi_S^{(k+1)} + [\underline{a_E\phi_E^{(k)} + a_W\phi_W^{(k+1)} + b}]$$

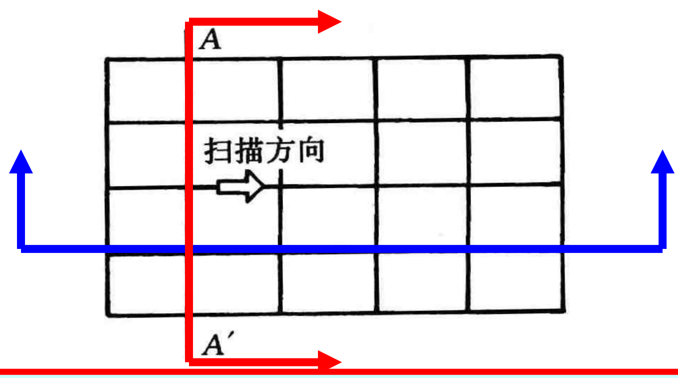


Scanning （扫描） in E-W direction

8.2.3 Alternative direction iteration－ADI

1. Basic idea

First direct solution for each row(行)（or column 列），then direct solution for each column （or row); The combination of the two updating of the entire domain consists of one cycle iteration:



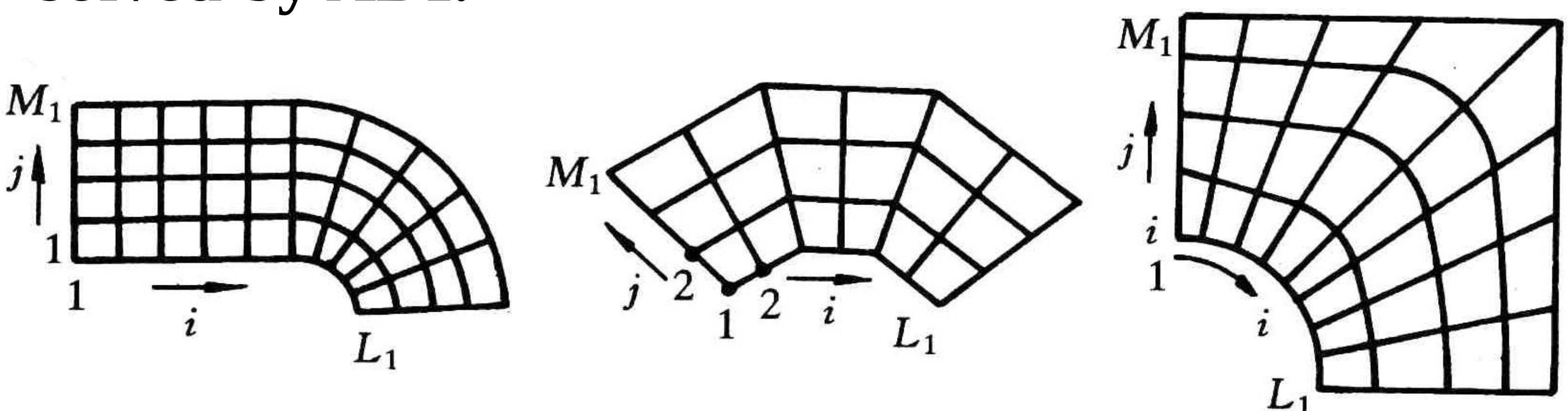
Alternative direction iteration (ADI) vs. alternative direction implicit (ADI):

It can be shown that: one-time step forward of transient problem is equivalent to one cycle iteration for steady problem (see appendix).

Therefore ADI-iteration of solving multi-dimensional steady problem for one iteration **(ADI-iteration)** is very similar to the ADI-implicit of solving multidimensional unsteady problem for one time step **(ADI-implicit)**.

2. ADI-line iteration is widely adopted in flow and heat transfer problem numerical solution.

ABEqs. generated on structured grid system can be solved by ADI.



7.3 Convergence Conditions and Acceleration Methods for Solving Linear ABEqs.

7.3.1 Sufficient condition for iteration convergence of Jakob and G-S iteration

7.3.2 Analysis of factors influencing iteration convergence speed

7.2.3 Methods for accelerating transferring boundary condition influence into solution domain

7.3.1 Sufficient condition for iteration convergence of Jakob and G-S iteration

1. Sufficient condition－Scarborough criterion

Coefficient matrix is non-reducible (不可约), and is diagonal predominant(对角占优）：

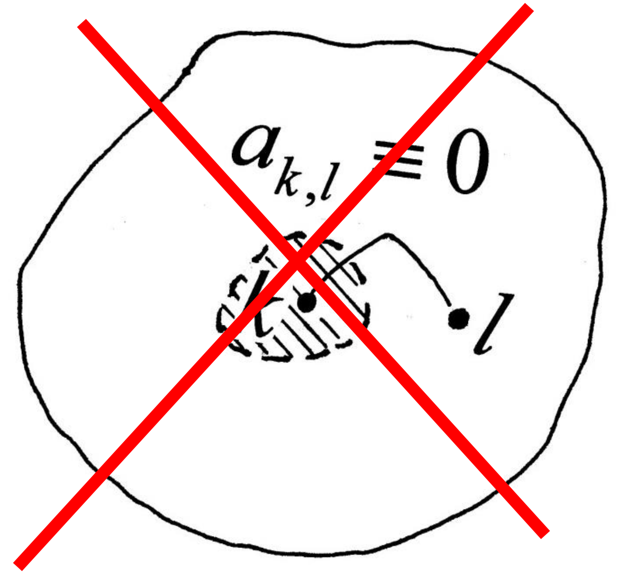
$$\frac{\sum |a_{nb}|}{|a_P|} \le 1 \begin{cases} \le 1 \text{ for all equations (a)} \\ < 1 \text{ at least for one equations (b)} \end{cases}$$

2. Analysis of coefficients of discretized diffusion-convection equation by recommended method

1) Matrix is non-reducible－If matrix is reducible then the set (集合） of coefficients subscript (矩阵下标) ,W , can be divided into two non-empty (非空) sub-sets, R and S，W＝R＋S，and taking any element from R and S, say k and l，we must have：$a_{k,l} \equiv 0$ ；If such condition does not exist, then the matrix is called non-reducible (不可约)

Analysis：Coefficient of discretized equation represents the influence of neighboring nodes. For nodes in elliptic region any one must has its effects on its neighbors; If matrix is reducible, it implies that the computational domain can be divided into two regions which do not affect each other---totally impossible .

Non-reducible matrix is determined by the physical fact that neighboring parts in flow and heat transfer are affected each other.



2) Diagonal predominant－Coefficients constructed in the present course must satisfy this condition:

(1) Transient and fully implicit scheme

$$a_P = \sum a_{nb} + a_P^0 - S_P \Delta V \quad , a_P^0 > 0, -S_P > 0 \quad , a_P > \sum a_{nb}$$

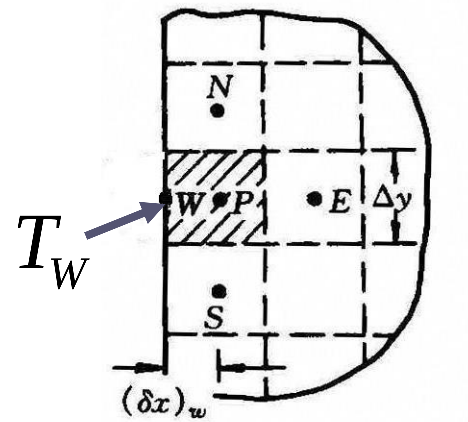
(2) Steady problem with non-constant source term

$$-S_P > 0 \quad , a_P > \sum a_{nb}$$

(3) Steady problem without source term

For inner grids: $a_P = \sum a_{nb}$

At least one node in the boundary can be found to satisfy : $a_P > \sum a_{nb}$



1)Assuming that Tw is known, then when the eq.

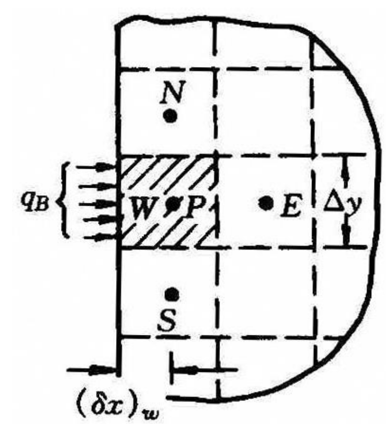
$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b$$

is solved, it becomes:

$$a_P T_P = a_E T_E + 0 + a_N T_N + a_S T_S + (b + a_W T_W)$$

Hence here: $a_P = \sum a_{nb} > a_E + 0 + a_N + a_S$

2) For 3rd kind boundary condition, additional source term helps



$$-S_P > 0, \; a_P = \sum a_{nb} - (-|S_P|) > \sum a_{nb}$$

It is impossible that all boundary nodes are of 2nd type, at least one node is of 1st or 3rd type. Otherwise there is no definite solution!

Thus numerical methods recommended by the present course must satisfy this sufficient condition.

7.3.2 Analysis of factors influencing iteration convergence speed

1. Transferring effects of B.C. into domain---View P.1

The steady state heat conduction with constant properties are governed by Laplace equation, $\nabla^2\phi = 0$ for which initial uniform field satisfies. However, it is not the solution because B.C. is not satisfied

Thus the transferring speed for the effects of boundary condition must affect iteration convergence speed.

2. Satisfaction of conservation condition---View P.2

For a problem with 1st kind boundary condition, it is possible to incorporate all the known boundary values into the initial field, but such an initial filed does not satisfy conservation condition. Thus techniques which is in favor of satisfying conservation condition can accelerate convergence speed;

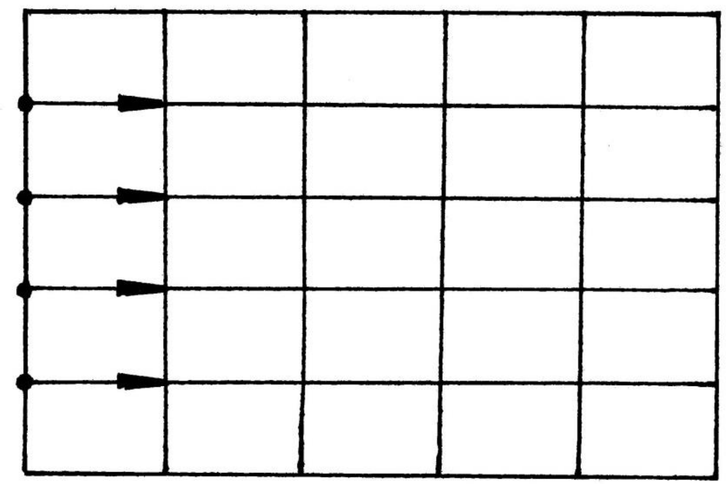
3. Attenuation (衰减） of error vector---View Point 3

The error vector is attenuated during iteration. Error vector is composed of components of different frequency. Techniques which can uniformly attenuate different components must can accelerate convergence speed.

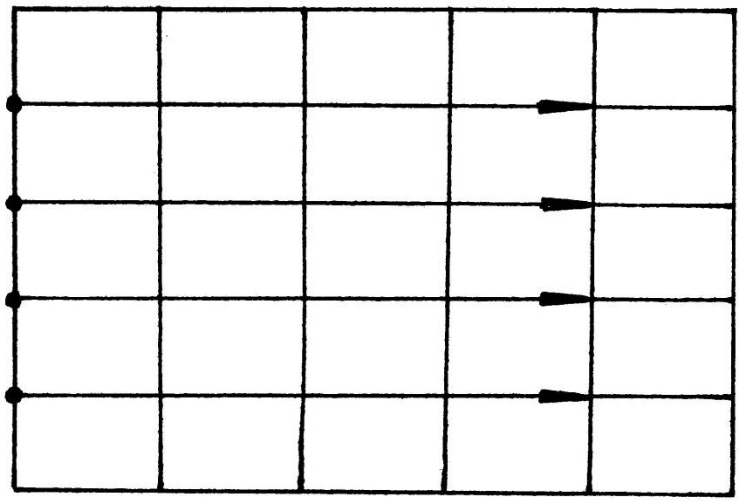
4. Increasing percentage of direct solution---View P.4

Direct solution is the most strong technique that both conservation and boundary condition can be satisfied. Thus appropriately increasing direct solution proportion (份额) is in favor of accelerating convergence speed.

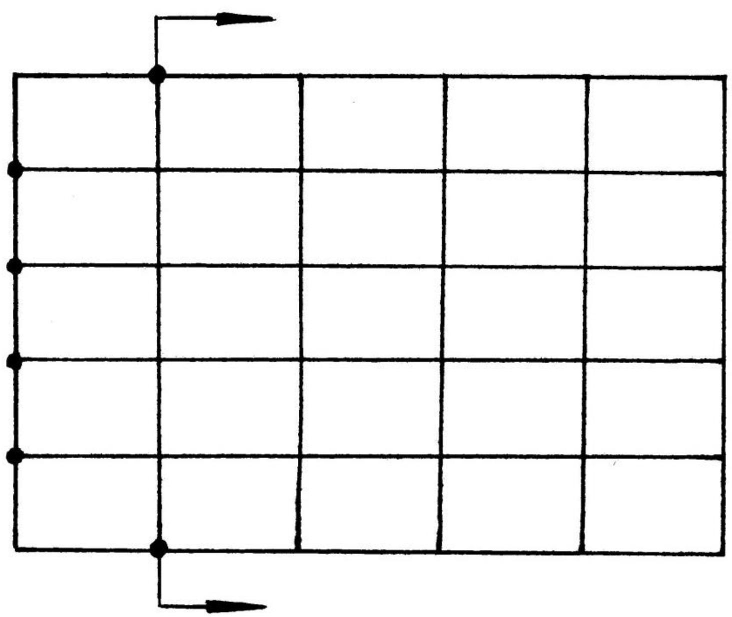
7.3.3 Techniques for accelerating transferring B.C. effects



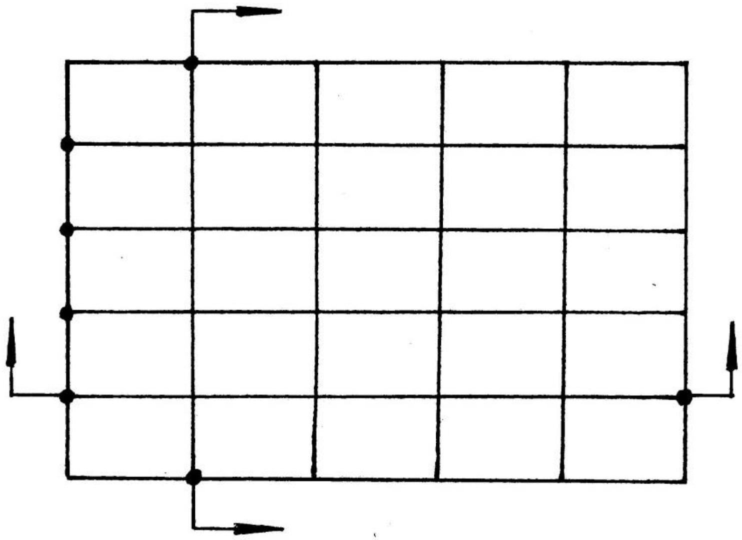
Jakob iteration: In each cycle the effect of B.P. can transfer into inner region by one space step. Very low convergence speed.



G-S iteration: The effects of the iteration starting boundary are transferred into the entire domain; Convergence speed is accelerated.



Line iteration: The effects of iteration starting boundary and the related two end boundaries are all transferred into the entire domain; convergence speed is further accelerated.



ADI line iteration: In every cycle iteration effects of all the boundaries are transferred into the entire domain. The fastest convergence speed.

ADI line itr.>Line itr.>G-S itr.>Jakob itr.

Jakob iteration has the slowest convergence speed. That is the change between two successive iterations is the smallest; This feature is in favor of convergence of outer iteration for highly non-linear problems when inner iteration cycle number is specified. In the SIMPLEST algorithm, Jakob iteration is used for the convective part of ABEqs.

7.4 Block Correction Method –Promoting Satisfaction of Conservation

7.4.1 Necessity for block correction technique

7.4.2 Basic idea of block correction

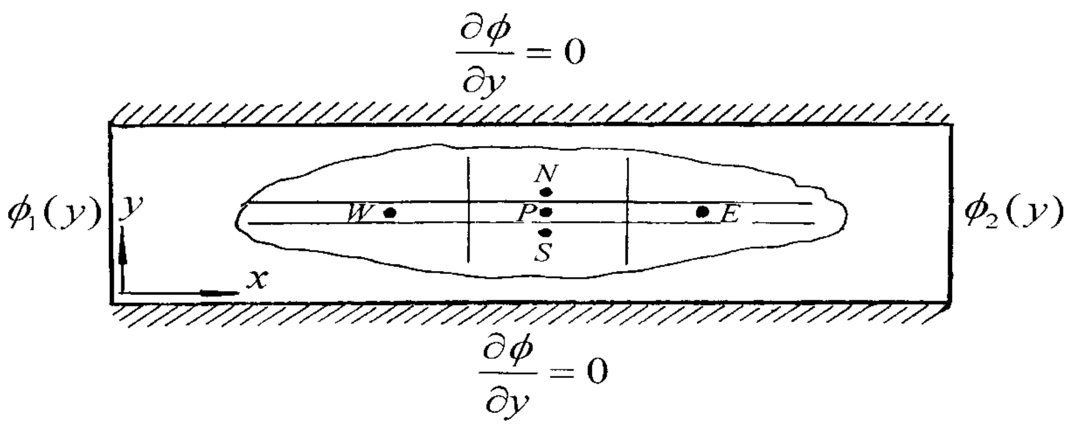
7.4.3 Single block correction and the boundary condition

7.4.4 Remarks of application of B.C. Technique

7.4 Block Correction Method –Promoting Satisfaction of Conservation

7.4.1 Necessity for block correction technique

For 2-D steady heat conduction shown below when ADI is used to solve the ABEqs. convergence speed is very low： EW boundaries have the strongest effect because of 1st kind boundary, but the influencing coefficient is small ； N-S boundary is adiabatic, no definite information can offer, but has larger coefficient — Thus to accelerate convergence of solving ABEqs., a special method is needed.



7.4.2 Basic idea of block correction

Physically, iteration is a process for satisfying conservation condition; In one cycle of iteration, a correction, ϕ', is added to previous solution, ϕ^* (which does not satisfy conservation condition), such that $(\phi^* + \phi')$ can satisfy conservation condition better. The process of solving ABEqs.of ϕ is actually the process of getting ϕ' .

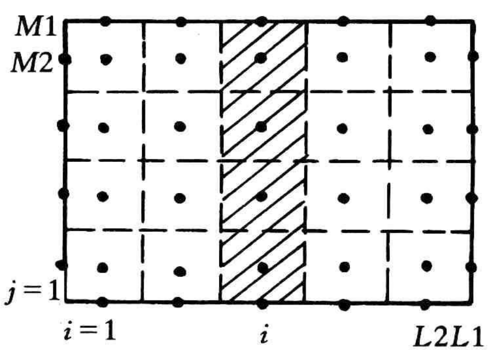
For 2-D problem, corrections are also of 2-D; In order that only 1-D corrections are solved, corrections are somewhat averaged for one block, denoted by $\bar{\phi}'_i$ or $\bar{\phi}'_j$, and it is required that $(\phi^*_{i,j} + \bar{\phi}'_i)$ or $(\phi^*_{i,j} + \bar{\phi}'_j)$ satisfies the conservation condition for the correspondent block.

7.4.3 Single block correction and the boundary condition

1.Equation for correction:

It is required that: $(\phi^*_{i,j} + \bar{\phi}'_i)$ **satisfy following eq.**

$$\sum_j \underline{AP}(\phi^*_{i,j} + \bar{\phi}_i) = \sum_j \underline{AIP}(\phi^*_{i+1,j} + \bar{\phi}_{i+1}) + \sum_j \underline{AIM}(\phi^*_{i-1,j} + \bar{\phi}_{i-1})$$



$$+ \sum_j (\underline{AJM})(\phi^*_{i,j-1} + \bar{\phi}_i)$$

$$+ \sum_j (\underline{AJP})(\phi^*_{i,j+1} + \bar{\phi}_i) + \sum_j \underline{CON}$$

$$(i = IST, \ldots. L2)$$

IST-starting subscript in X-direction; L2-last but one.

Rewrite into ABEqs. of $\overline{\phi}'_{i-1}, \overline{\phi}'_{i}, \overline{\phi}'_{i+1}$ **:**

$$(BL)\underline{\overline{\phi}'_{i}} = (BLP)\underline{\overline{\phi}'_{i+1}} + (BLM)\underline{\overline{\phi}'_{i-1}} + BLC, i = IST, \ldots. L2$$

where

$$BL = \sum_{j=JST}^{M2}(AP) - \sum_{j\neq M2}(AJP) - \sum_{j\neq JST}(AJM)$$

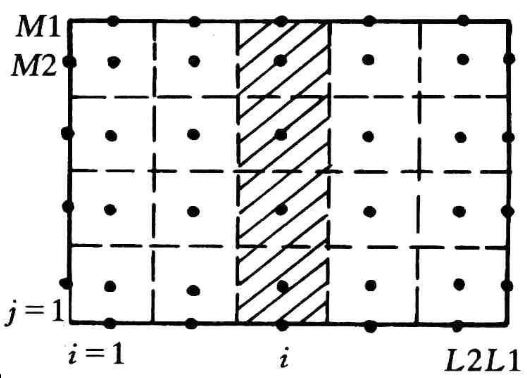
$$BLP = \sum_{j=JST}^{M2}(AIP) \qquad BLM = \sum_{j=JST}^{M2}(AIM)$$

$$BLC = \sum_{j=JST}^{M2} CON + \sum_{j=JST}^{M2}(AJP)\phi^{*}_{i,j+1} + \sum_{j=JST}^{M2}(AJM)\phi^{*}_{i,j-1}$$

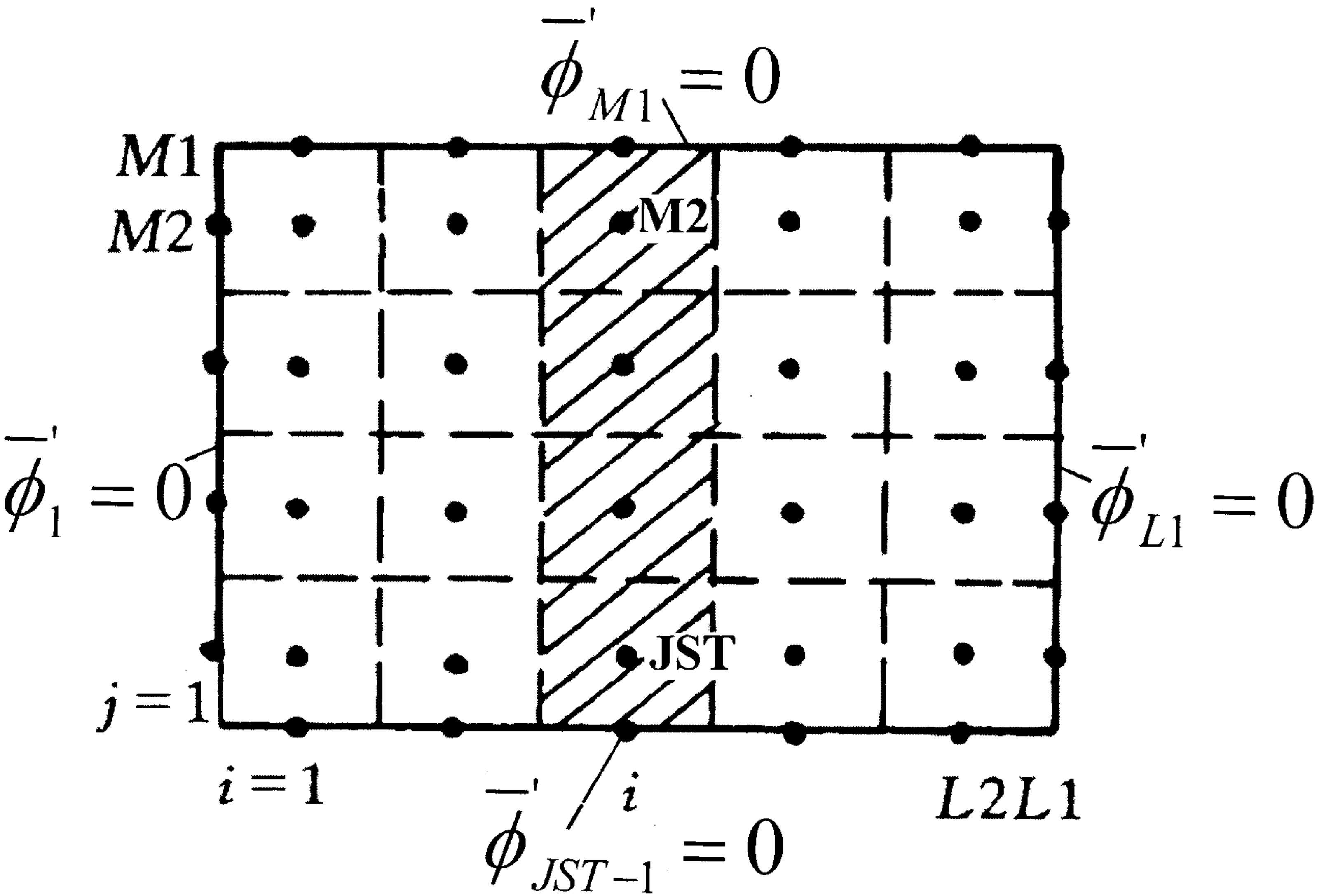
$$+ \sum_{j=JST}^{M2}(AIP)\phi^{*}_{i+1,j} + \sum_{j=JST}^{M2}(AIM)\phi^{*}_{i-1,j} - \sum_{j=JST}^{M2}(AP)\phi^{*}_{i,j}$$

$$BL = \sum_{j=JST}^{M2}(AP) - \sum_{j \neq M2}(AJP) - \sum_{j \neq JST}(AJM)$$

ASTM is adopted to deal with 2nd and 3rd kind boundary condition， this is equivalent to that all boundaries are of 1st kind, and the correction for boundary nodes is zero； Thus when summation is conducted in y-direction the 1st term and the last term corrections are zero. Hence, for AJM term JST is not needed， and for AJP M2 is not needed.



2. Boundary condition for the correction ---zero

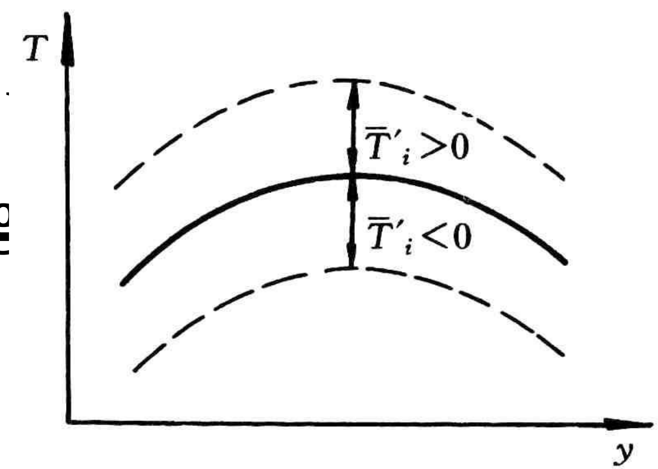


7.4.4 Remarks of application of B.C. Technique

1.BCT is not an independent solution method. It should be combined with some other method, such as ADI.;

2. For further accelerating convergence ADI block correction may be used.;

3. For variables of physically larger than zero values the B.C.T. may not used (such as turbulent kinetic energ component of a mixed gas). Because B.C.T. adds or subtracts a constant correction within the entire block, which may lead to minus values.



7.5 Multigrid Techniques –Promoting Simultaneous Attenuation of Different Wave-length Components

7.5.1 Error vector is attenuated(衰减) in the iteration process of solving ABEqs.

7.5.2 Basic idea and key issue of multigrid technique

7.5.3 Transferring solutions between different grid systems

7.5.4 Cycling patterns between different grid systems

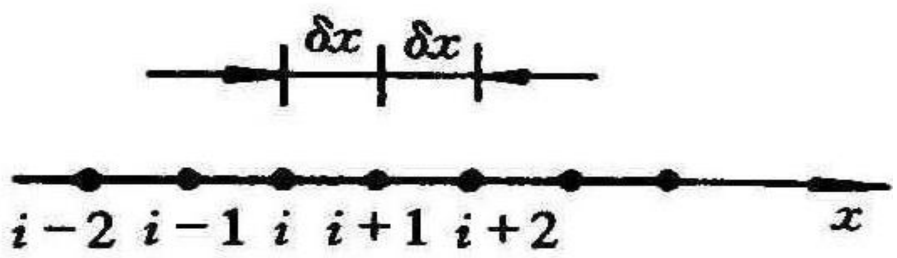
7.5 Multigrid Techniques –Promoting Simultaneous Attenuation of Different Wave-length Components

7.5.1 Error vector is attenuated in the iteration process of solving ABEqs

1. How error vector is attenuated during iteration?

von Neumann method is adopted to analyze the attenuation of error vector. For 1-D steady heat conduction problem:

$$\frac{d^2T}{dx^2} + f(x) = 0$$



Discretized at a uniform grid system, yielding:

$$T_{i-1} - 2T_i + T_{i+1} = -(\delta x)^2 f_i$$

Adopting G-S iteration method from left to right：

$$T_{i-1}^{(k)} - 2T_i^{(k)} + T_{i+1}^{(k-1)} = -(\delta x)^2 f_i$$

In the kth cycle iteration error vector is denoted by, $\vec{\varepsilon}^{(k)}$ **and its component is denoted by** $\varepsilon_i^{(k)}$ **, then we have:**

$$T_i = T_i^{(k)} + \varepsilon_i^{(k)}$$

According to the discussion in Chapter 3, variation of error with iteration is as follows

$$\varepsilon_{i-1}^{(k)} - 2\varepsilon_i^{(k)} + \varepsilon_{i+1}^{(k-1)} = 0$$

2. Analysis of attenuation of harmonic components

Expressing $\varepsilon_i^{(k)}$ **as:** $\psi(k)e^{Ii\theta}$ **and substituting，yielding**

$$\frac{\psi(k)}{\psi(k-1)} = \frac{e^{I\theta}}{2-e^{-I\theta}} = \mu$$

Amplifying factor (增长因子)

Analyzing amplifying factor for different phase angles:

$\theta = \pi,$

$$|\mu| = \frac{|\cos\pi + I\overset{0}{\sin\pi}|}{|2-\cos\pi + I\overset{0}{\sin\pi}|} = \frac{1}{2+1} = \frac{1}{3}, \boxed{\text{Ite.5 times}} \quad 0.333^5 = 4.09\times10^{-3}$$

$\theta = \pi/2,$

$$|\mu| = \frac{\left|\overset{0}{\cos\frac{\pi}{2}} + I\sin\frac{\pi}{2}\right|}{\left|2-\overset{0}{\cos\frac{\pi}{2}} + I\sin\frac{\pi}{2}\right|} = \frac{1}{\sqrt{2^2+1}} = \frac{1}{\sqrt{5}}, \boxed{\text{Ite5 times}} \quad 0.447^5 = 0.0178$$

$\theta = \pi/10,$

$$|\mu| = \frac{\left|\cos\frac{\pi}{10} + I\sin\frac{\pi}{10}\right|}{\left|2-\cos\frac{\pi}{10} + I\sin\frac{\pi}{10}\right|} = \frac{|0.9510+0.3090I|}{|2-(0.9510+0.3090I)|} = \frac{1}{1.094}, \boxed{\text{Ite.5 times}} \quad 0.914^5 = 0.658$$

According to von Neumann method: $\theta = k_x \Delta x = \dfrac{2\pi}{\lambda}\Delta x$

At a fixed space step, short wave has a larger phase angle, and is attenuated very fast; while long wave component has small phase angle and converges very slowly.

From above calculation phase angle can be an indicator for short/long wave components.

Generally components with phase angle within following range are regarded as short wave ones:

$$\pi / 2 \le \theta \le \pi$$

This phase angle is dependent on space step length $(\theta = k_x \Delta x = \frac{2\pi}{\lambda}\Delta x)$. If after several iterations the length step is amplified then originally long wave component may be behaved as a short wave and can be attenuated very fast at that grid system.

In such a way by amplifying space step length several times the entire ABEqs. may be converged much faster than that iteration is just conducted at a single grid system.

This is the major idea of multigrid technique for solving ABEqs.

7.5.2 Major idea and key issue of multigrid technique

1. Major idea－Solving ABEqs. is conducted at several grid systems with different step length such that error components with different frequencies can be attenuated simultaneously.

2. Key issues－

(1) How to transfer solutions at different grid systems?

(2) How to cycle the solutions between several grid systems?(怎样在不同网格间做循环?）

7.5.3 Transferring solutions between two gird systems

Basic concept－solution transferred between different grid system is the one of the finest grid.

Taking two grid systems, one coarse and one fine, as an example to show the transferring of solutions.

1.From fine grid to coarse grid

$$\vec{A}^{(k-1)}\phi^{(k-1)}=\vec{b}^{(k-1)}+I_k^{k-1}(\underline{\vec{b}^{(k)}-\vec{A}^{(k)}\phi^{(k)}})$$

Residual of fine grid

Operator for transferring form kth grid to (k-1)th grid

Matrix at (k-1)th grid determined from solution of kth grid.

Source term at (k-1)th grid determined from solution of kth grid

Solution of the fine grid solved at (k-1)th grid

2. Transferring from coarse grid to fine grid

$$\phi_{rev}^{(k)} = \phi_{old}^{k} + I_{k-1}^{k}(\phi^{(k-1)} - I_{k}^{k-1}\phi_{old}^{(k)})$$

Improved solution at fine grid

Original solution at fine grid

Operator for transferring correction part of solution at (k-1) th grid to kth grid

New solution of fine grid obtained at (k-1)th grid

Solution of kth grid expressed at (k-1)th grid

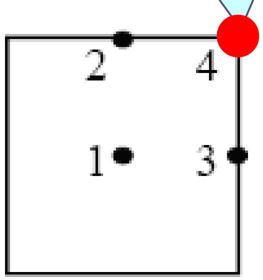
3. Restriction and prolongation operators

1) Restriction operator(**限定算子**) I_k^{k-1}

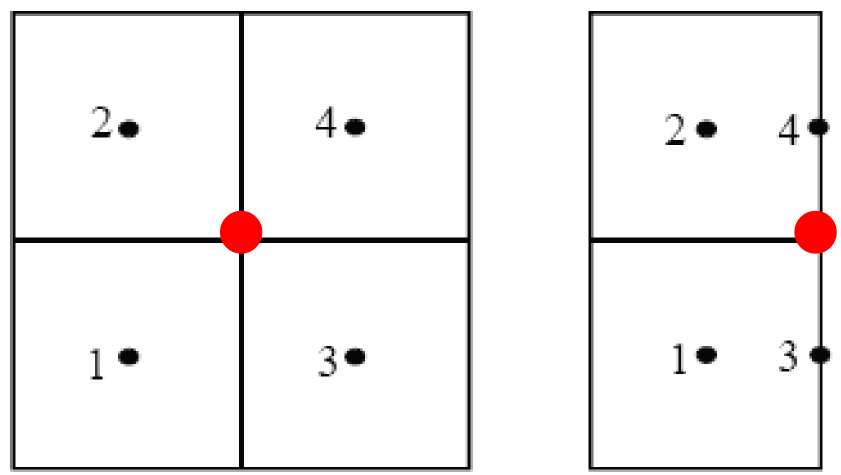
- **Direct injection(直接注入)**
- **Nearby average(就近平均)**, Linear interpolation

(**From fine to course**)

For node 4
direct injection



Near average

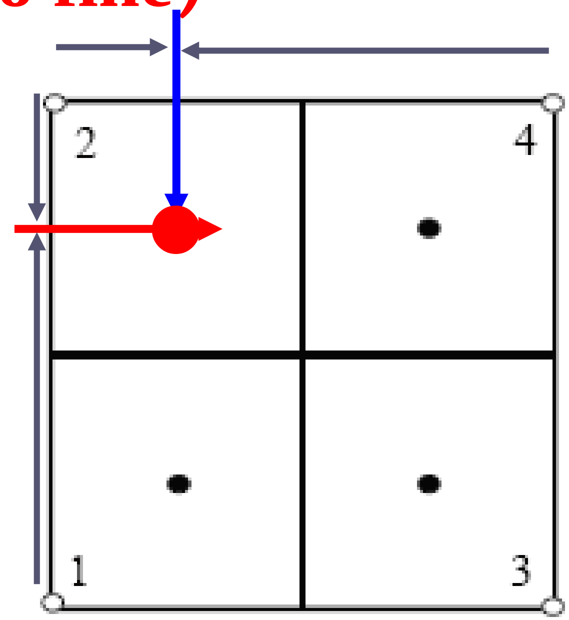


● _fine → □ course

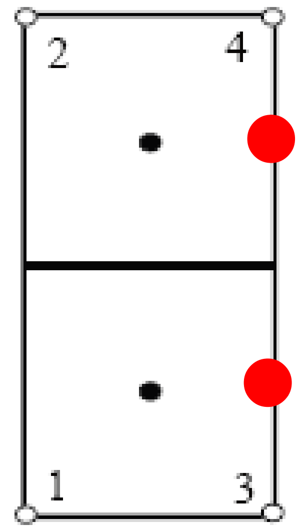
2) Prologation operator (延拓算子) I_{k-1}^{k}

- Direct injection
- linear interpolation
- Quadratic interpolation(二次插值)

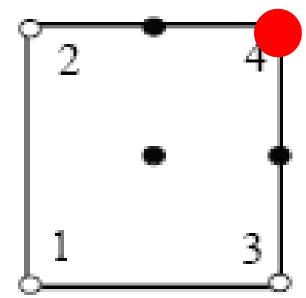
(From course to fine)



Quadratic interpolation



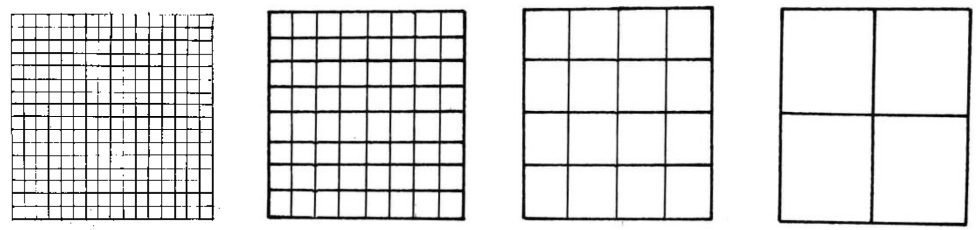
Linear interpolation between nodes 3，4



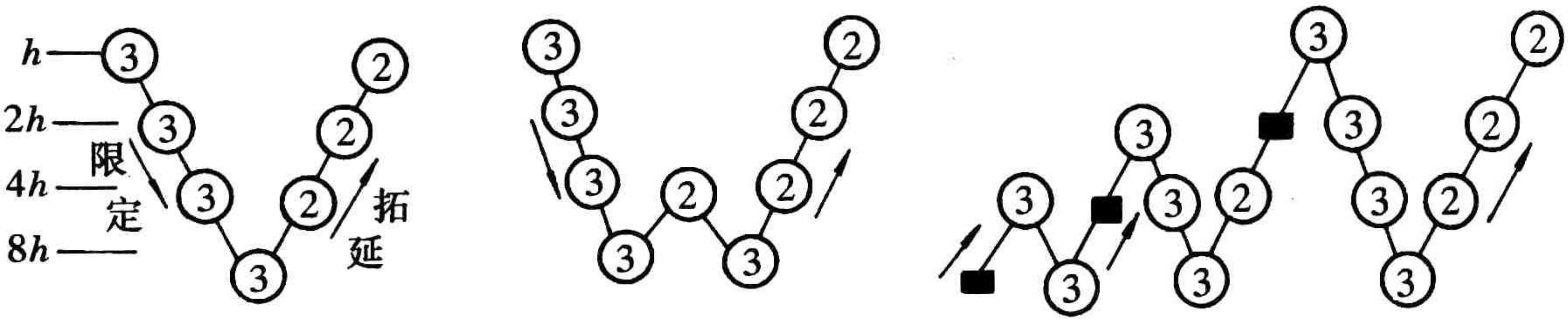
Node 4-Direct injection

□ Course → ● —Fine

7.5.4 Cycling method between several grids



Three cycling patterns:



(a) **V-cycle** (b) **W -cycle** (c) **FMG-cycle**

Number in the circle shows times of iteration. FMG cycle is widely adopted in fluid flow and heat transfer problems. Black symbol represents converged solution.

Home work (pp.294-295)

7－1　7－3　7－4　7－6

Due in November 13

Problem # 7-1

Try to calculate and prove that the following equations are convergent for Jacobi iterative method, whereas are divergent for GS point iterative method.

$$x_1 + 2x_2 - 2x_3 = 1$$

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 + 2x_2 + x_3 = 5$$

Problem# 7-3 Solve following algebraic equations

$$4x_1 + x_2 = -1$$

$$x_1 + 6x_2 + 2x_3 = 0$$

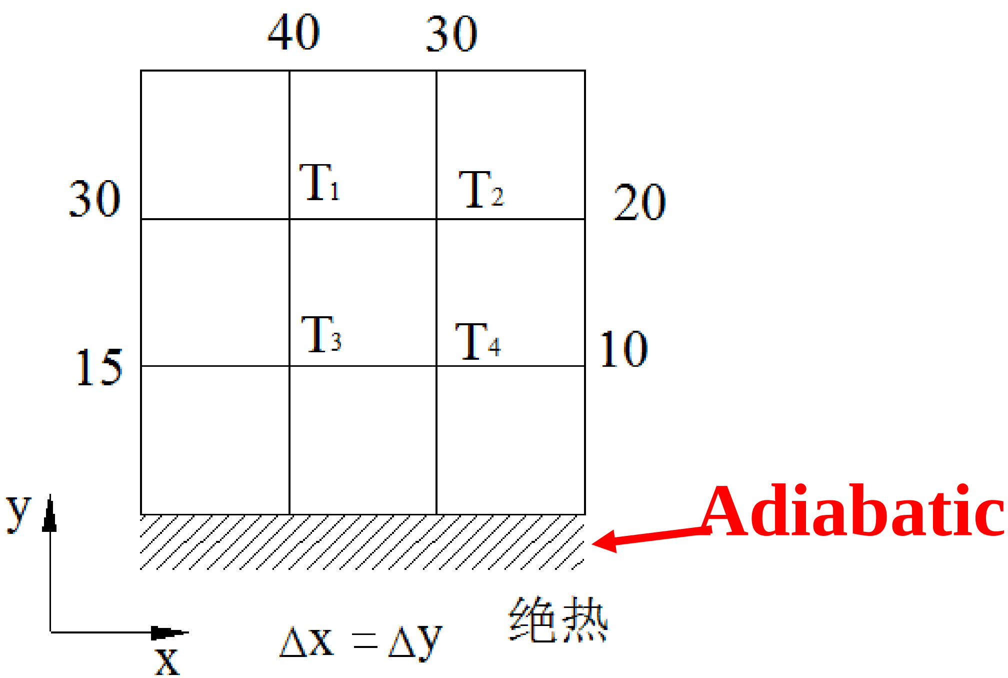
$$2x_2 + 4x_3 = 0$$

by Jakob point ieration, G-S point iteration and SOR point iteration (alfa=1.8), respectively and compare their convergece speed.

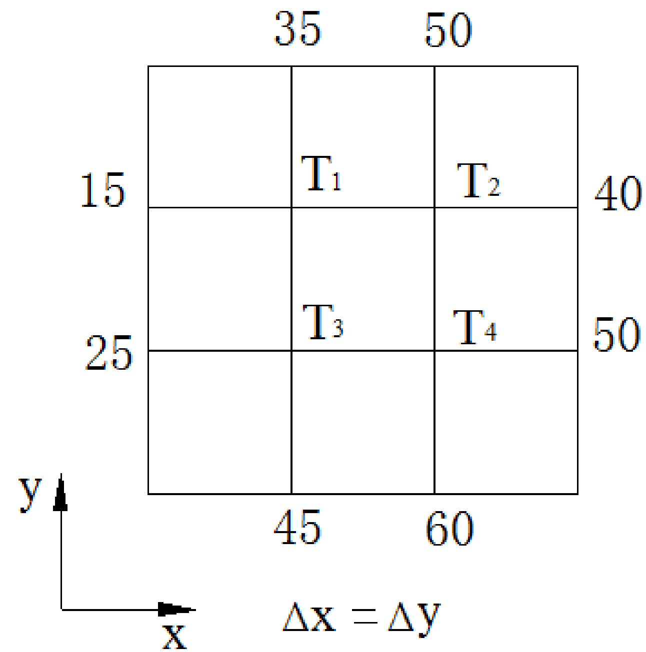
Problem # 7-4

Bottom of a square object is thermally insulated, and the temperature of remaining three sides is shown in figure given below. Determine the temperature inside the square nodes 1,2,3,4. Thermal properties of the object are constants, and without internal heat source.

(See figure in the next page)



Problem 7-4



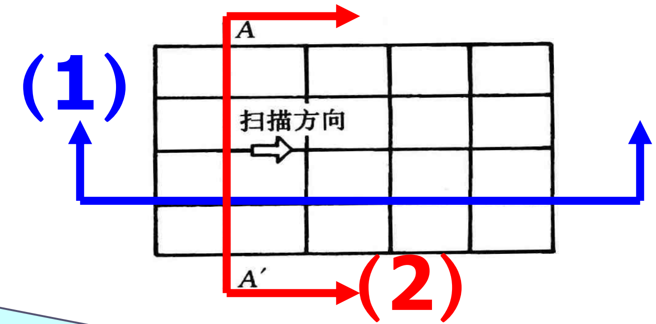
Problem 7-6

Problem # 7-6

A physical square, as shown in figure above, is the steady state heat conduction problem. Calculate the temperature of internal nodes 1,2,3,4 using the GS point iterative method and linear iterative method and compare their convergence rate. Also compare the results with example 1 and explain the observed facts.

Appendix of 8-2-3 (ADI for iteration vs. ADI for implicit)

ADI-Jakob iteration can be expressed as：：



$b^{(k+1/2)}$

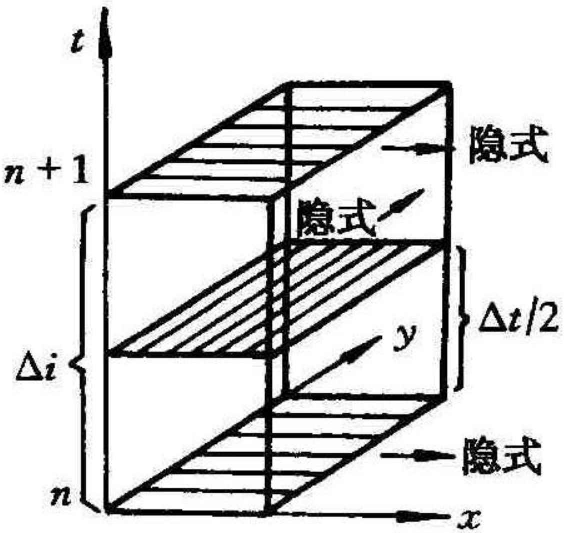
$$a_P\phi_P^{(k+1/2)} = a_E\phi_E^{(k+1/2)} + a_W\phi_W^{(k+1/2)} + \underline{[a_N\phi_N^{(k)} + a_S\phi_S^{(k)} + b]}$$

$$a_P\phi_P^{(k+1)} = a_N\phi_N^{(k+1)} + a_S\phi_S^{(k+1)} + \underline{[a_E\phi_E^{(k+1/2)} + a_W\phi_W^{(k+1/2)} + b]}$$

$b^{(k+1)}$

This expression is very similar to Peaceman-Rachford ADImplicit method for transient problem：

2-D Peaceman-Rachford method



2－DADImplicit

Dividing Δt into two sub-periods.

In the 1st sub-period $\Delta t/2$ x-direction is implicit, y-direction is explicit;

In the 2nd $\Delta t/2$ y-direction is implicit, and x is explicit.

Let $\phi^{(k+1/2)}$ represent temporary values at middle time

$\delta_x^2\phi_{i,j}^k$ represent CD for 2nd-order x-direction derivative at time level k ; then we have:

1st sub-period:

$$\frac{\phi_{i,j}^{k+1/2}-\phi_{i,j}^{k}}{\Delta t/2}=a(\delta_x^2\phi_{i,j}^{k+1/2}+\delta_y^2\phi_{i,j}^{k}) \quad \textbf{(1)}$$

2nd sub-period:

$$\frac{\phi_{i,j}^{k+1}-\phi_{i,j}^{k+1/2}}{\Delta t/2}=a(\delta_x^2\phi_{i,j}^{k+1/2}+\delta_y^2\phi_{i,j}^{k+1}) \quad \textbf{(2)}$$

Rewrite Eq.(1):

$$\underbrace{(1+\frac{a\Delta t}{\Delta x^2})}_{a_P}\phi_{i,j}^{k+1/2}=\underbrace{(\frac{a\Delta t}{2\Delta x^2})}_{a_E,a_W}(\phi_{i+1,j}^{k+1/2}+\phi_{i-1,j}^{k+1/2})+\underbrace{(\frac{a\Delta t}{2\Delta x^2})(\phi_{i,j+1}^{k}+\phi_{i,j-1}^{k})+(1-\frac{a\Delta t}{\Delta x^2})\phi_{i,j}^{k}}_{\boxed{b^{k+1/2}}}$$

$$a_S,a_N \qquad b$$

Thus one-time step forward of transient problem is equivalent to one cycle iteration for steady problem.

同舟共济
渡彼岸！

People in the same boat help each other to cross to the other bank, where….