# Numerical Heat Transfer
## (数值传热学)
### Chapter 4 Discretized Schemes of Diffusion and Convection Equation (2)



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# Chapter 4 Discretized diffusion-convection  equation

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# 4.5 Discussion on false diffusion

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# 4.5 Discussion on false diffusion

## 4.5.1 Meaning and reasons of false diffusion

False diffusion (假扩散), also called numerical viscosity (数值黏性)，is an important numerical character of discretized convective scheme.

## 1. Original meaning

Numerical errors caused by discretized scheme with $1^{st}$ order accuracy is called false diffusion；

The $1^{st}$ term in the TE of such scheme contains $2^{nd}$ order derivative, thus the diffusion action is somewhat magnified at the sense of second-order accuracy, hence the numerical error is called "false diffusion".

Taking 1-D unsteady advection eq. as an example. The two 1st-order derivatives are discretized by 1st-order accuracy schemes.

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} \quad \xrightarrow[\quad u>0 \quad]{\text{1st-oder scheme}} \quad \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -u\frac{\phi_i^n - \phi_{i-1}^n}{\Delta x}$$

Expanding $\phi_{i-1}^n, \phi_i^{n+1}$ at $(i,n)$ by Taylor series，and substituting into the above equation：

$$\frac{\phi_i^n + \Delta t\frac{\partial \phi}{\partial t})_{i,n} + \frac{1}{2}\Delta t^2\frac{\partial^2 \phi}{\partial t^2})_{i,n} + \frac{1}{6}\Delta t^3\frac{\partial^3 \phi}{\partial t^3})_{i,n}... - \phi_i^n}{\Delta t} =$$

$$-u\frac{\phi_i^n - [\phi_i^n - \Delta x\frac{\partial \phi}{\partial x})_{i,n} + \frac{1}{2}\Delta x^2\frac{\partial^2 \phi}{\partial x^2})_{i,n} - \frac{1}{6}\Delta x^3\frac{\partial^3 \phi}{\partial x^3})_{i,n}...}{\Delta x}$$

$$\frac{\partial \phi}{\partial t})_{i,n} = -u \frac{\partial \phi}{\partial x})_{i,n} - \frac{\Delta t}{2} \frac{\partial^2 \phi}{\partial t^2})_{i,n} + \frac{u \Delta x}{2} \frac{\partial^2 \phi}{\partial x^2})_{i,n} + O(\Delta x^2, \Delta t^2)$$

where the transient 2$^{nd}$ derivative can be re-written as follows:

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}(\frac{\partial \phi}{\partial t}) \cong \frac{\partial}{\partial t}(-u \frac{\partial \phi}{\partial x}) = -u \frac{\partial}{\partial x}(\frac{\partial \phi}{\partial t}) \cong -u \frac{\partial}{\partial x}(-u \frac{\partial \phi}{\partial x}) = u^2 \frac{\partial^2 \phi}{\partial x^2}$$

substituting into above equation

$$\frac{\partial \phi}{\partial t})_{i,n} = -u \frac{\partial \phi}{\partial x})_{i,n} + [\frac{u \Delta x}{2}(1 - \frac{u \Delta t}{\Delta x})](\frac{\partial^2 \phi}{\partial x^2})_{i,n} + O(\Delta x^2, \Delta t^2)$$

Thus at the sense of 2$^{nd}$-order accuracy above discretized equation simulates a convective-diffusive process , rather than an advection process（平流，纯对流）.

Only when $1 - \dfrac{u\Delta t}{\Delta x} = 0$ this error disappears.

$\dfrac{u\Delta t}{\Delta x}$ is called **Courant** number，in memory of a German mathematician Courant.

$$\frac{\partial \phi}{\partial t}\big)_{i,n} = -u\frac{\partial \phi}{\partial x}\big)_{i,n} + [\frac{u\Delta x}{2}(1-\frac{u\Delta t}{\Delta x})](\frac{\partial^2 \phi}{\partial x^2})_{i,n} + O(\Delta x^2, \Delta t^2)$$

**Remark：** We only study the false diffusion at the sense of 2$^{nd}$-order accuracy；i.e., if inspecting at the 2nd-order accuracy the above discretized equation actually simulates a convection-diffusion process. For most engineering problems 2$^{nd}$-oder accuracy solutions are satisfied.

## 2. Extended meaning

In most existing literatures almost all numerical errors are called false diffusion，which includes：

(1) 1st-order accuracy schemes of the 1st order derivatives (original meaning)；

(2) Oblique intersection(倾斜交叉) of flow direction with grid lines；

(3) The effects of non-constant source term which are not considered in the discretized schemes.

## 4.5.2 Examples caused by 1st-order accuracy schemes

## 1. 1-D steady convection-diffusion problem

When convection term is discretized by FUD，diffusion term by CD, numerical solutions will severely deviate from analytical solutions:
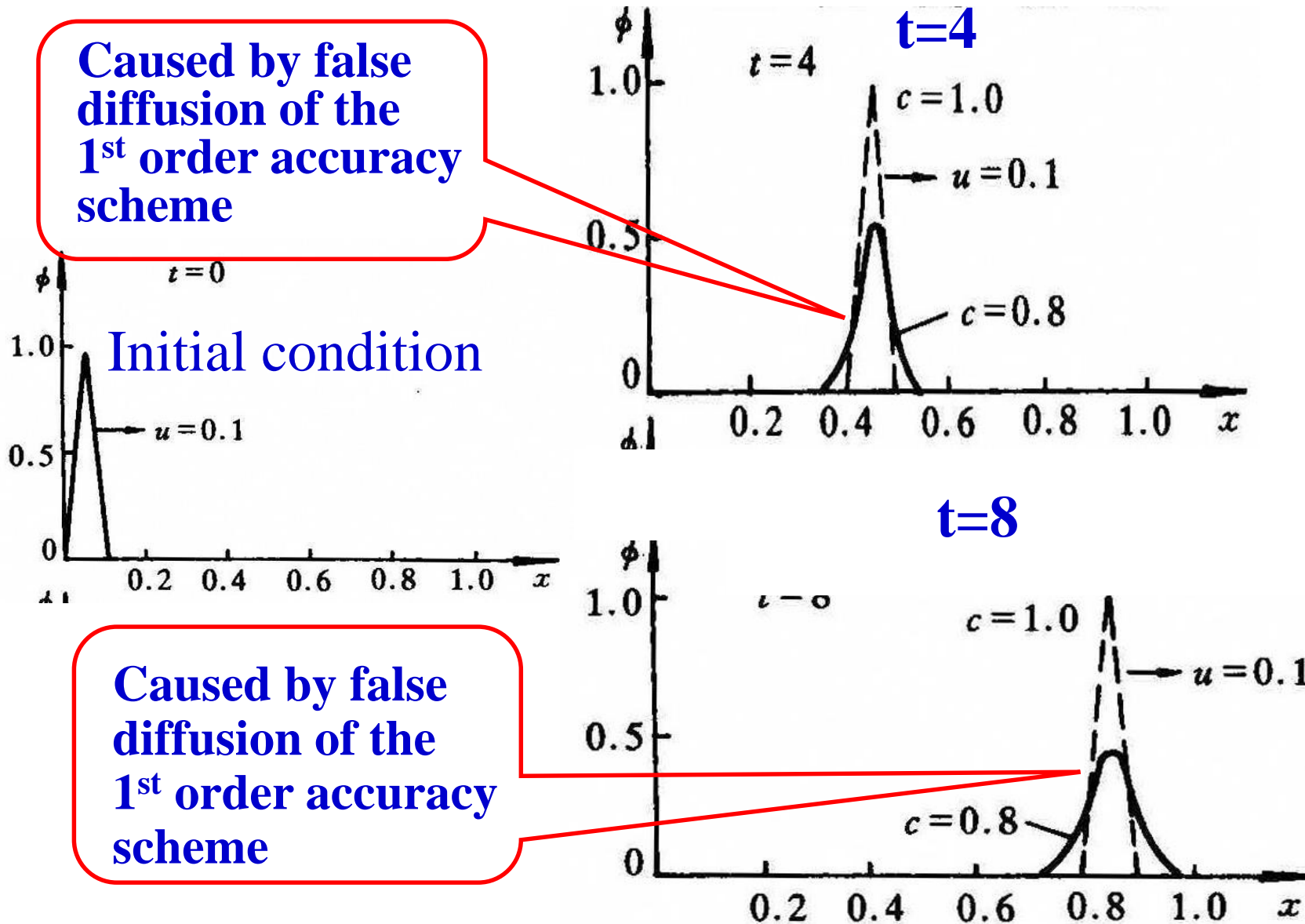
FUD: Physically plausible solution

FUD: severe error

CD: oscillating solution
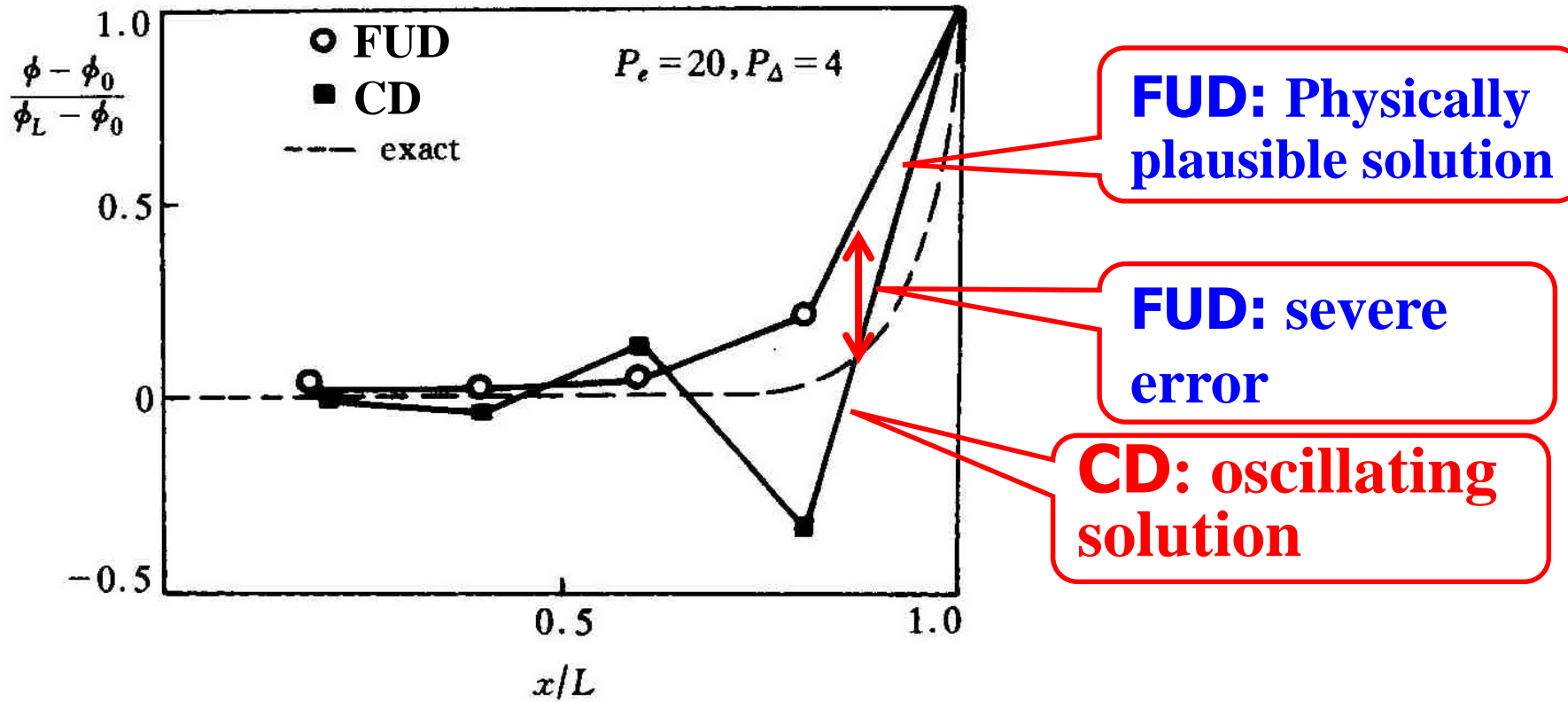
## 2. 1-D unsteady advection problem (Noye,1976)

$$\frac{\partial \phi}{\partial t} = -u \frac{\partial \phi}{\partial x}, \ 0 \leq x \leq 1, u = 0.1, \quad \phi(0,t) = \phi(1,t) = 0$$

In the range of $x \in [0,0.1]$ initial distribution is an triangle，others are zero. The two derivatives are discretized

the 1ˢᵗ –order accuracy schemes.  The results are as follows.

**t=4**

Caused by false diffusion of the 1ˢᵗ order accuracy scheme

Initial condition

**t=8**

Caused by false diffusion of the 1ˢᵗ order accuracy scheme

When Courant number is less than 1 ,severe error occurs，which erases （抹平）the sharp peak(抹平尖峰) and magnify the base (放大基底) gradually. Such error is called streamwise false diffusion(流向假扩散).

## 4.5.3 Errors caused by oblique intersection (倾斜交叉)

Two gas streams with different tempera-tures meet each other. Assuming zero gas dif-fusivities. If the flow direction is obliquely with respect to the grid lines, big numerical errors will be introduced.

$\Gamma = 0$  $\Gamma \neq 0$
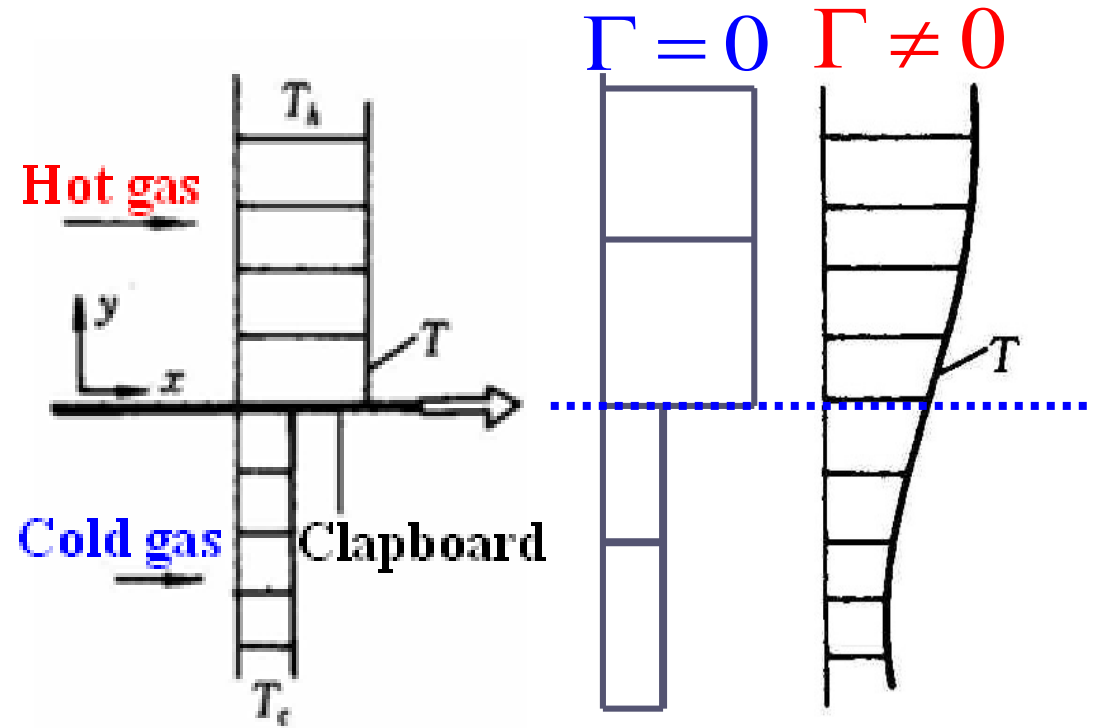


Gas flow with 0 and non-0 Gamma

# 1. Case 1: with x-y coordinates either parallel or perpendicular to flow direction

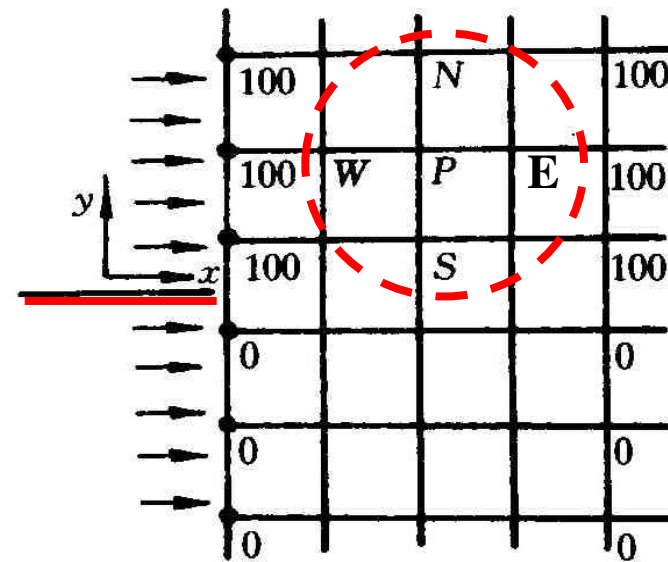Adopting FUD, then $A(|P_\Delta|) = 1$ ; For CV. P:

$$a_E = D_e + [\![ -F, 0 ]\!] \qquad \underset{\xrightarrow{\hspace{3cm}}}{U > 0, \Gamma = 0} \; 0$$

$$a_W = D_W + [\![ F_W, 0 ]\!] \qquad \underset{\xrightarrow{\hspace{3cm}}}{U > 0, \Gamma = 0} \; F_W$$

$$a_N = D_n + [\![ -F, 0 ]\!] \qquad \underset{\xrightarrow{\hspace{3cm}}}{V = 0, \Gamma = 0} \; 0$$

$$a_S = D_s + [\![ F, 0 ]\!] \qquad \underset{\xrightarrow{\hspace{3cm}}}{V = 0, \Gamma = 0} \; 0 \qquad \text{Upstream velocity} \; U$$

Thus we have: $a_P = a_E + a_W = a_W \qquad \phi_P = \phi_W \; !$

The upstream temperature is kept downstream!

## 2. Case 2: x-y coordinates intersect the on coming flow with 45 degree
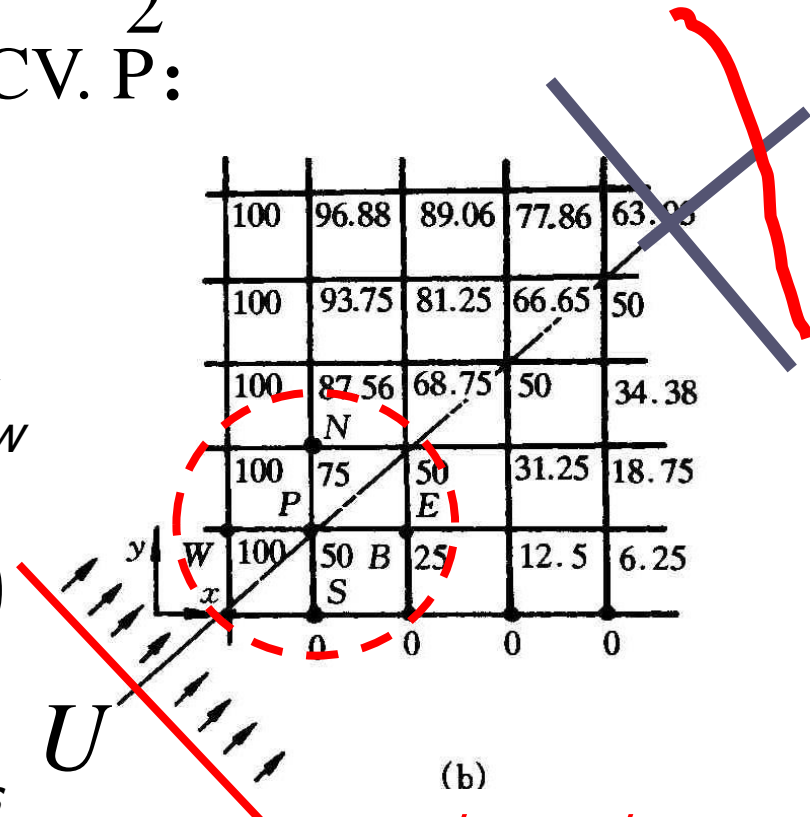
From upstream velocity $U$, $u = v = \dfrac{\sqrt{2}}{2} U$,

Again FUD is adopted, then for CV. P:

$$a_E = D_e + [\![ -F, 0 ]\!] \quad \xrightarrow{u>0, \Gamma=0} \quad 0$$

$$a_W = D_W + [\![ F_W, 0 ]\!] \quad \xrightarrow{u>0, \Gamma=0} \quad F_W$$

$$a_N = D_n + [\![ -F, 0 ]\!] \quad \xrightarrow{v>0, \Gamma=0} \quad 0$$

$$a_S = D_s + [\![ F, 0 ]\!] \quad \xrightarrow{v>0, \Gamma=0} \quad F_S$$

| 100 | 96.88 | 89.06 | 77.86 | 63. |
|-----|-------|-------|-------|------|
| 100 | 93.75 | 81.25 | 66.65 | 50 |
| 100 | 87.56 | 68.75 | 50 | 34.38 |
| 100 | 75 | 50 | 31.25 | 18.75 |
| 100 | 50 | 25 | 12.5 | 6.25 |

(b)

$$F_w = F_s, \quad a_P \phi_P = a_W \phi_W + a_S \phi_S, \quad a_P = a_W + a_S, \quad \phi_P = \frac{\phi_W + \phi_S}{2}!$$

Fluid temperatures across the diagonal become smooth and continuous. This is caused by the **cross-diffusion.**

**Discussion**：For case 1 where velocity is parallel to x coordinate，the FUD scheme also produces false diffusion, but compared with convection it can not be exhibited(展现): the zero diffusivity corresponds to an extremely large Peclet number，i.e., convection is so strong that false diffusion can not be exhibited. When chances come (有机会时) it will take action. Example 1 of this section is such a situation.

**4.5.4 Errors caused by non-constant source term**

Given：$\begin{cases} \dfrac{d(\rho u\phi)}{dx} = \dfrac{d}{dx}(\Gamma\dfrac{d\phi}{dx}) + S, \\ x = 0, \phi = \phi_0 ; x = L, \phi = \phi_L \end{cases}$   $S$ non-constant, distribuiton is specified.

For cases with such non-constant source term neither one of the five 3-point schemes can get accurate solution.

Taking hybrid scheme as an example. When grid Peclet number is less than 2，numerical results agree with analytical solution quite well; However, when grid Peclet number is larger than 2 ,deviations become large. Its coefficient is defined by:

$$a_E = D_e A(|P_{\Delta e}|) + [\![ -F, 0 \ , \ A(|P_{\Delta e}|) = [\![ 0, 1 - 0.5|P_{\Delta e}| ]\!]$$

Assuming that variation of Peclet number is implemented via changing diffusion coefficient while flow rate is remained unchanged then when

$P_{\Delta e} \geq 2$, hybrid: $A(|P_{\Delta e}|) = \left[\kern-0.15em\left[ 0, 1 - 0.5|P_{\Delta e}| \right]\kern-0.15em\right] = 0$ thus $a_E = 0$ leading to the same numerical solutions for all cases with $P_{\Delta e} \geq 2$.

Analytical solutions for grid Peclect number larger than 2

Numerical solutions for grid Peclet number equal and larger than 2



Given source term
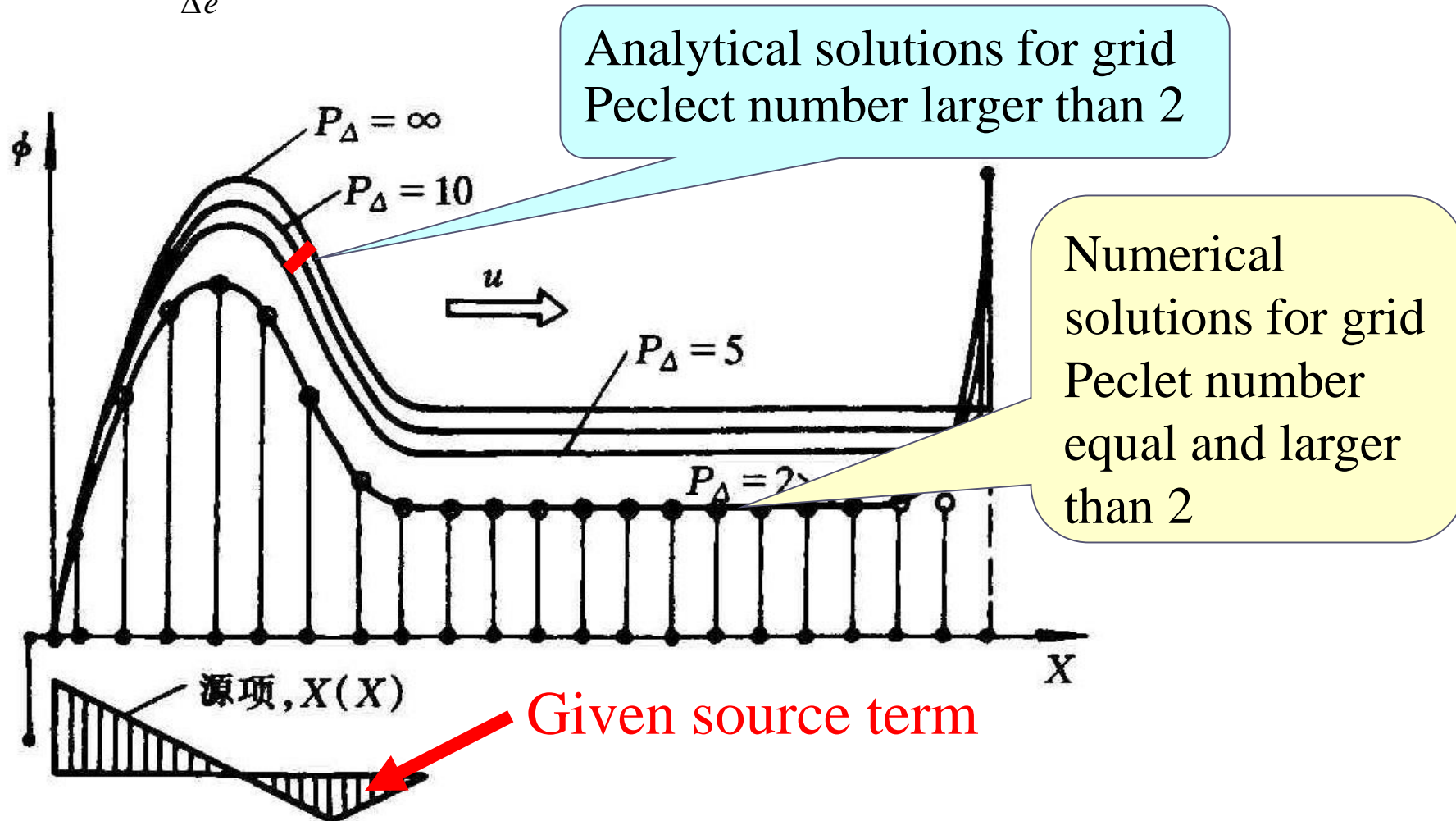
## 4.5.5 Two famous examples

## 1. Smith-Hutton problems（1982）

Solution for temp. distribution with a known flow field
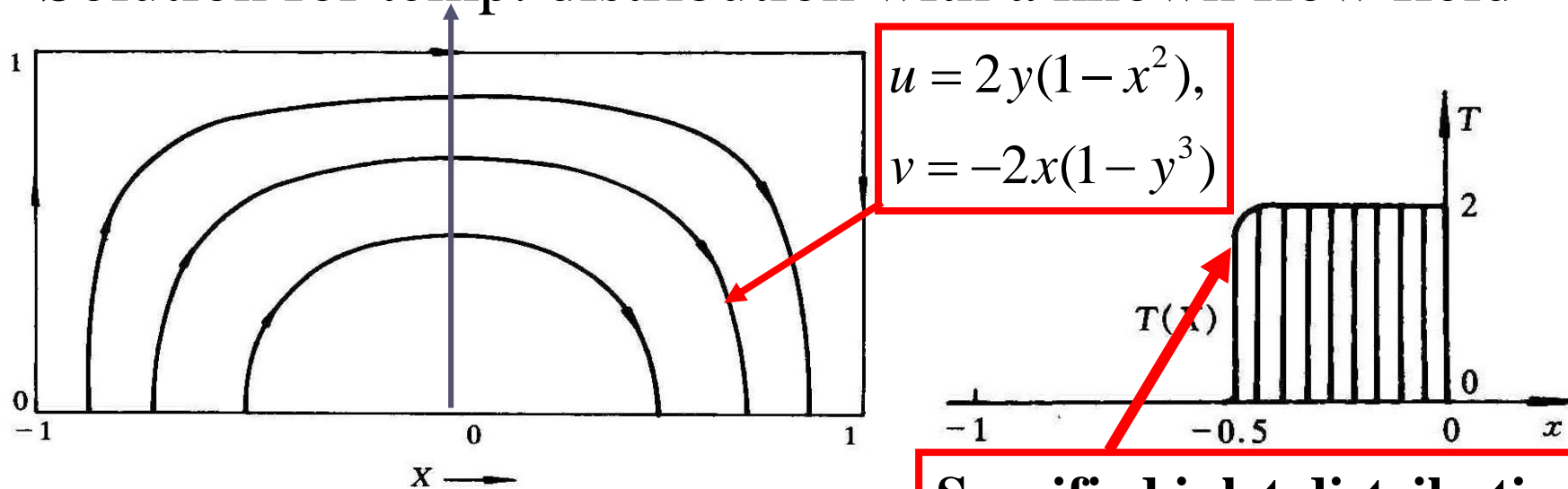
$$u = 2y(1-x^2),$$
$$v = -2x(1-y^3)$$

**Known flow field**

**Specified inlet distribution**

$$T_{in}(x) = 1 + \tanh[\alpha(1+2x)]$$

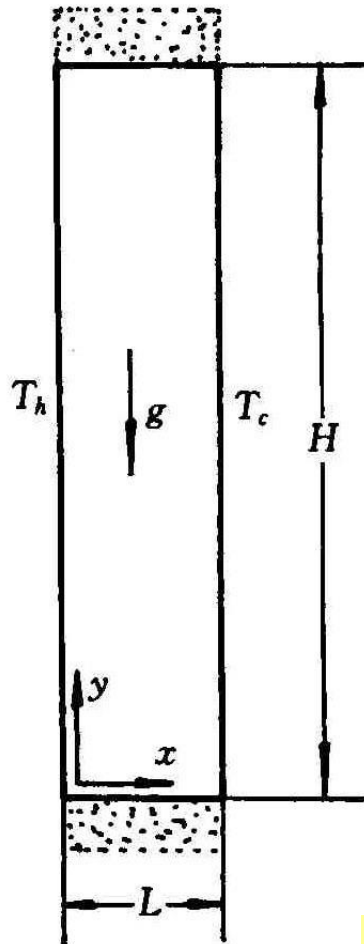The larger the coefficient $\alpha$ the sharper the profile.

Solved by 2-D D-C eq., convection term is discretized by the scheme studied.

**Reference solution**

**Solution by QUICK with 20X10 is close the one by Power law with 80X40**

**Power law**    **x**

**QUICK**

REF
80 *by* 40
40 *by* 20
20 *by* 10
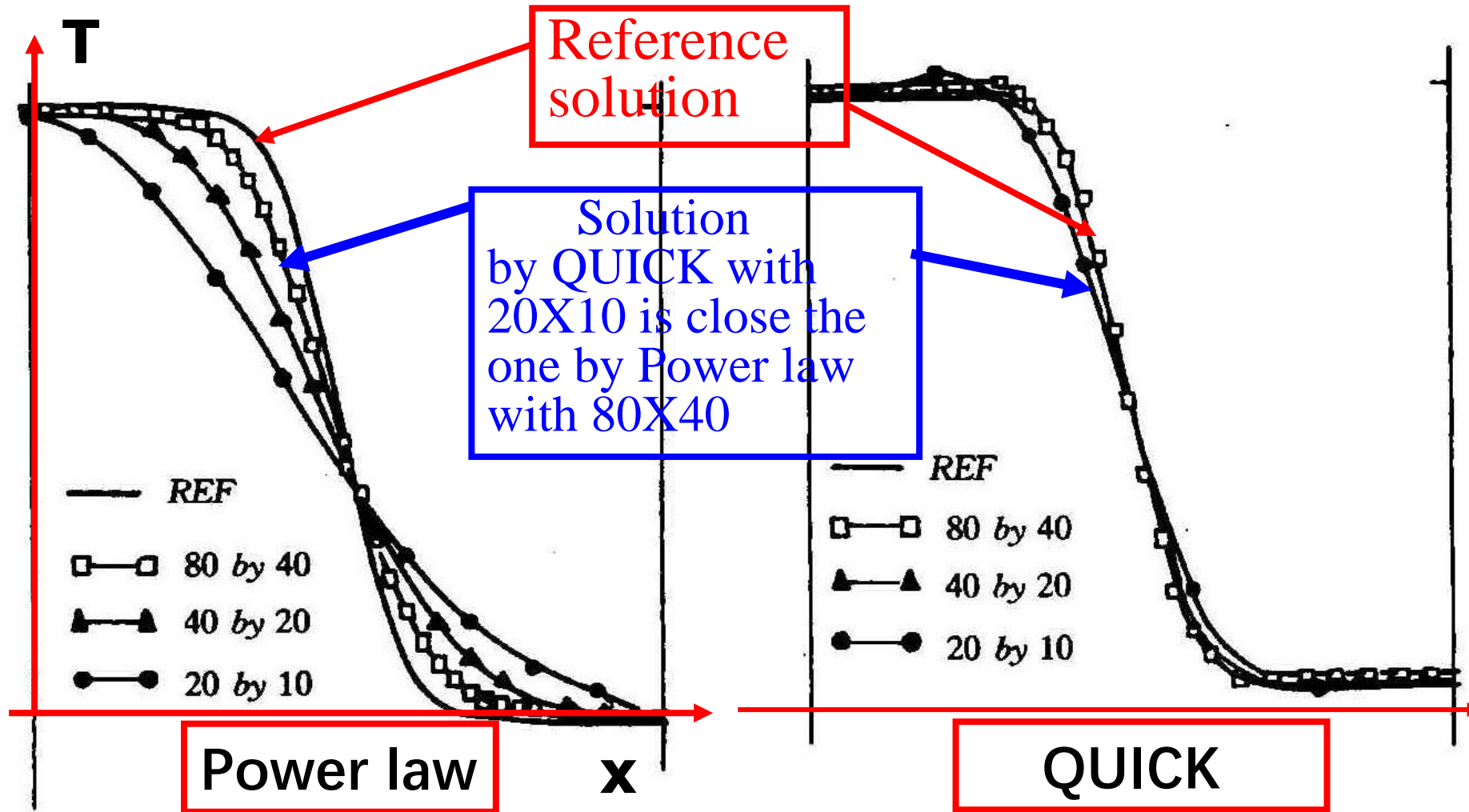
Solution from QUICK by 20X10 grids has the same accuracy as that from power law by 80X40 grids.

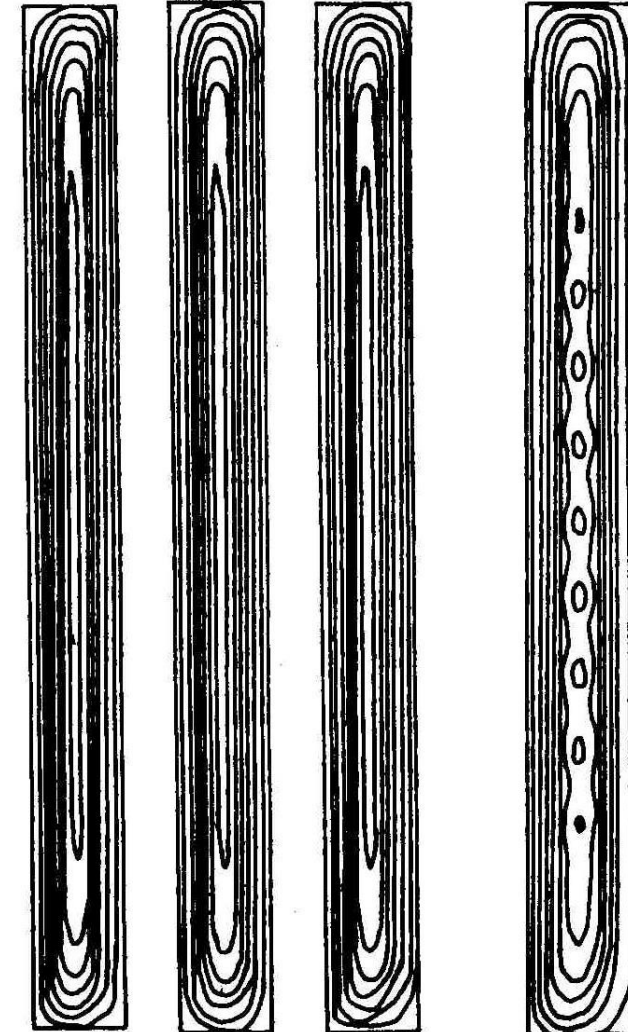## 2) Leonard problem (1996)

Natural convection in a tall cavity

$H / L = 33$
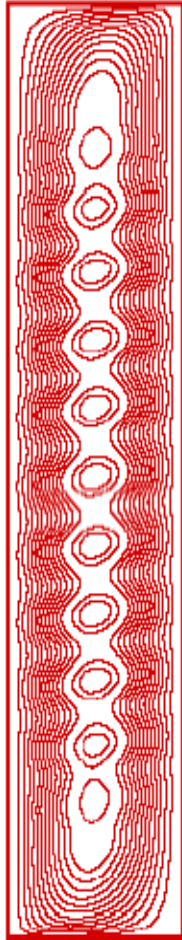
$$Gr = \frac{gL^3\alpha\Delta T}{v^2}$$

$$= 9500,$$

$$\mathrm{Pr} = 0.71$$

$$32 \times 129 = 4128$$



(a)     (b)     (c)     (d)

FUS     HS     PLS     QUICK

# PWL scheme

# QUICK scheme



**Table 5 Dimensionless cell coordinate calculated with PWL**

| | X | Y |
|---|---|---|
| 1 | 0.498515248326 | 27.8179321847 |
| 2 | 0.498515248326 | 25.7841071708 |
| 3 | 0.498515248326 | 23.4863048689 |
| 4 | 0.498515248326 | 21.1045097935 |
| 5 | 0.498515248326 | 18.6927172991 |
| 6 | 0.498515248326 | 16.2869242885 |
| 7 | 0.498515248326 | 13.8871307617 |
| 8 | 0.498515248326 | 11.4933367187 |
| 9 | 0.498515248326 | 9.1415390625 |
| 10 | 0.498515248326 | 6.89773211496 |
| 11 | 0.498515248326 | 4.92390193917 |

Note: Nu = 39.0

Grid number 102×3102



**Table 8 Dimensionless cell coordinate calculated with QUICK**

| | x | y |
|---|---|---|
| 1 | 0.518501419014 | 29.1039634016 |
| 2 | 0.490007077493 | 27.4006482603 |
| 3 | 0.499915660431 | 24.67564866 |
| 4 | 0.499997148246 | 21.9077572869 |
| 5 | 0.499991534052 | 19.1825723813 |
| 6 | 0.499886807287 | 16.4151439754 |
| 7 | 0.499878758708 | 13.6898093029 |
| 8 | 0.499990193278 | 10.9220760437 |
| 9 | 0.50007191963 | 8.19718832227 |
| 10 | 0.500120639936 | 5.47165901886 |
| 11 | 0.479889934259 | 3.81172796021 |

Note: Nu = 42.61

Grid number 102×3102

## Grids=316404

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Solutions from lower-order scheme can not resolute small vortices if mesh is not fine enough.

At coarse grid system, solution differences by different schemes are often significant!

Solution from higher order scheme with a less grid number can reach the same accuracy as that from lower order scheme with a larger grid number.

With increased grid number power law can also resolute small vortices.

The differences between different schemes are gradually reduced with increasing grid number.

Jin WW, He YL, TaoWQ. How many secondary flows are in Leonard's vertical slot? Progress in Computational Fluid Dynamics, 2009, 9(3/4):283-291

# 4.6 Methods for overcoming or alleviating effects of false diffusion

## 4.6.1 Higher order schemes to overcome streamwise false diffusion

**1. Second order upwind scheme (SUD)**

**2. Third order upwind scheme (TUD)**

**3. QUICK**

**4. SGSD**

## 4.6.2 Methods for alleviating cross false diffusion

**1. Effective diffusivity method**

**2. Self-adaptive grid method**

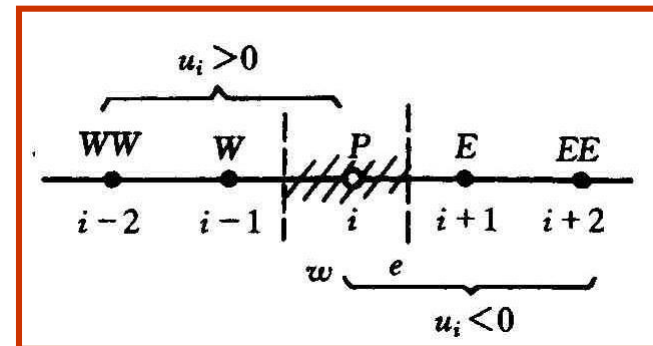# 4.6 Methods for overcoming or alleviating effects of false diffusion

## 4.6.1 Higher order schemes to overcome stream-wise false diffusion

### 1. SUD — Taking two upstream points for scheme

**(1)** Taylor expansion definition — 2$^{nd}$ order one side UD

$$u \frac{\partial \phi}{\partial x})_i = \frac{u_i}{2\Delta x}(3\phi_i - 4\phi_{i-1} + \phi_{i-2}), \ u > 0$$

Rewriting it into the form of interface CD + an additional term:



$$u \frac{\partial \phi}{\partial x})_P = u_P(\frac{\phi_P - \phi_W}{\Delta x} + \frac{\phi_P - 2\phi_W + \phi_{WW}}{2\Delta x})$$

This is equivalent to CD＋curvature correction: slope at grid P = slope at w-interface + a correction term：

$$(\frac{\phi_P - 2\phi_W + \phi_{WW}}{2\Delta x})$$
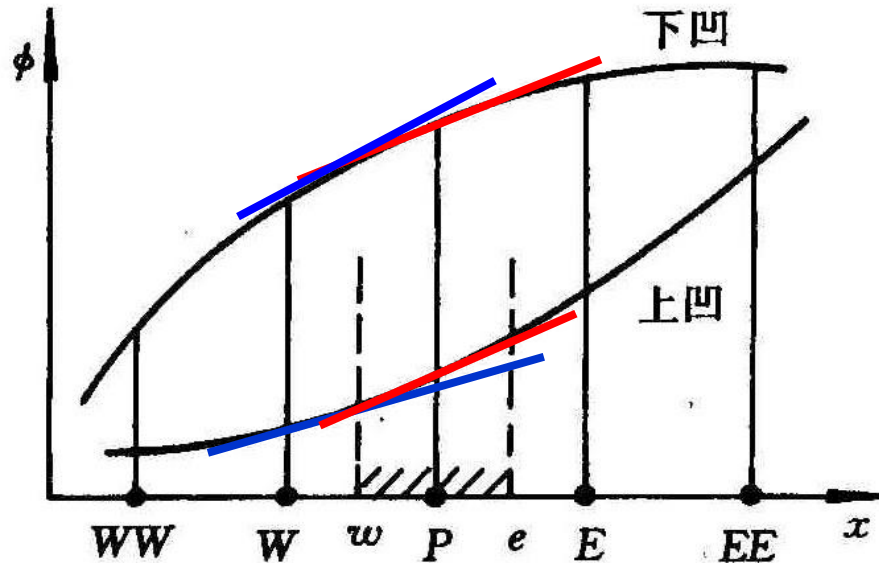


Check the sign (plus or minus) of the correction term to see if it is consistent with the curvature.
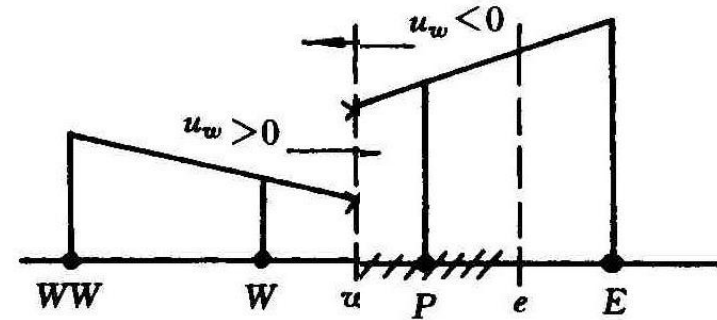
| Concave upward(上凹)， | $(\phi_P - 2\phi_W + \phi_{WW}) > 0$ **Correction>0**； |
| Concave Downward(下凹) | $(\phi_P - 2\phi_W + \phi_{WW}) < 0$ **Correction<0** |

**(2) FVM－**Interface interpolation takes two upstream points.

$$\phi_w = \begin{cases} 1.5\phi_W - 0.5\phi_{WW}, & u > 0 \\ \\ 1.5\phi_P - 0.5\phi_E, & u < 0 \end{cases}$$



Equivalence of the two definitions:

$$\frac{1}{\Delta x}\int_{w}^{e}\frac{\partial \phi}{\partial x}dx = \frac{\phi_e - \phi_w}{\Delta x} = \frac{(1.5\phi_P - 0.5\phi_W) - (1.5\phi_W - 0.5\phi_{WW})}{\Delta x}$$
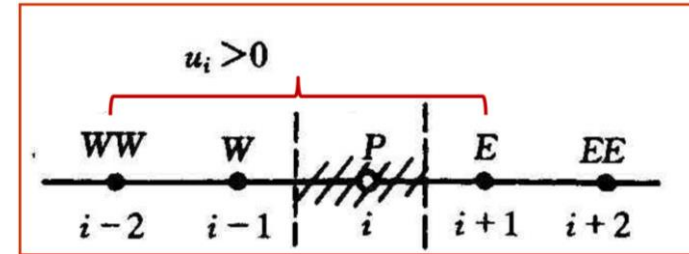
$$= \frac{3\phi_P - 4\phi_W + \phi_{WW}}{2\Delta x}$$

**FVM:** Integral averaged value over a CV;

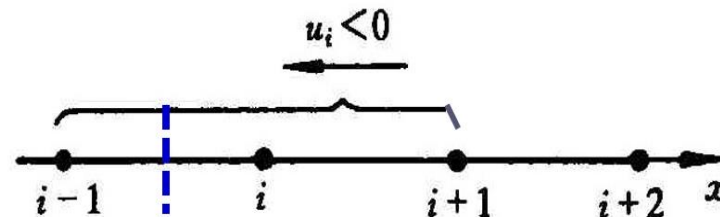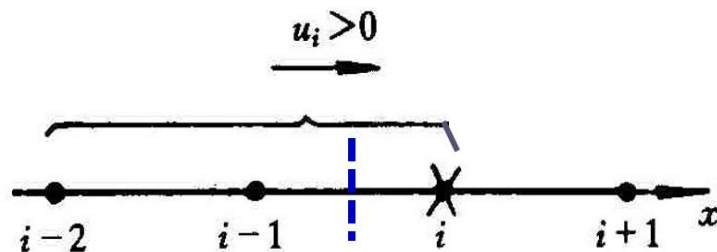**FDM:** Discretized value at a node

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## 2. TUD (三阶迎风)

**(1) Taylor expansion**－3rd-order scheme of 1st derivative with biased positions of nodes （节点偏置）.

$$u\frac{\partial \phi}{\partial x}\Big)_i \overset{u>0}{=} \frac{u_i}{6\Delta x}(2\phi_{i+1}+3\phi_i-6\phi_{i-1}+\phi_{i-2})$$

**Remark：** one downstream node is adopted, which improves the accuracy but weakens the stability.

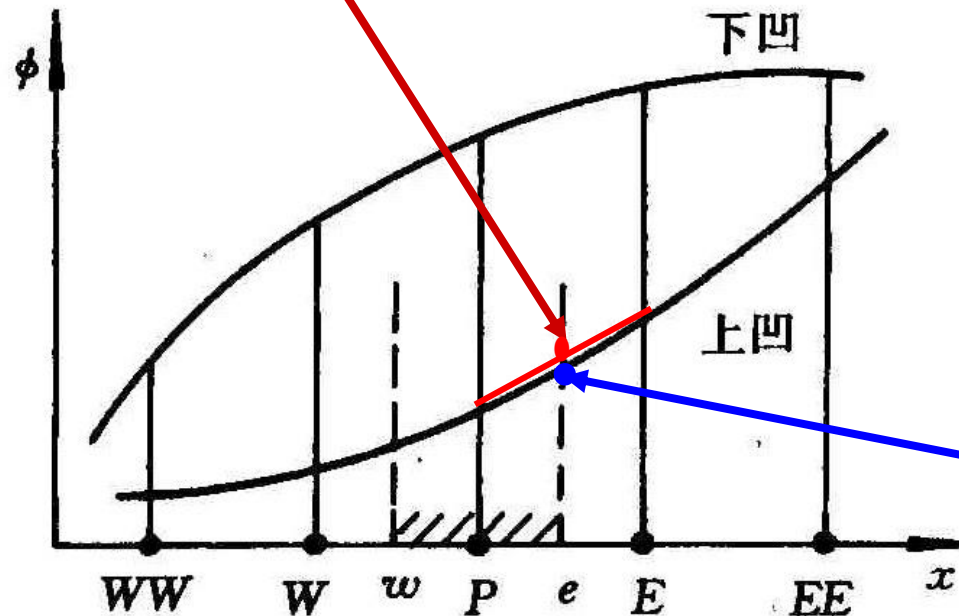**(2) FVM**－interface interpolation is implemented by two upstream nodes and one downstream node

# 3.QUICK scheme (FVM definition)

**1) Position definition**—CD at interface with a curvature correction

$$\phi_e = \frac{\phi_E + \phi_P}{2} - \frac{1}{8}Cur$$

**CD at interface**

**curvature correction**

**Actual interface value**

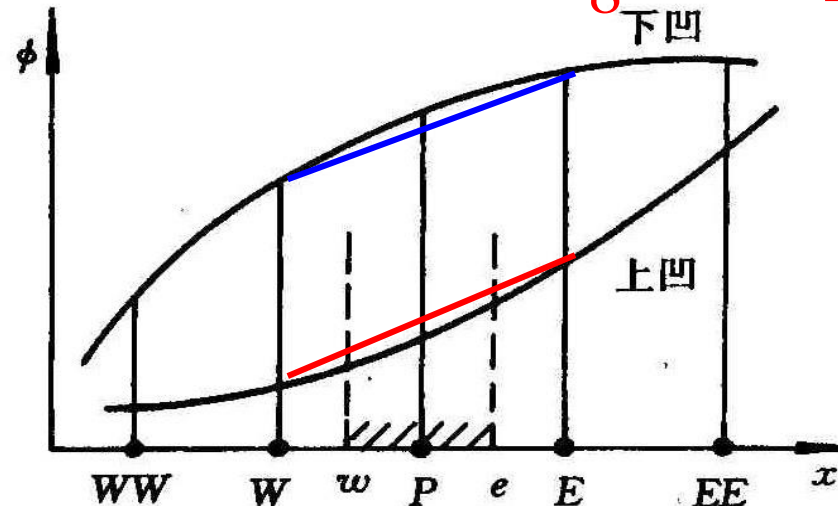Curvature correction

**How to determine CUR?** Two considerations：

(1) Reflecting concave（凹）upward（向上凸）or
concave downward （向下凹）curvature
automatically

Concave upward

$$(\phi_W - 2\phi_P + \phi_E) > 0, \quad -\frac{1}{8}Cur$$

Decreasing the
Interface value a bit!

Concave downward

$$(\phi_W - 2\phi_P + \phi_E) < 0 \quad -\frac{1}{8}Cur$$

Increasing the
Interface value a bit!

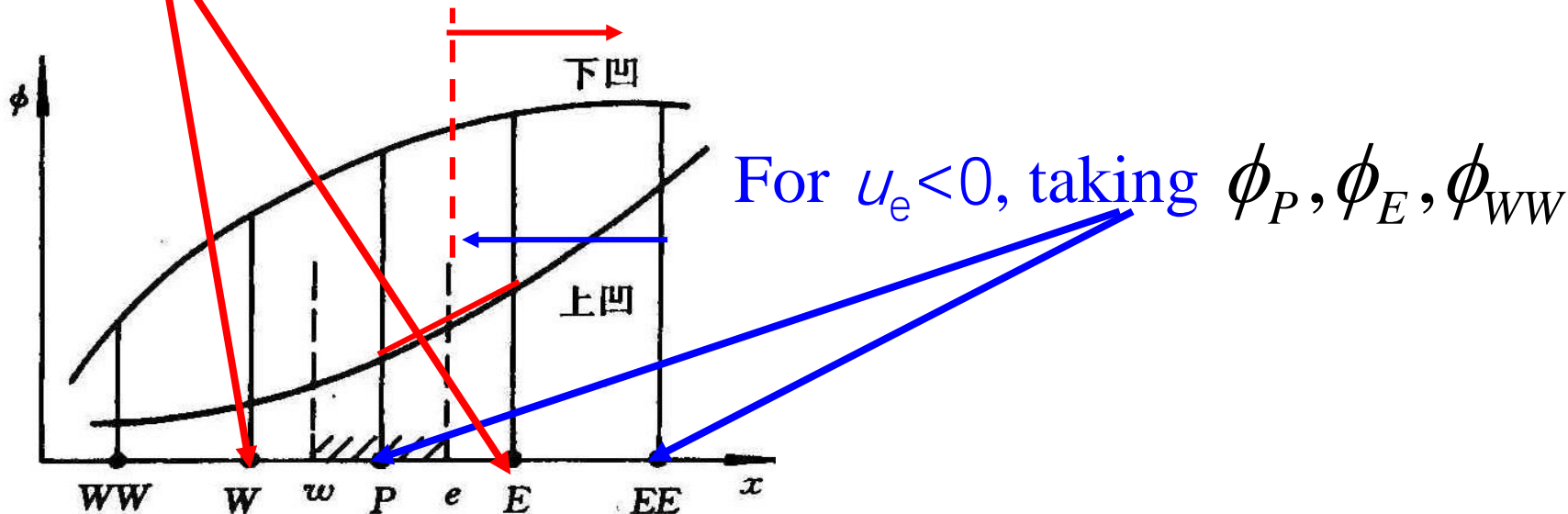(2) Adopting upwind idea for enhancing stability：

For interface e

When $u > 0$, taking $\phi_W, \phi_P, \phi_E$

When $u < 0$, taking $\phi_P, \phi_E, \phi_{EE}$

For $u_e > 0$, taking $\phi_W, \phi_P, \phi_E$

For $u_e < 0$, taking $\phi_P, \phi_E, \phi_{WW}$

下凹

上凹

WW    W    w    P    e    E    EE    x

Curvature correction for QUICK：

$$\text{Cur} = \begin{cases} \phi_W - 2\phi_P + \phi_E, & u > 0 \\ \phi_P - 2\phi_E + \phi_{EE}, & u < 0 \end{cases}$$
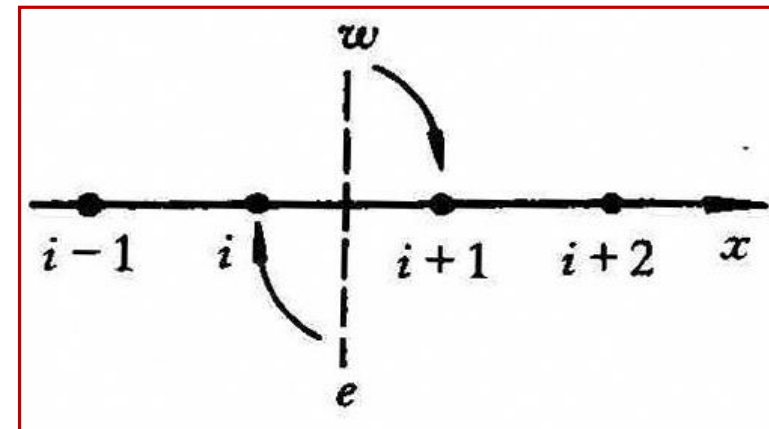
QUICK＝quadratic interpolation of convective
kinematics

**Two remarks**

1) **QUICK** possesses conservative character(守恒特性)—
Interface interpolation and discretized 1$^{st}$ derivative
are continuous at interface:

(1)（i+1/2）interface
value depends on flow
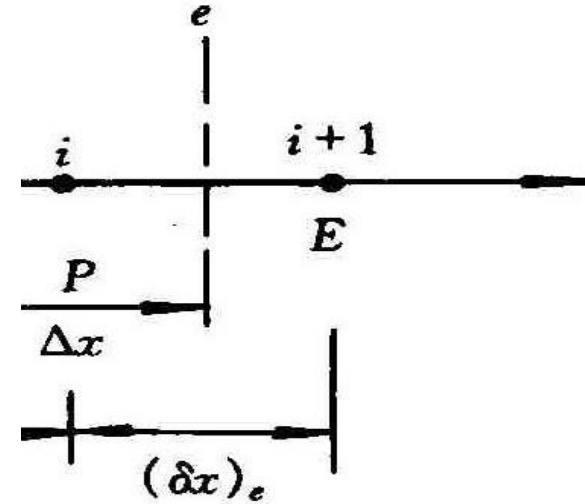direction，for both i
and ( i+1) is the same；

(2) (i+1/2) interface discretized 1$^{st}$ derivative is

$$\frac{\phi_E - \phi_P}{(\delta x)_e},$$ for both $P$ or $E$ is the same.

Thus QUICK possesses conservative character

2) QUICK − subscript definition,
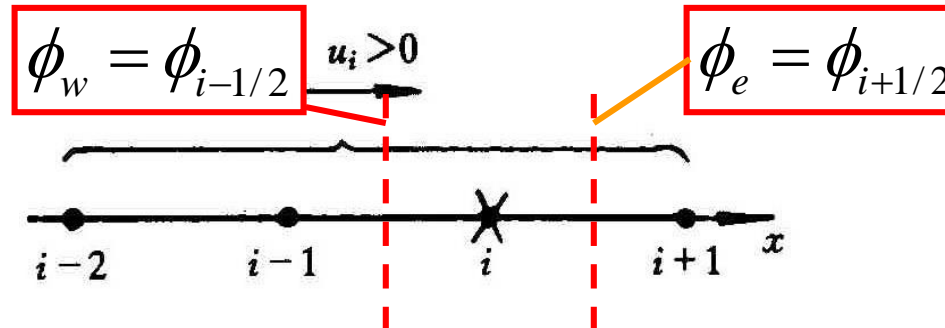
For $u > 0$:
$$\phi_e = \phi_{i+1/2} = \frac{1}{8}(3\phi_{i+1} + 6\phi_i - \phi_{i-1})$$

$$8 \rightarrow 6 \rightarrow 3$$

$$\phi_w = \phi_{i-1/2} = \frac{1}{8}(3\phi_i + 6\phi_{i-1} - \phi_{i-2})$$

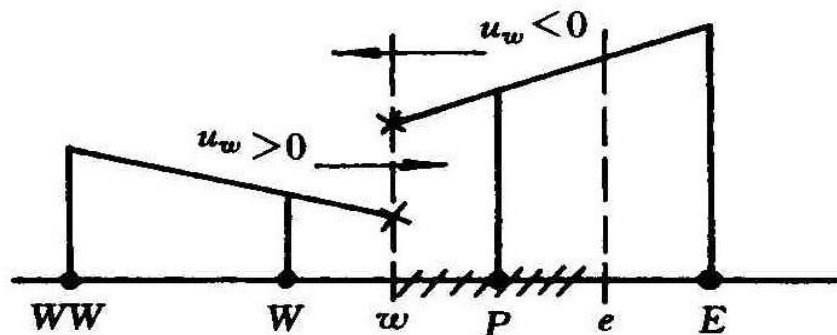$$\phi_w = \phi_{i-1/2} \qquad u_i > 0 \qquad \phi_e = \phi_{i+1/2}$$

# 4. SGSD－A kind of composite (组合)scheme

## 1) SCSD scheme (1999) (Uniform grid)

CD: $\phi_e = 0.5(\phi_P + \phi_E)$

No false diffusion ($2^{nd}$ order)，but only conditionally stable!

SUD: $\phi_e = \begin{cases} 1.5\phi_W - 0.5\phi_{WW}, & u > 0 \\ 1.5\phi_P - 0.5\phi_E, & u < 0 \end{cases}$

Absolutely stable（discussed later），but has appreciable(显著的) numerical errors.

Thus combining the two schemes in such a way maybe useful:

When Pe number is small, CD predominates;
when Pe number is large, SUD predominates:

$$\phi_e^{SCSD} = \beta\phi_e^{CD} + (1-\beta)\phi_e^{SUD}, \ 0 \le \beta \le 1$$

$$\beta = 1, \phi^{SCSD} \equiv \phi^{CD}; \quad \beta = 0, \phi^{SCSD} \equiv \phi^{SUD}; \quad \beta = 3/4, \phi^{SCSD} \equiv \phi^{QUICK}$$

It can be shown:

$$P_{\Delta,cr} = (\frac{\rho u \delta x}{\Gamma})_{cr} = \frac{2}{\beta}$$

Beyond which the scheme is unstable!

By adjusting Beta value its critical Peclet number can vary from 0 to infinite! Therefore it is called:
**s**tability-**c**ontrollable **s**econd-order **d**ifference－SCSD（倪明玖，1999）.

Ni M J, Tao W Q. J. Thermal Science, 1998, 7(2):119-130

**Question:** how to determine Beta? Especially how to calculate Beta based on the flow field automatically?

## 2）SGSD格式（2002）

From $P_{\Delta,cr} = \dfrac{2}{\beta}$ $\longrightarrow$ $\beta = \dfrac{2}{P_{\Delta,cr}}$, replace $P_{\Delta,cr}$ in denominator by $(2 + P_\Delta)$:

$$\beta = \frac{2}{2 + P_\Delta} \begin{cases} P_\Delta \to 0, \beta \to 1, & \text{CD dominates；} \\ P_\Delta \to \infty, \beta \to 0, & \text{SUD dominates} \end{cases}$$

1) It can be determined from flow field with different effects of diffusion and convection being considered automatically!

2) Three coordinates can have their own Peclet numbers!

Li ZY, Tao WQ. A new stability-guaranteed second-order difference scheme. **NHT-Part B**, 2002，42 (4): 349-365

# 5. Discussion on implementing higher-order schemes

## 1) Near boundary point：

Taking practice A as an example：  For the interface between nodes 1 and 2, if $u_f>0$，how to implement higher order schemes?

**Boundary: solid or fluid**
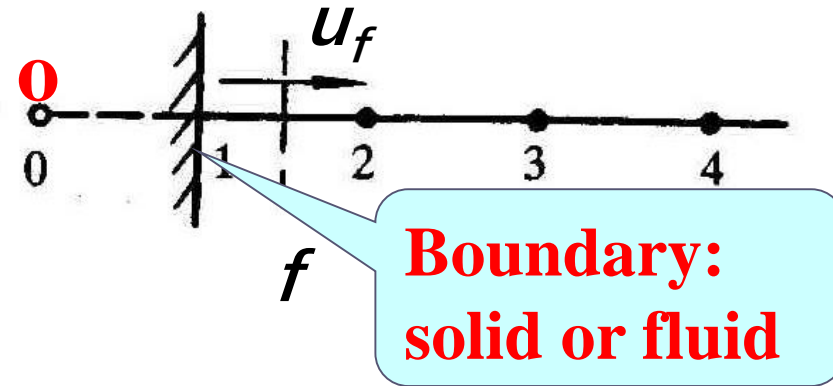
**Two ways can be adopted:**

(1) Fictitious point method (**虚拟点法**)：  Introducing a fictitious point O and assuming:

$$\phi_o + \phi_2 = 2\phi_1 \quad \Longrightarrow \quad \phi_o = 2\phi_1 - \phi_2$$

(2) Order reduction (**降阶**) method:  $\phi_f = \phi_1, u_f > 0$
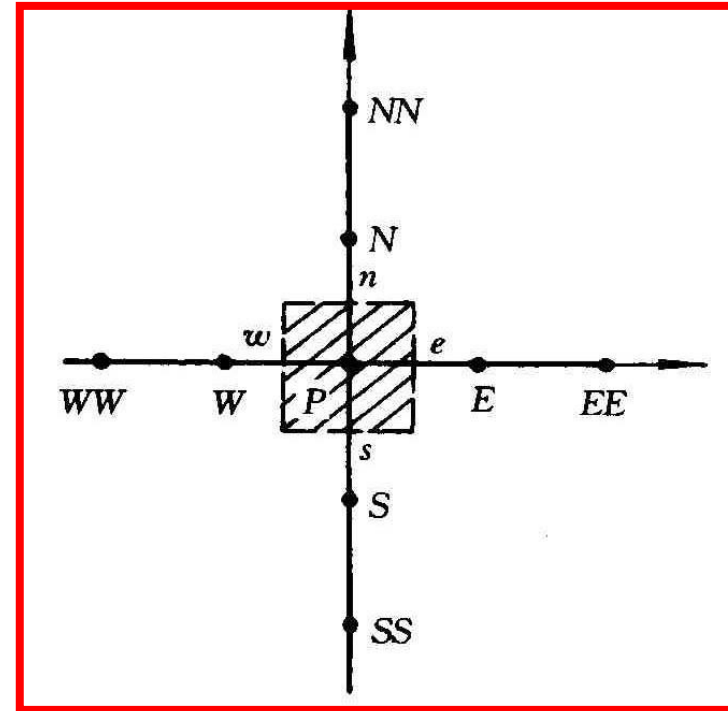
## 2) Solution of ABEqs. :

When QUICK，TUD etc. are used，the matrix of 2-D problem is nine-diagonal and the ABEqs. may be solved by
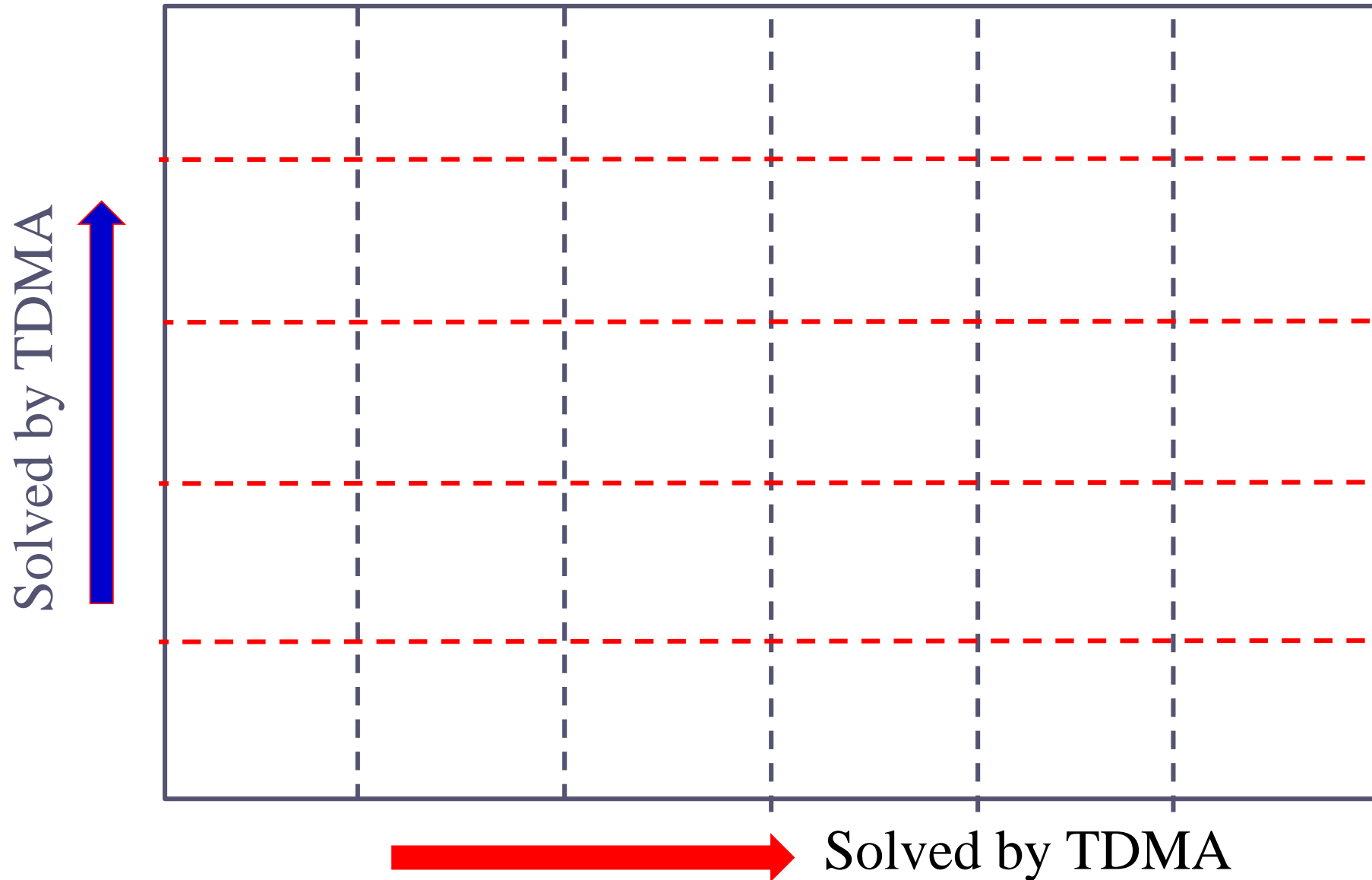
**(1) Penta-diagonal matrix (五对角阵算法) PDMA;**

**(2) Deferred correction(延迟修正)。**

$$\phi_e^H = \phi_e^L + (\phi_e^H - \phi_e^L)^* \qquad * - \text{previous iteration}$$

The lower-order part $\phi_e^L$ forms ABEqs.；those with * go to source part，and **ADI method** is used. The converged solution is the one of higher-order scheme.

# ADI ---Alternative Direction Iteration



Solved by TDMA

Solved by TDMA

## 4.6.2 Methods for alleviating (减轻) effects of cross-diffusion

## 1. Adopting effective diffusivity for FUD

$$(\Gamma_{\phi,x})_{eff} = \llbracket 0, (\Gamma_\phi - \Gamma_{cd,x}) \rrbracket$$

$\Gamma_\phi$ — diffusivity of physical problem;

$\Gamma_{cd,x}$ — diffusivity from cross false diffusion

By reducing diffusivity used in simulation the cross diffusion effect can be alleviated.
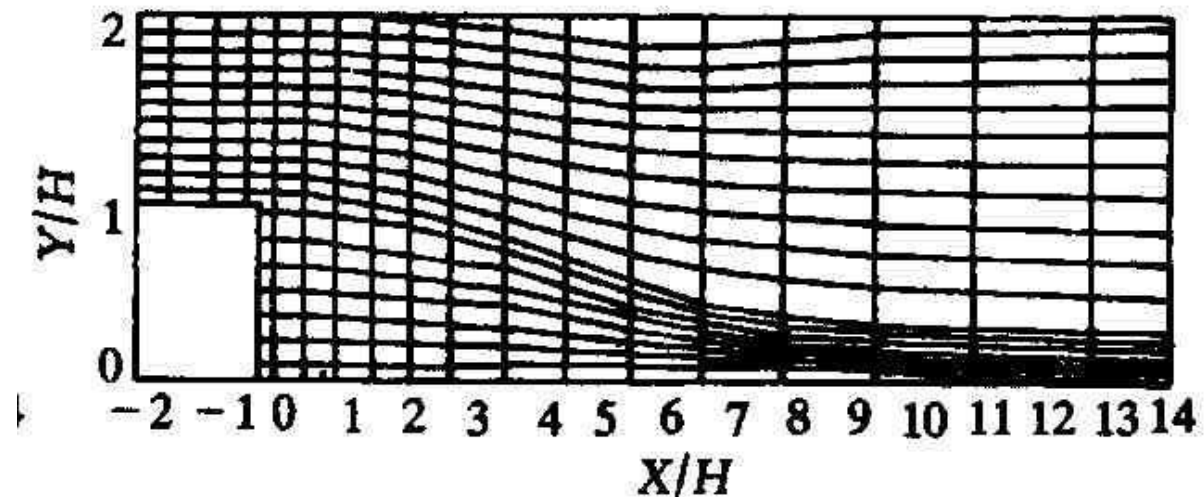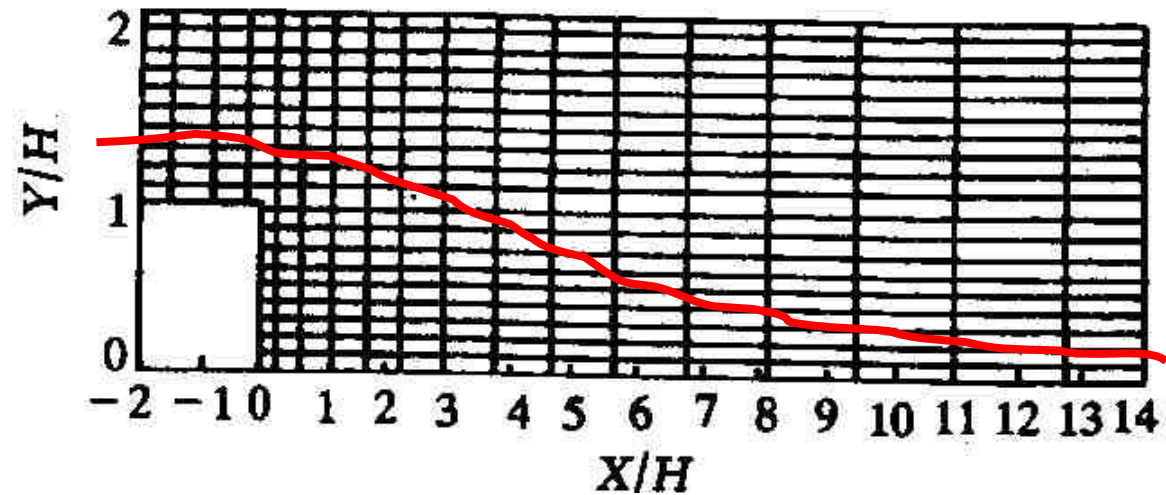
$$\Gamma_{cd,x} = u\Delta x(1 - \frac{u\delta t}{\Delta x})$$

$$\delta t = \frac{1}{\dfrac{u}{\Delta x} + \dfrac{v}{\Delta y} + \dfrac{w}{\Delta z}}$$

（Inspired(启发) from Noye problem)

# 2. Adopting self-adaptive grids (SAG-自适应网格）

SAG can alleviate (减轻)cross-diffusion caused by oblique intersection of streamline to grid line

## 4.6.3 Summary of convective scheme

1. For conventional fluid flow and heat transfer problems，in the debugging process (**调试过程**) FUD or PLS may be used；For the final computation QUICK or SGSD is recommended, and defer correction is used for solving the ABEqs.

2. For direct numerical simulation (DNS) of turbulent flow，fourth order or more are often used;

3. When there exists a sharp variation of properties, higher order and bounded schemes (**高阶有界格式**) should be used.

Recent advances can be found in:

Jin W W, Tao W Q. NHT, Part B, 2007, 52(3): 131-254
Jin W W, Tao W Q. NHT, Part B, 2007, 52(3): 255-280

## 4.7 Discretization of multi-dimensional problem and B.C. treatment

### 4.7.1 Discretization of 2-D diffusion-convection equation

1. Governing equation expressed by $J_x$, $J_y$

2. Results of disctretization

3. Ways for adopting other schemes

### 4.7.2 Treatment of boundary conditions

1. Inlet boundary

2. Solid boundary

3. Central line

4. Outlet boundary

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# 4.7 Discretization of multi-dimensional problem and B.C. treatment

## 4.7.1 Discretization of 2-D diffusion-convection equation

### 1. Governing equation expressed by $J_x$, $J_y$

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u\phi)}{\partial x} + \frac{\partial(\rho v\phi)}{\partial y} = \frac{\partial}{\partial x}(\Gamma\frac{\partial\phi}{\partial x}) + \frac{\partial}{\partial y}(\Gamma\frac{\partial\phi}{\partial y}) + S$$

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial}{\partial x}\underbrace{(\rho u\phi - \Gamma\frac{\partial\phi}{\partial x})}_{J_x} + \frac{\partial}{\partial y}\underbrace{(\rho v\phi - \Gamma\frac{\partial\phi}{\partial y})}_{J_y} = S$$

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = S$$

## 2. Results of discretization

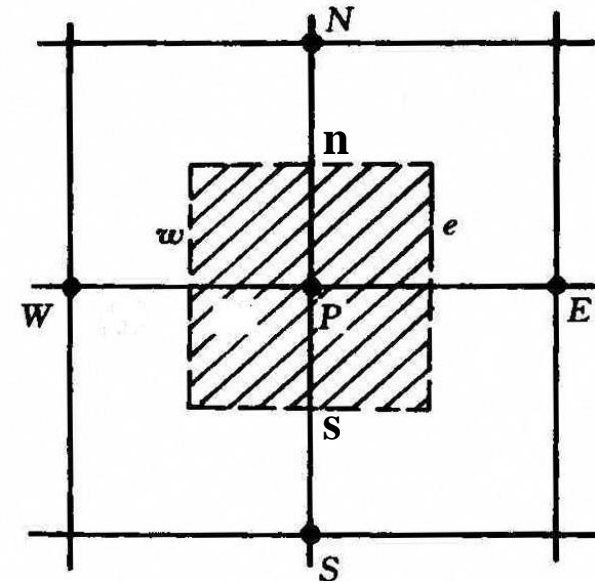In order to extend the results of 1-D discussion, introducing $J_x$, $J_y$ to 2-D case

Integrating above equations for CV. P

$$\iiint \frac{\partial(\rho\phi)}{\partial t} dt dx dy = [(\rho\phi)_P - (\rho\phi)_P^0]\Delta V$$

$$\int\int\int_w^e \frac{\partial J_x}{\partial x} dx dy dt = \int_t^{t+\Delta t}\int_s^n (J_x^e - J_x^w) dy dt$$

$$\int\int\int_s^n \frac{\partial J_y}{\partial y} dx dy dt = \int_t^{t+\Delta t}\int_w^e (J_y^n - J_y^s) dx dt$$

$$\iiint S dx dy dt = (S_C + S_P\phi_P)\Delta V \Delta t$$

Assuming that at the interface $J_x^e, J_x^w$ are constant, then:

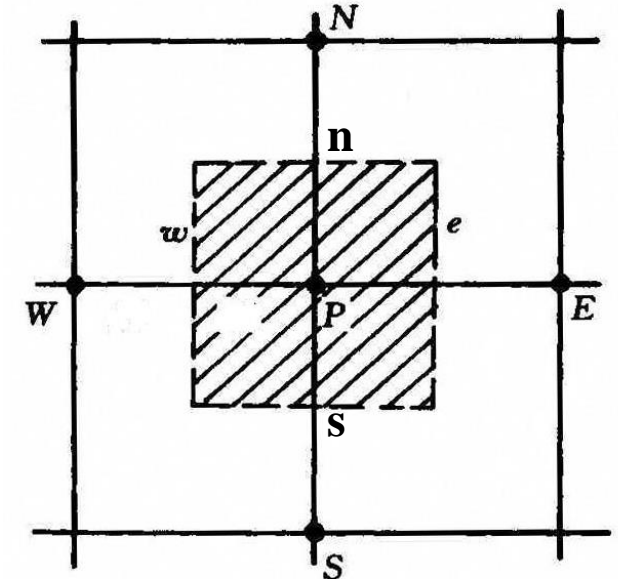$$(J_x^e - J_x^w)\Delta y\Delta t = (J_x^e\Delta y\text{-}J_x^w\Delta y)\Delta t = (J_e - J_w)\Delta t$$

**Expressing $J$ via $J^*$:**

$$J_e = J_e^*D_e = D_e[B(P_{\Delta e})\phi_P - A(P_{\Delta e})\phi_E]$$

**Add-sub**

$$J_e = J_e^*D_e = D_e[\{A(P_{\Delta e}) + P_{\Delta e}\}\phi_P - A(P_{\Delta e})\phi_E]$$

$$J_e = J_e^*D_e = \{D_eA(P_{\Delta e}) + F_e\}\phi_P - D_eA(P_{\Delta e})\phi_E$$

$$J_e = J_e^*D_e = D_eA(P_{\Delta e})\phi_P + F_e\phi_P - D_eA(P_{\Delta e})\phi_E$$

$$a_E$$

$$a_E$$

$$D_e = \frac{\Gamma\Delta y}{\delta x},$$

$$F_e = \rho u\Delta y$$

The same derivation can be done for three other terms，$J_w$, $J_n$, $J_s$ .

Finally the general discretization equation for 2-D five-point scheme can be obtained:

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_S\phi_S + a_N\phi_N + b$$

$$a_P = a_E + a_W + a_N + a_S + a_P^0 - S_P\Delta V$$

$$b = S_C\Delta V + a_P^0\phi_P^0 \qquad a_P^0 = \frac{\rho_P\Delta V}{\Delta t}$$

$$a_E = D_e A(|P_{\Delta e}|) + [\![-F_e, 0]\!] \qquad a_W = D_w A(|P_{\Delta w}|) + [\![F_w, 0]\!]$$

$$a_N = D_n A(|P_{\Delta n}|) + [\![-F_n, 0]\!] \qquad a_S = D_s A(|P_{\Delta s}|) + [\![F_s, 0]\!]$$

## 3. Ways for adopting other schemes

Adopting defer correction method, and putting the additional part of the other scheme into source term (b) of the algebraic equation.

## 4.7.2 Treatment of boundary conditions

**1. Inlet boundary** — usually specified;

**2. Center line** — symmetric boundar**y:**

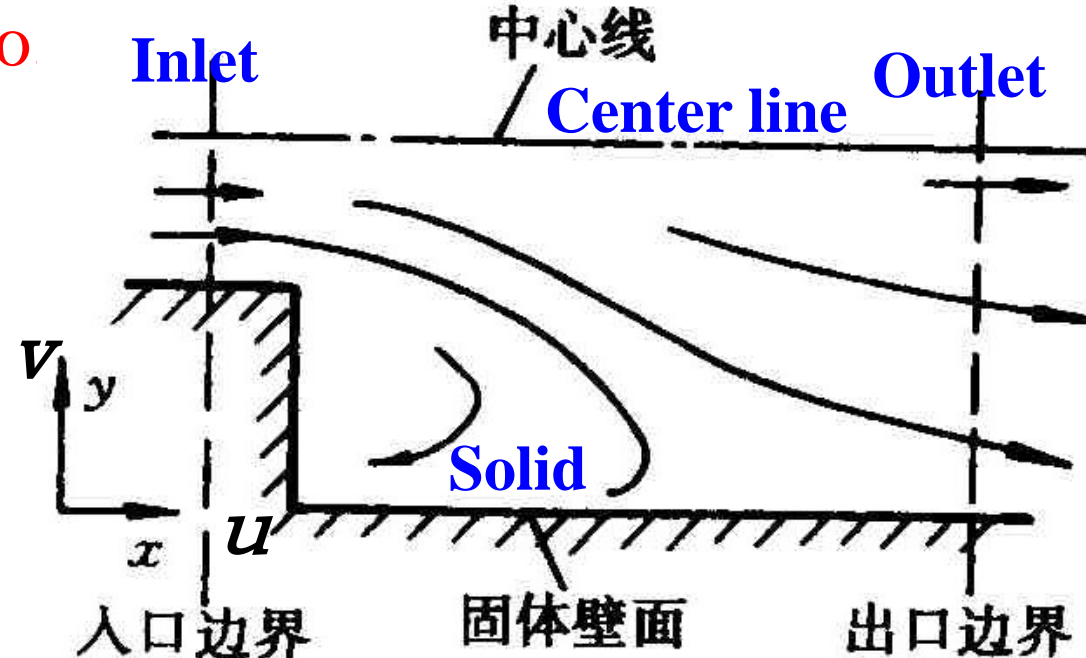Velocity component normal to the center line is equal to zero;

First derivative normal to the lcenter ine of other variable is equal to zero

$$v = 0; \frac{\partial \phi}{\partial n} = 0$$
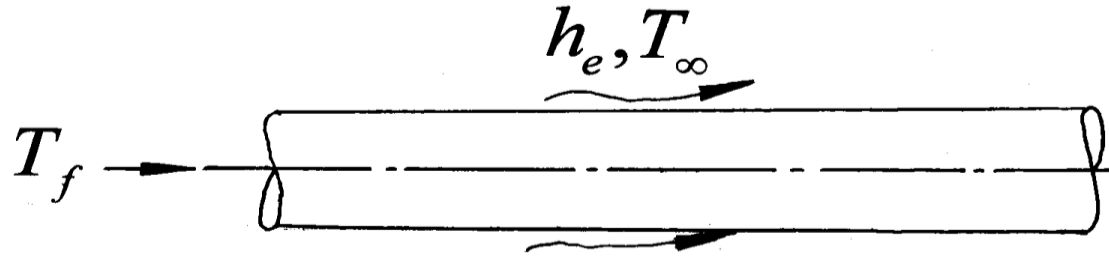
**3. Solid boundary**

No slip for *u,v* **;**

Three types for *T*.

Inlet

中心线
Center line
Outlet

$v$
$y$

Solid

$x$
$u$

入口边界     固体壁面     出口边界

Known temp.**－1**;　Given heat flux**－2**;

External convective heat transfer**－3**;

$h_e, T_\infty$

$T_f$

<span style="color:red">Inner convective heat transfer coefficient is solved</span>
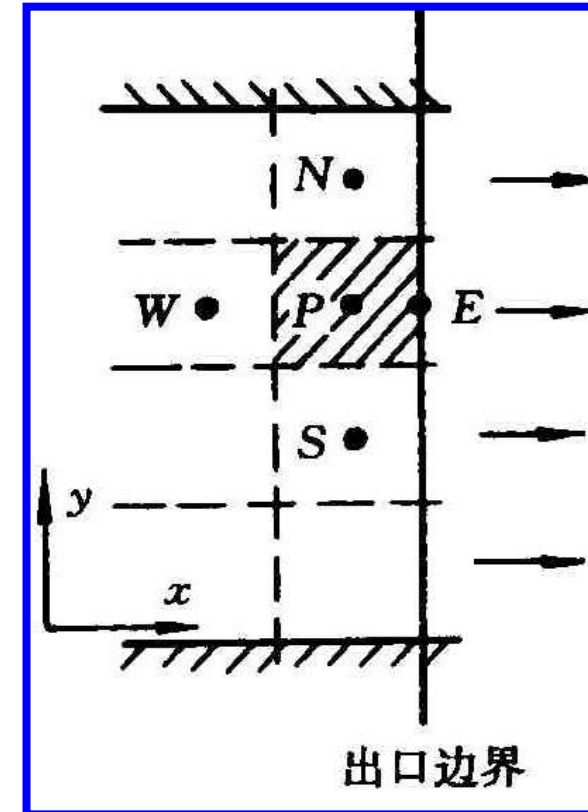
**4. Outlet boundary**

Conventional methods:

**(1) Local one-way** (局部单向化)

$$a_E = 0$$

**(2) Fully developed** (充分发展)

$$\frac{\partial \phi}{\partial x} = 0 \quad \longrightarrow \quad \phi_E = \phi_P^*$$

出口边界

**Home work of Chapter 4**

**5−2    5−3    5−5    5−9**

**In  5−5  Taking**  $\Delta x = \Delta y = 0.2$

**Home work due on 10-27**

## Problem # 5-2

One-dimensional steady-state convection diffusion equation without source term，whereas boundary conditions are $x = 0$, $\phi = \phi_0$, $x = L$, $\phi = \phi_L$ 。Taking 10 to 20 nodes for range $x/L = 0 \sim 1$, using the following 4 methods：Central difference, first order upwind , Hybrid scheme and QUICK scheme，then draw the plot between $(\phi - \phi_0)/(\phi_L - \phi_0)$ and $x/L$ using three values of Peclet number i.e. $P_\Delta = 1, 5, 10$ and compare the results with exact values.

(Note: take care the difference between gird Peclet number, $P_\Delta$, whole Peclet number and $P_\Delta = \dfrac{\rho u L}{\Gamma}$ )

**Problem # 5-3**

For one-dimensional unsteady convection - diffusion equation

$$\frac{\partial(\rho\phi)}{\partial(t)} = -\frac{\partial(\rho u\phi)}{\partial x} + \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right),$$ using power law scheme for the discretization and find

the values of followings constants $a_E$, $a_W$, $a_P^0$ and $a_P : \Delta t = 0.05$, where $\rho u = 1$,

$P_\Delta = 0.1$, $10$。 All units are the same.

**Problem # 5-5**

Consider a two-dimensional steady-state convection diffusion equation, where $\rho u = 5$,
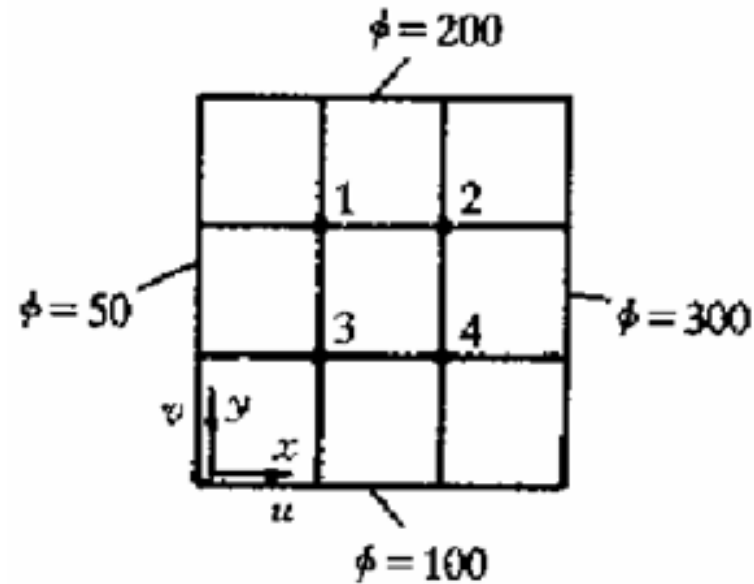
$\rho v = 3$, $\Gamma = 0.5$, the boundary values are shown in figure given below,

also $\Delta x = \Delta y = 0.2$。 By using ( a ) first order upwind scheme ; ( 2 ) hybrid scheme ; ( 3 )

power law scheme ; ( 4 ) second-order upwind scheme, try to get the values of $\phi$ at

four nodes (1,2,3,4 ).

**Figure of Problem 5-5-**

**Problem # 5-7**

Discretize the equation (5-1) for uniform grid and $u > 0$ by using the QUICK scheme and get the discretization equation. Then use the sign preservation rule as in section 5.7 to analyze the stability of this scheme.

**Problem # 5-9**

Define the third-order upwind scheme using the interface function interpolation method and verify the consistence in form with the definition of derivative expression for given nodes.

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同舟共济
渡彼岸！

**People in the same boat help each other to cross to the other bank, where....**

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