Numerical Heat Transfer
(数值传热学)
Chapter 7 Mathematical and Physical Characteristics of Discretized Equations (Chapter 3 of Textbook)

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7.3 Conservation of Discretized Equations

7.3.1 Definition and analyzing model

7.3.2 Direct summation method

7.3.3 Conditions for guaranteeing conservation of discretized equations

7.3.4 Discussion—expected but not necessary (期待而非必须)

7.3 Conservation of Discretized Equations

7.3.1 Definition and analyzing model

1. Definition

If the summation of a certain number of discretized equations over a finite volume （有限大小体积）satisfies conservation requirement , these discretized equations are said to possess conservation (**离散方程具有守恒性**).

2. Analyzing model---advection equation

It is easy to show that CD of diffusion term possesses conservation. Discussion is only performed for the equation which only has transient term and convective term (advection equation, **平流方程**).

Advection equation

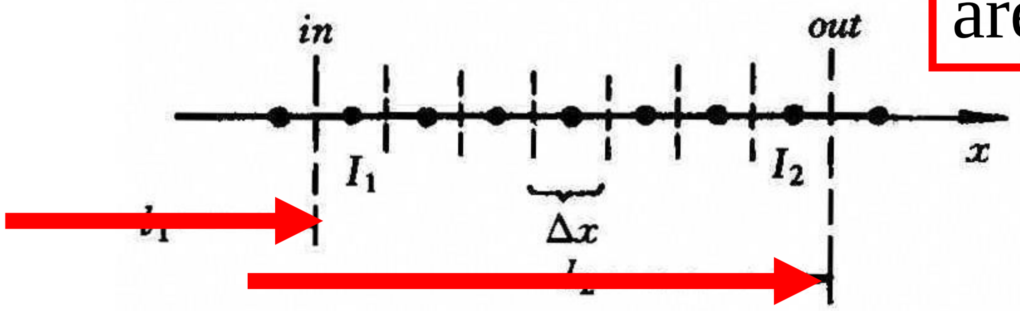
$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{\partial (u\phi)}{\partial x} = 0 & \text{(Conservative)} \\[2ex] \dfrac{\partial \phi}{\partial t} + u\dfrac{\partial \phi}{\partial x} = 0 & \text{(Non-conservative)} \end{cases}$$

7.3.2 Direct summation method (直接求和法)

Summing up FTCS scheme of advection eq. of conservative form over the region of $[I_1, I_2]$:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -\frac{u_{i+1}\phi_{i+1} - u_{i-1}\phi_{i-1}}{2\Delta x}$$

Time level of the spatial terms are not shown



$$\sum_{I_1}^{I_2}\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=-\sum_{I_1}^{I_2}\frac{u_{i+1}\phi_{i+1}-u_{i-1}\phi_{i-1}}{2\Delta x}=-\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2\Delta x}$$

$$\sum_{I_1}^{I_2}(\phi_i^{n+1}-\phi_i^n)\Delta x=-\Delta t\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2}$$

Increment(**增值**) of ϕ within Δt and $[I_1, I_2]$

Is it equal to the net amount of ϕ entering the space region by convection within the same time period?

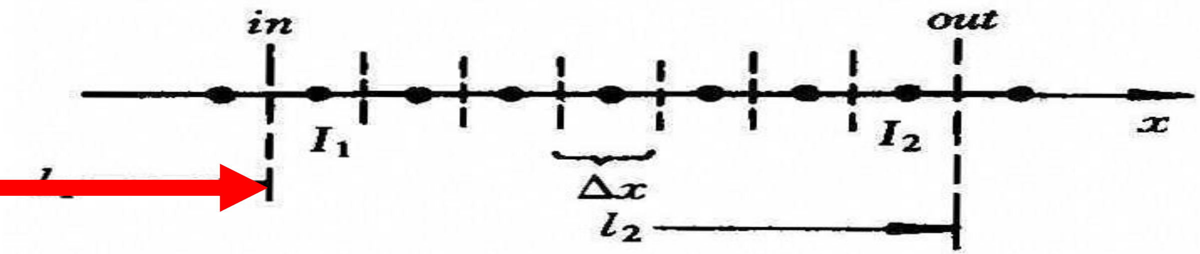
Analyzing should be made for the right hand terms of the equation to see whether this is true:

$$-\Delta t\sum_{I_1}^{I_2}\frac{(u\phi)_{i+1}-(u\phi)_{i-1}}{2}=\frac{\Delta t}{2}\sum_{I_1}^{I_2}[(u\phi)_{i-1}-(u\phi)_{i+1}]$$

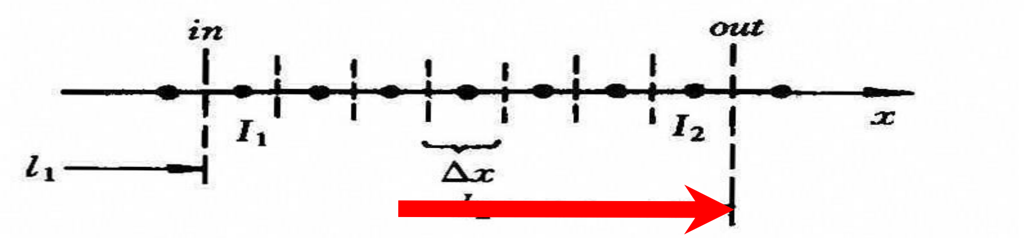
For the term $\sum_{I_1}^{I_2}[(u\phi)_{i-1}-(u\phi)_{i+1}]$ directly summing up: for the left end, we have:

$i=I_1$	$(u\phi)_{I_1-1}$	$-(u\phi)_{I_1+1}$			
$i=I_1+1$	$(u\phi)_{I_1}$		$-(u\phi)_{I_1+2}$		
$i=I_1+2$		$(u\phi)_{I_1+1}$		$-(u\phi)_{I_1+3}$	
$i=I_1+3$			$(u\phi)_{I_1+2}$		$-(u\phi)_{I_1+4}$
$i=I_1+4$				$(u\phi)_{I_1+3}$	
......		......			

$$(u\phi)_{I_1-1}+(u\phi)_{I_1}$$



For the right end:



$$i = I_2 - 3 \qquad (u\phi)_{I_2-4} \qquad -(u\phi)_{I_2-2}$$

$$i = I_2 - 2 \qquad (u\phi)_{I_2-3} \qquad -(u\phi)_{I_2-1}$$

$$i = I_2 - 1 \qquad (u\phi)_{I_2-2} \qquad -(u\phi)_{I_2}$$

$$i = I_2 \qquad (u\phi)_{I_2-1} \qquad -(u\phi)_{I_2+1}$$

$$-[(u\phi)_{I_2} + (u\phi)_{I_2+1}]$$

Then: $$\frac{\Delta t}{2}\sum_{I_1}^{I_2}[(u\phi)_{i-1} - (u\phi)_{i+1}]$$

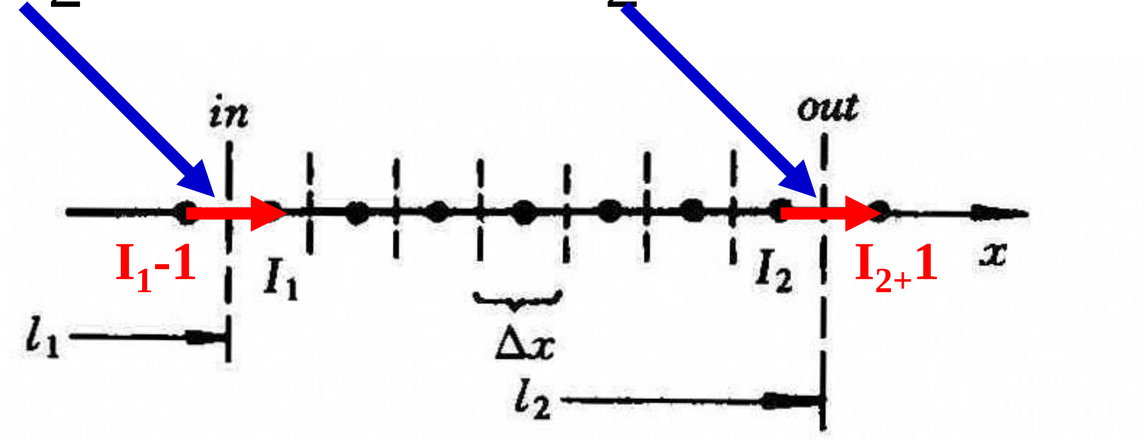
$$= \frac{\Delta t}{2}\{[(u\phi)_{I_1-1} + (u\phi)_{I_1}] - [(u\phi)_{I_2} + (u\phi)_{I_2+1}]\}$$

Left end of domain   Right end of domain

Further: $$\frac{\Delta t}{2}\{[(u\phi)_{I_1-1}+(u\phi)_{I_1}]-[(u\phi)_{I2}+(u\phi)_{I_2+1}]\}=$$

$$\Delta t\{[\frac{(u\phi)_{I_1-1}+(u\phi)_{I_1}}{2}]-[\frac{(u\phi)_{I2}+(u\phi)_{I_2+1}}{2}]\}$$

CD-uniform grid



$$=\Delta t(\phi\, flowin-\phi\, flowout)$$

Thus the central difference discretization of the convective term possesses conservative feature.

7.3.3 Conditions for guaranteeing conservation

1.Governing equation should be conservative

For non-conservative form: $$\frac{\partial \phi}{\partial t}+u\frac{\partial \phi}{\partial x}=0$$

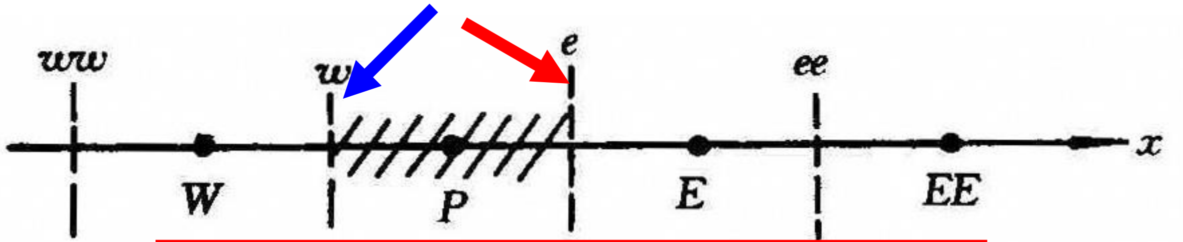
Its FTCS scheme is $$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t}=-u_i\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x}$$

By direct summation, the above results do not possess conservation because of no cancellation (**抵消**) can be made for the product terms. Only when u and ϕ have the same subscript , the cancellation of inner terms can be done.

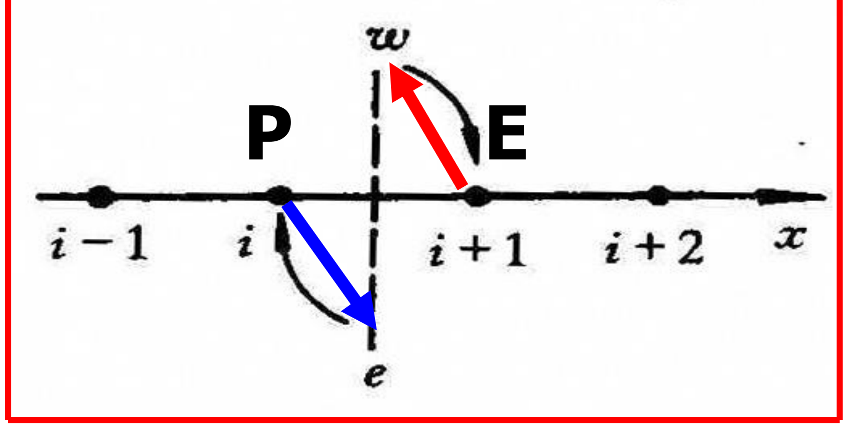
2. Dependent variable and its 1st derivative are continuous at interface

Meaning of “Continuous”

Different interfaces viewed from point P



The same interface viewed from two points P and E



By “Continuous” we mean:

$$(\phi_e)_P = (\phi_w)_E ; \qquad [(\frac{\delta\phi}{\delta x})_e]_P = [(\frac{\delta\phi}{\delta x})_w]_E$$

The piecewise linear profile can meet this condition.

Interface-biased quadratic (界面偏向的二次插值) can not satisfy such requirement

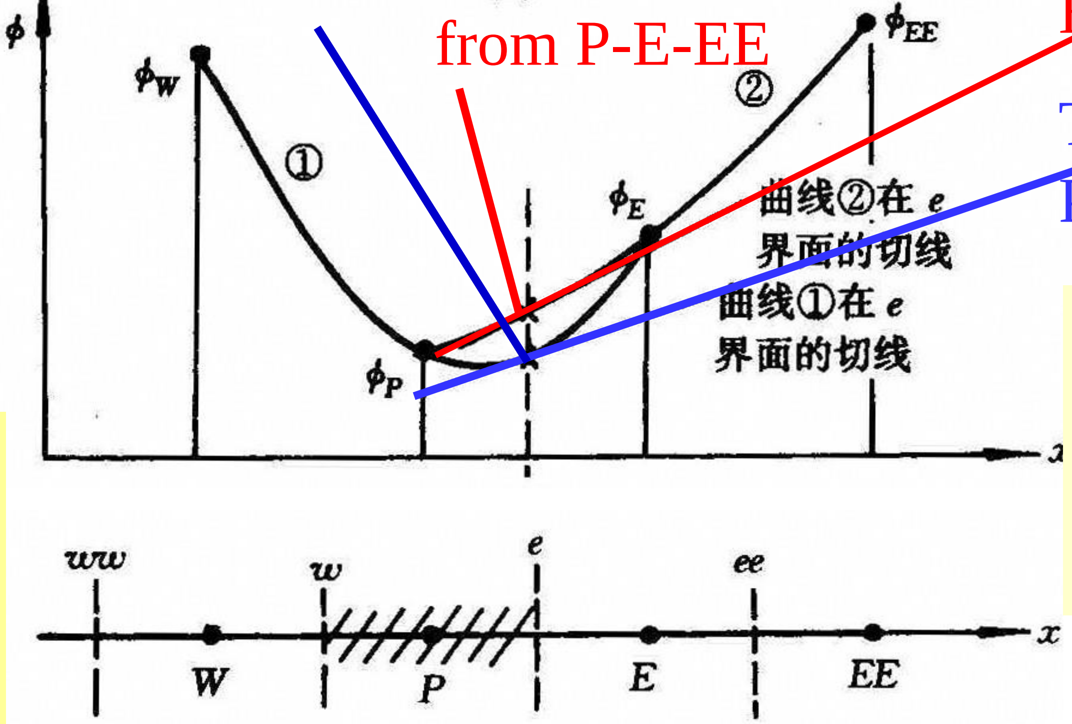
Interface value from W-P-E

Interface value from P-E-EE

Tangential to P-E-EE at interface

Tangential to W-P-E at interface

For west side of the interface, W, P and E are used for interpolation



For east side of the interface, P , E and EE are used for interpolation

Both first order derivatives and interface values are not equal!

7.3.4 Discussion－Conservation is expected but not necessary for all simulation.(希望而非必须)

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7.4 Transportive (迁移)Character of Discretized Equations

7.4.1 Essential (基本的) difference between convection and diffusion

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7.4.3 Analysis of transport character of discretized scheme of convection term

7.4.4 Upwind scheme of convection term possesses transport character

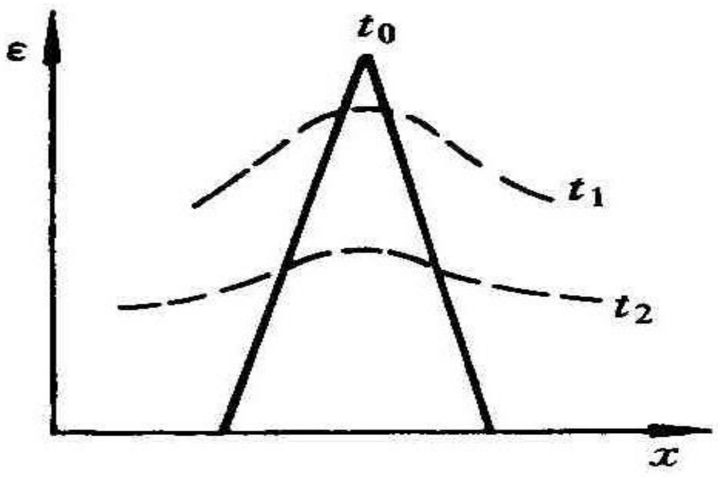
7.4.5 Discussion on transport character of discretized convection term

7.4 Transportive Property of Discretized Equations

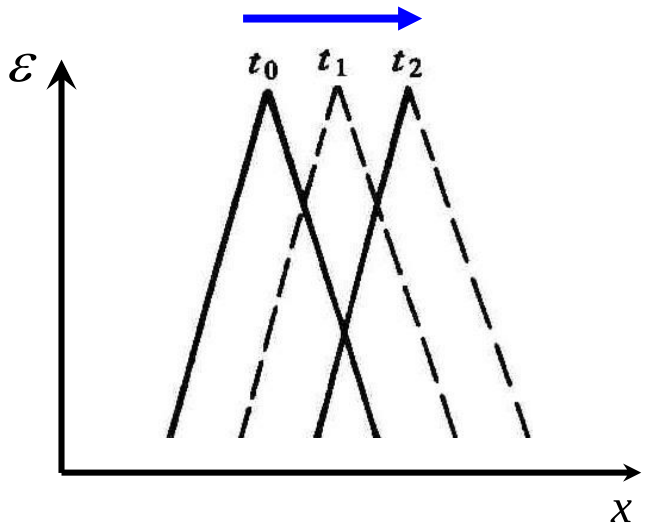
7.4.1 Essential difference between convection and diffusion

Diffusion—Random thermal motions of molecules, no bias(偏向) in direction;

Convection—Directional moving of fluid element, always from upstream to downstream(从上游到下游)



(a)



(b)

7.4.2 CD of diffusion term can propagate disturbances all around (四面八方) uniformly

1. FTCS scheme of diffusion eq.

$$\frac{\partial \phi}{\partial t} = \Gamma \frac{\partial^2 \phi}{\partial x^2} \longrightarrow \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \Gamma \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}$$

$$\phi_i^{n+1} = \phi_i^n \left(1 - 2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right) + \frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\left(\phi_{i-1}^n + \phi_{n+1}^n\right)$$

2. Discrete disturbance analysis (离散扰动分析法)

（1） Assuming a uniform and zero initial field ;

（2） Assuming that a disturbance ε occurs at a point i, at some instant, n, while at all other points and all subsequent time levels no any disturbances;

（3）Analyzing the transfer of the disturbance by the studied scheme.

3. Implementation of discrete disturbance analysis

For point i at (n+1) instant:

Known: $\phi_i^n = \varepsilon,\ \phi_{i-1}^n = \phi_{i+1}^n = 0,$

$$\phi_i^{n+1} = \underbrace{\phi_i^n}_{=\varepsilon}\left(1 - 2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right) + \frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\left(\underbrace{\phi_{i-1}^n}_{0} + \underbrace{\phi_{i+1}^n}_{0}\right)$$

$$\phi_i^{n+1} = \varepsilon\left(1 - 2\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right) \xrightarrow{\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2} \le 0.5} 0 < \phi_i^{n+1} < \varepsilon$$

Stability requires

Physically reasonable

For Point (i ＋1) at (n+1) instant：

$$\frac{\phi_{i+1}^{n+1}-\phi_{i+1}^{n}}{\Delta t}=\frac{\Gamma}{\rho}\frac{\phi_{i+2}^{n}-2\phi_{i+1}^{n}+\phi_{i}^{n}}{\Delta x^{2}}$$

(ϕ_{i+1}^{n} = 0, ϕ_{i+2}^{n} = 0, ϕ_{i+1}^{n} = 0, $\phi_{i}^{n}=\varepsilon$)

$$\phi_{i+1}^{n+1}=\varepsilon\left(\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^{2}}\right)$$

← **Physically reasonable**

For Point (i －1) at (n+1) instant：

$$\frac{\phi_{i-1}^{n+1}-\phi_{i-1}^{n}}{\Delta t}=\frac{\Gamma}{\rho}\frac{\phi_{i}^{n}-2\phi_{i-1}^{n}+\phi_{i-2}^{n}}{\Delta x^{2}}$$

(ϕ_{i-1}^{n} = 0, $\phi_{i}^{n}=\varepsilon$, ϕ_{i-1}^{n} = 0, ϕ_{i-2}^{n} = 0)

$$\phi_{i-1}^{n+1} = \varepsilon\left(\frac{\Gamma}{\rho}\frac{\Delta t}{\Delta x^2}\right)$$

Physically reasonable

$$\phi_{i+1}^{n+1} = \phi_{i-1}^{n+1}$$

Disturbance is transported onto two directions uniformly by diffusion term

7.4.3 Analysis of transport character (迁移特性) of discretized convective term

1. Definition－If a scheme can only transfer disturbance towards the downstream (下游) direction, then it possesses the transport character (具有迁移特性) ;
2. Analysis－Applying discrete disturbance analysis to an advection equation with the studied scheme;
3. CD does not possess transport character.

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} \rightarrow \frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -u\frac{\phi_{i+1}^n-\phi_{i-1}^n}{2\Delta x}$$

$(u>0)$

For Point (i+1) at (n+1) instant:

$$\frac{\phi_{i+1}^{n+1}-\phi_{i+1}^{n}}{\Delta t} = -u\frac{\phi_{i+2}^n-\phi_i^n}{2\Delta x} \rightarrow \phi_{i+1}^{n+1} = \left(\frac{u\Delta t}{2\Delta x}\right)\varepsilon$$

($\phi_{i+1}^{n} \rightarrow 0$, $\phi_{i+2}^{n} \rightarrow 0$, $\phi_i^n = \varepsilon$)

Disturbance is transferred downstream!

Physically reasonable!

For Point (i-1) at (n+1) instant:

$$\frac{\phi_{i-1}^{n+1}-\phi_{i-1}^{n}}{\Delta t} = -u\frac{\phi_i^n-\phi_{i-2}^n}{2\Delta x} \rightarrow \phi_{i-1}^{n+1} = -\left(\frac{u\Delta t}{2\Delta x}\right)\varepsilon \, ?$$

($\phi_{i-1}^{n} \rightarrow 0$, $\phi_i^n = \varepsilon$, $\phi_{i-2}^{n} \rightarrow 0$)

Disturbance is transferred upstream， and its sign is the opposite to the original one!

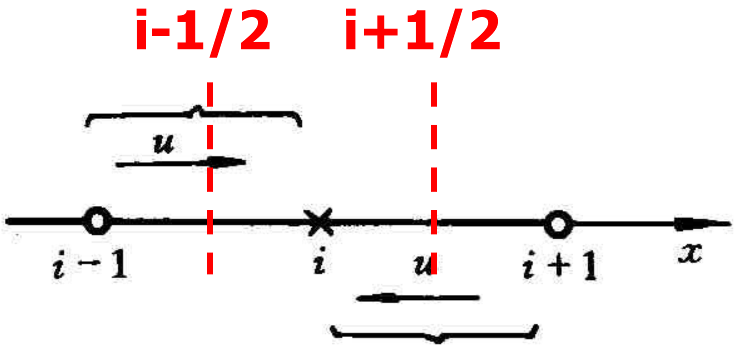
CD of convective term does not possess transport character!

7.4.4 Upwind scheme (迎风格式) of convective term possesses transport character

1. Definitions in FVM and FDM

$$\text{FDM}: \left.\frac{\partial \phi}{\partial x}\right)_i = \begin{cases} \dfrac{\phi_i - \phi_{i-1}}{\delta x}, & u > 0 \\ \dfrac{\phi_{i+1} - \phi_i}{\delta x}, & u < 0 \end{cases}$$

$$\text{FVM}: \phi_{i+1/2} = \begin{cases} \phi_i, & u > 0 \\ \phi_{i+1}, & u < 0 \end{cases}$$



2. FUD possesses transport character

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} \xrightarrow[\text{FDM}]{u>0} \frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -u\frac{\phi_i^n - \phi_{i-1}^n}{\Delta x}$$

For point (i+1) at (n+1) instant

$$\frac{\phi_{i+1}^{n+1}-\phi_{i+1}^{n}}{\Delta t} = -u\frac{\phi_{i+1}^{n}-\phi_{i}^{n}}{\Delta x}$$

0 0 $=\varepsilon$

Thus: $\phi_{i+1}^{n+1} = \varepsilon\left(\frac{u\Delta t}{\Delta x}\right)$ ← **Physically reasonable**

For point (i-1) at (n+1) instant:

$$\frac{\phi_{i-1}^{n+1}-\phi_{i-1}^{n}}{\Delta t} = -u\frac{\phi_{i-1}^{n}-\phi_{i-2}^{n}}{\Delta x}$$

0 0 0

Thus $\phi_{i-1}^{n+1} = 0$ ← **Physically required**

Disturbance is not transferred upstream; FUD possesses transport character.

7.4.5 Discussion on transportive character of discretized convective term

1. Transportiv character (T.C.) is an important property of discretized convective term; Those who possess T.C. are absolutely stable;

2. Within the stable range, CD is superior to (优于) FUD; Strong convection may lead solution by CD oscillating while solution by FUD is always physically plausible!

3. For those schemes who do not possess T.C. in order to get an absolutely stable solution the coefficients of the scheme should satisfy certain conditions. （替代教材73页4－5行的“凡是不具有迁移特性的对流项…因而只是条件地稳定）；

4. Numerical solution with FUD often has large FALSE-diffusion error; FUD is not recommended for the final solution; while in the debugging (调试) stage it may be used for its absolutely stability. Upwind idea once was widely used to construct higher-order schemes.

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7.5 Stability analysis of discretized diffusion-convection equation

7.5.1 Three kinds of instability in numerical simulation

1. Instability of explicit scheme for initial problem

Too large time step of explicit scheme will introduce oscillating results; Purpose of stability study is to find the allowed maximum time step; for 1-D diffusion problem:

$$\frac{a\Delta t}{\Delta x^2} \le 0.5$$

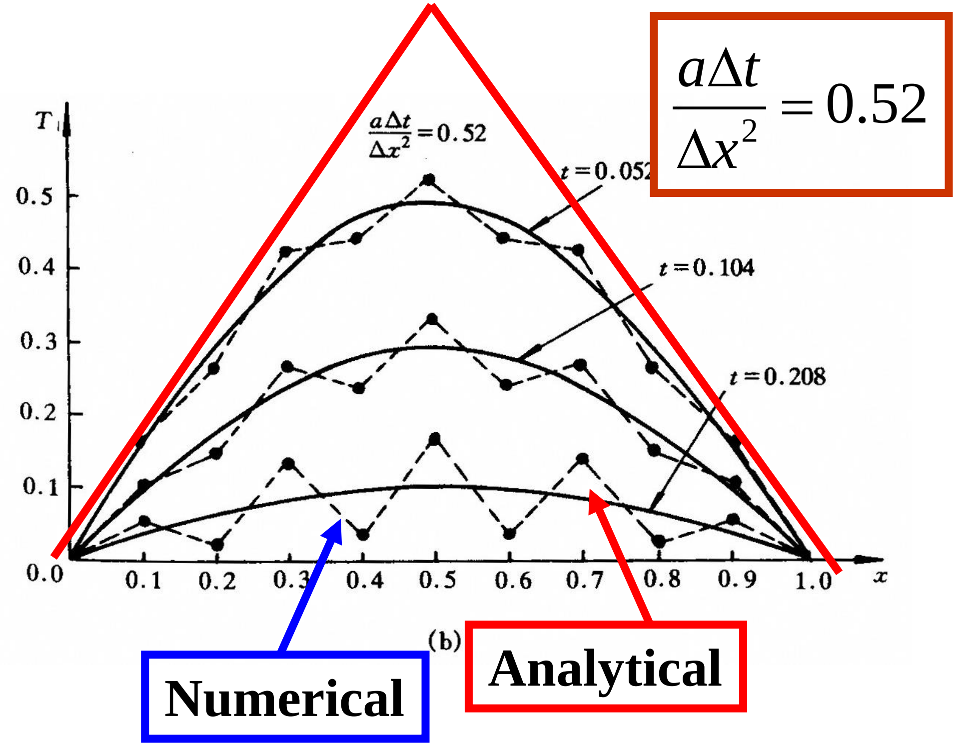
2. Instability of iterative solution procedure of ABEqs.

If iterative procedure can not converge, such procedure is called unstable! Unstable procedure can not get a solution!

3. Instability caused by discretized convective term

For CD, QUICK,TUD large space step, high velocity may cause to oscillating (wiggling) (振荡的) results. It is called **convective instability**. The purpose of stability study for convection scheme is to find the related critical Peclet number. The consequence (后果） of the three instabilities:

1. Transient instability of explicit scheme: oscillating solutions , and these are the actual solutions of the ABEqs. solved.



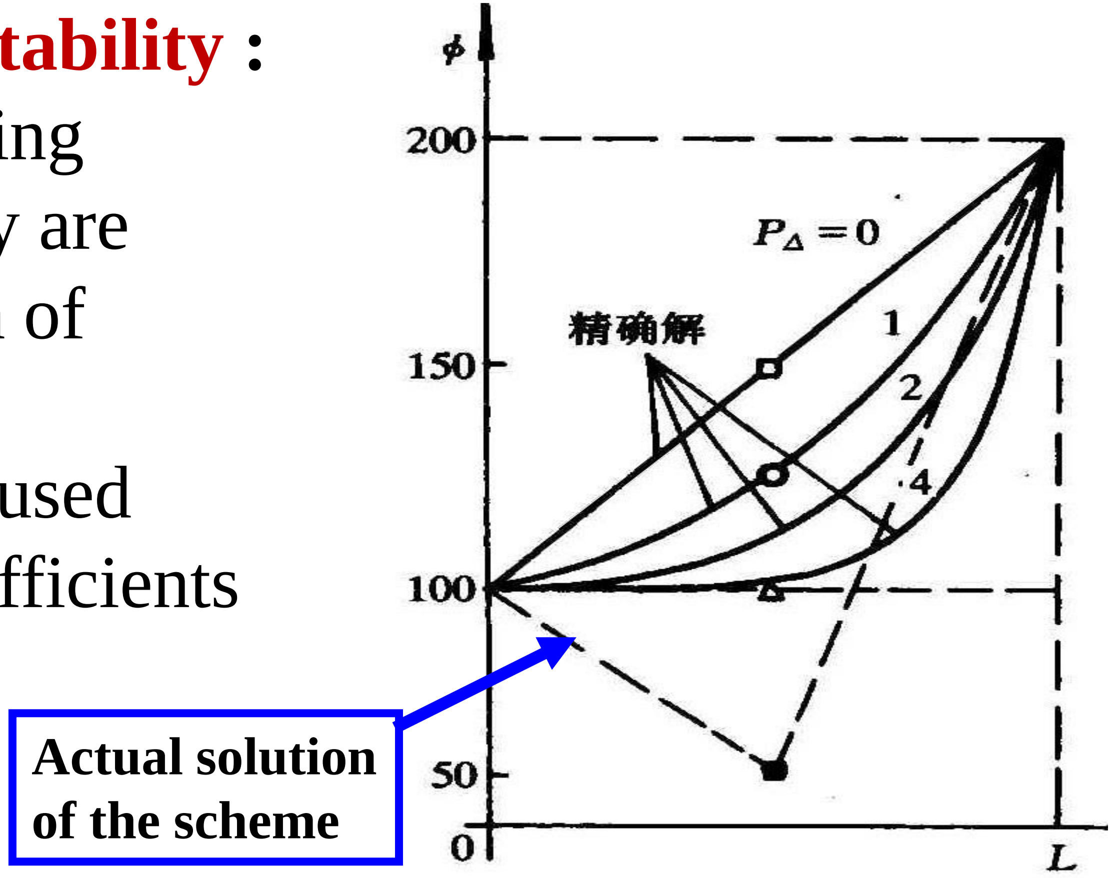
2. Instability of solution procedure for ABEqs.: no solution at all.

von Neumann method can be adopted to analyze such instability, see

Ni MJ, Tao WQ, Wang SJ .Stability analysis for discretized steady convective-diffusion equation. Numerical Heat Transfer, Part B,1999, 35 (3): 369-388

3. Convective instability :

leading to oscillating solutions and they are the actual solution of the ABEqs.
The problem is caused by unphysical coefficients of the discretized equations.



5.7.2 Sign preservation principle for analyzing convective instability

1. Basic idea:

An iterative solution procedure of the ABEqs. of diffusion-convection problem is a marching process (步进过程), from step to step, like the solution procedure of the explicit scheme of an initial problem;

If any disturbance (扰动) at a node is transported in such a way that its effect on the neighboring node is of the opposite sign（符号相反）then the final solution will be oscillating.

Tao W Q, Sparrow EM. The transportive property and convective numerical stability of the steady-state diffusion-convection finite difference equation. Numerical Heat Transfer, 1987, 11:491-497

Thus to avoid oscillating results we should require that any disturbance at a node should be transported in such a way that its effect on the neighboring nodes must have the same sign as the original disturbance, i.e., sign is preserved (符号不变).

2. Analysis method:

(1) The iterative solution procedure of the discretized diffusion-convection equation is modeled by the marching process of the explicit scheme of an initial problem;

(2) Stability is an inherent (固有的) character, which can be tested by adding any disturbance；

(3) The studied scheme is used to discretize the convection term of 1-D transient diffusion-convection

equation, and diffusion term is by CD; The transfer of a disturbance to the next time level is determined by the discrete disturbance analysis method.

(4) Stability of the scheme requires that the effect of any disturbance at any tine level on the neighboring point at the next time level **must has the same sign.**

3. Implementation procedure

(1) Applying the studied scheme to the explicit scheme of 1-D transient diffusion-convection equation ;

(2) Adopting the discrete disturbance analysis method to determine the transportation of disturbance ε_i^n introduced any time level n and node i ;

(3) Stability of the studied scheme requires:

$$\frac{\phi_{i\pm1}^{n+1}}{\varepsilon_i^n} \geq 0$$ **(Sign preservation principle，SPP)**

If above equation is unconditionally valid, the scheme is absolutely stable; Otherwise the condition that makes the above equation valid gives the critical Peclet number, beyond which the scheme will lead to osillating solution.

(4) We have shown that disturbance transportation by diffusion via CD is $\Gamma\Delta t / \rho\Delta x^2$ ， then discrete disturbance analysis can be only conducted for the studied convection scheme, and then adding the two effect terms together.

4. Implementation example

Stability analysis for TUD scheme:

$$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x} \quad \boxed{u \ge 0} \rightarrow \quad \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -u\frac{2\phi_{i+1}^n + 3\phi_i^n - 6\phi_{i-1}^n + \phi_{i-2}^n}{6\Delta x}$$

Disturbance analysis for the convection term

For node(i+1) (downstream):

$$\frac{\phi_{i+1}^{n+1} - \phi_{i+1}^n}{\Delta t} = -u\frac{2\phi_{i+2}^n + 3\phi_{i+1}^n - 6\phi_i^n + \phi_{i-1}^n}{6\Delta x}$$

($\phi_{i+1}^n \to 0$, $\phi_{i+2}^n \to 0$, $\phi_{i+1}^n \to 0$, $\phi_i^n \to \varepsilon_i^n$, $\phi_{i-1}^n \to 0$)

Thus: $$\phi_{i+1}^{n+1} = \left(\frac{u\Delta t}{\Delta x}\right)\varepsilon_i^n$$ ← **Physically reasonable**

Disturbance is transported by convection downstream!

For node (i－1) (upstream):

$$\frac{\phi_{i-1}^{n+1}-\phi_{i-1}^{n}}{\Delta t}=-u\frac{2\phi_i^n+3\phi_{i-1}^n-6\phi_{i-2}^n+\phi_{i-3}^n}{6\Delta x}$$

($\phi_i^n \to \varepsilon_i^n$; $\phi_{i-1}^n \to 0$, $\phi_{i-1}^n \to 0$, $\phi_{i-2}^n \to 0$, $\phi_{i-3}^n \to 0$)

Thus: $\phi_{i-1}^{n+1}=-\frac{1}{3}(\frac{u\Delta t}{\Delta x})\varepsilon_i^n$

Disturbance is transported upstream with opposite sign!

For node(i+1): $\dfrac{\frac{\Gamma\Delta t}{\Delta x^2}+(\frac{u\Delta t}{\Delta x})\varepsilon_i^n}{\varepsilon_i^n}>0$ **Automatically satisfied!**

For node (i−1):

$$\dfrac{\frac{\Gamma\Delta t}{\Delta x^2}-\frac{1}{3}(\frac{u\Delta t}{\Delta x})\varepsilon_i^n}{\varepsilon_i^n}\geq 0$$

Valid only when $\frac{\rho u\Delta x}{\Gamma}\leq 3$

$$\Rightarrow \frac{\rho u\Delta x}{\Gamma}=P_{\Delta cr}\equiv 3!$$

Leonard (1981) once analyzed the stability character of TUD and concluded that it is inherently stable, However numerical practice shows it is only conditionally stable.

5. Summary of analysis results

Stability of seven schemes (Table 5-3 of Textbook)

No	Scheme	Definition of scheme	Transferred by convection		Stability condition
			Up	Down	
1	**FUD**	$\left.\frac{\partial \phi}{\partial x}\right\|_i \cong \frac{\phi_i - \phi_{i-1}}{\Delta x}, u>0$ $\frac{\phi_{i+1} - \phi_i}{\Delta x}, u<0$	0	$\left(\frac{u\Delta t}{\Delta x}\right)\epsilon$	**Abs. stable**
2	**CD**	$\left.\frac{\partial \phi}{\partial x}\right\|_i \cong \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}$	$-\frac{1}{2}\left(\frac{u\Delta t}{\Delta x}\right)\epsilon$	$\frac{1}{2}\left(\frac{u\Delta t}{\Delta x}\right)\epsilon$	$P_\Delta \leqslant 2$

No	Scheme	Definition of scheme	Transferred by convection		Stability condition
			Up	Down	
3	**SUD**	$\left.\frac{\partial\phi}{\partial x}\right\|_i \cong \frac{\phi_i-\phi_{i-1}}{\Delta x}+\frac{\phi_i-2\phi_{i-1}+\phi_{i-2}}{2\Delta x}, u>0$ $\frac{\phi_{i+1}-\phi_i}{\Delta x}+\frac{\phi_i-2\phi_{i+1}+\phi_{i+2}}{2\Delta x}, u<0$	0	$2\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	**Abs. stable**
4	**TUD**	$\left.\frac{\partial\phi}{\partial x}\right\|_i \cong \frac{2\phi_{i+1}+3\phi_i-6\phi_{i-1}+\phi_{i-2}}{6\Delta x}, u>0$ $\frac{-\phi_{i+2}+6\phi_{i+1}-3\phi_i-2\phi_{i-1}}{6\Delta x}, u<0$	$-\frac{1}{3}\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	$\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	$P_\Delta \leqslant 3$
5	**Fromm**	$\phi_{i+1/2}=\frac{1}{4}(\phi_{i+1}+4\phi_i-\phi_{i-1})$	$-\frac{1}{4}\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	$\frac{1}{4}\left(\frac{u\Delta t}{\Delta x}\right)\varepsilon$	$P_\Delta \leqslant 4$

<table>
<tr><th rowspan="2">No</th><th rowspan="2">Scheme</th><th rowspan="2">Definition of scheme</th><th colspan="2">Transferred by convection</th><th rowspan="2">Stability condition</th></tr>
<tr><th>Up</th><th>Down</th></tr>
<tr><td>6</td><td>QUICK</td><td>$\phi_{i+1/2}=\frac{\phi_i+\phi_{i+1}}{2}-\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{8}, u>0$
$\frac{\phi_i+\phi_{i+1}}{2}-\frac{\phi_{i+2}-2\phi_{i+1}+\phi_i}{8}, u<0$</td><td>$\frac{-3}{8}$
$\left(\frac{u\Delta t}{\Delta x}\right)_\varepsilon$</td><td>$\frac{7}{8}$
$\left(\frac{u\Delta t}{\Delta x}\right)_\varepsilon$</td><td>$P_\Delta \leqslant \frac{8}{3}$</td></tr>
<tr><td>7</td><td>Expon. scheme</td><td>Discretized form of 1-D diffusion-convection eq.
$a_P\phi_P = a_E\phi_E + a_W\phi_W$
$a_E=\frac{\rho u}{\exp(P_\Delta)-1}, \; a_W=\frac{\rho u \exp(P_\Delta)}{\exp(P_\Delta)-1}$
$a_P=a_E+a_W+a_P^0, \; b=a_P^0\phi_P^0, \; a_P^0=\frac{\rho\Delta x}{\Delta t}$</td><td>Total effects of Dif-Con
$\frac{a_E}{a_P^0}\varepsilon(\geqslant 0)$</td><td>$\frac{a_W}{a_P^0}\varepsilon(\geqslant 0)$</td><td>Abs. stable</td></tr>
</table>

7.5.3 Discussion on the analysis results of schemes

1) For those schemes possessing transportive property the SPP is always satisfied, and the schemes are absolutely stable, such as **FUD**, **SUD**;

2) For those schemes containing downstream node they do not possess transportive property, and are often conditionally stable. Only when the coefficients in the interpolation satisfy certain conditions they can be absolutely stable: **CD**, **TUD**, **QUICK**, **FROMM**;

3) For conditionally stable schemes, the larger the coefficients of the downstream nodes the smaller the critical Peclet number.

CD: $\phi_e = \dfrac{\phi_E + \phi_P}{2}$; For situation of positive velocity,

Coefficient of downstream node is 1/2, $P_{\Delta cr} = 2$

QUICK: $\phi_{i+1/2} = \dfrac{1}{8}(3\phi_{i+1} + 6\phi_i - \phi_{i-1})$

Coefficient of downstream node is 3/8, $P_{\Delta cr} = 8/3$

TUD: $\dfrac{\partial \phi}{\partial x})_i = \dfrac{2\phi_{i+1} + 3\phi_i - 6\phi_{i-1} + \phi_{i-2}}{6\Delta x}$

Coefficient of downstream node is 2/6, $P_{\Delta cr} = 6/2$

$= 3$

FROMM: $\phi_e = \dfrac{1}{4}(\phi_{i+1} + 4\phi_i - \phi_{i-1})$

Coefficient of downstream node is 1/4, $P_{\Delta cr} = 4$

There is some inherent relationship!

4) All the above analyses for convective stability are based on the following conditions:

(1) 1-D problem;

(2) Linear problem（u, Γ are known constants)；

(3) Two-point boundary problem；

(4) No non-constant source term；

(5) Uniform grid system；

(6) Diffusion term is discretized by CD.

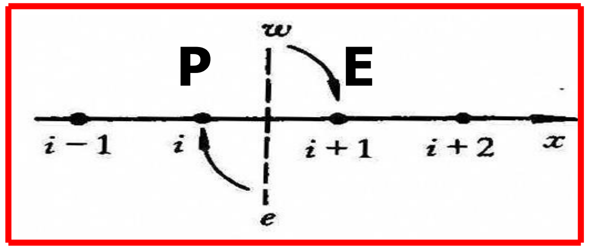
The resulted critical Peclet is the smallest； Violation (违反) of any above condition will enhance stability.(20221102)

Key Points of Last Lecture

1. Conservation of the discretization equation

If the summation of a certain number of discretized equations satisfies conservation requirement , then they are said to possess conservation.

2. Condition for satisfaction of conservation feature



$$(\phi_e)_P = (\phi_w)_E; \;\; [(\delta\phi/\delta x)_e]_P = [(\delta\phi/\delta x)_w]_E$$

3. Transportive property of a convection scheme

If a convection scheme can only transfer disturbance towards the downstream direction, then it possesses the transport character.

4. Discrete disturbance analysis

Assuming that a disturbance ε occurs at a point *i* and some instant *n*, while at all other points and all subsequent time levels no any disturbances; Analyzing the transfer of the disturbance by the studied scheme.

5. Sign preservation principle $\phi_{i\pm1}^{n+1} / \varepsilon_i^n \geq 0$

7.5.4 Summary of discussion on convective scheme

1. For conventional fluid flow and heat transfer problems, in the debugging process (调试过程) FUD or PLS may be used; For the final computation QUICK or SGSD is recommended, and defer correction may be used for solving the ABEqs.

2. For direct numerical simulation(DNS) of turbulent flow, schemes of fourth order or more are often used;

3. When there exists a sharp variation of properties, higher order and bounded schemes (高阶有界格式) should be used.

Recent advances in scheme construction of FVM can be found in:

Jin W W, Tao W Q. Numerical Heat Transfer, Part B, 2007, 52(3): 131-254
Jin W W, Tao W Q. Numerical Heat Transfer, Part B, 2007, 52(3): 255-280

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Question & Answer

Time： From 8:30-9:40 PM November 08,2022

Tencent (腾讯会议)：**880-621-895**

Welcome you to attend!

Home Work 7（2022-2023）

Please finish your homework independently !!!

Please hand in on Nov. 16

Problem 7-1

A general Crank-Nicolson scheme of the 1-D transient heat conduction is as follows,

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=a\left[\theta\frac{T_{i+1}^{n+1}-2T_i^{n+1}+T_{i-1}^{n+1}}{(\Delta x)^2}+(1-\theta)\frac{T_{i+1}^n-2T_i^n+T_{i-1}^n}{(\Delta x)^2}\right]$$

where θ is the weighting factor, $0\le\theta\le1$.

Using von-Neumann analysis method to find the initial stability conditions for the scheme.

Problem 7-2（Problem 3-3 in Textbook）

In the 2-D diffusion-convection equation:

$$\rho\frac{\partial\phi}{\partial t}+\rho(u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y})=\Gamma(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2})$$

u,v,ρ,Γ all are known constants.

Its one discretized scheme is as follows:

$$\rho\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n}}{\Delta t}+\rho u\frac{\phi_{i,j}^{n}-\phi_{i-1,j}^{n}}{\Delta x}+\rho v\frac{\phi_{i,j}^{n}-\phi_{i,j-1}^{n}}{\Delta y}=$$

$$\Gamma\frac{\phi_{i+1,j}^{n}-2\phi_{i,j}^{n}+\phi_{i-1,j}^{n}}{\Delta x^2}+\Gamma\frac{\phi_{i,j+1}^{n}-2\phi_{i,j}^{n}+\phi_{i,j-1}^{n}}{\Delta y^2}$$

By applying von Neumann analysis method to show that the stability condition is :

$$\Delta t \le \frac{1}{\frac{2a}{\Delta x^2}+\frac{2a}{\Delta y^2}+\frac{u}{\Delta x}+\frac{v}{\Delta y}},\ a=\frac{\Gamma}{\rho}$$

Problem 7-3

Discuss the transportive property for the 2nd-order and 3rd-order upwind schemes.

Problem 7-4

Apply the sign-preservation principle to analyze the convective stability for the QUICK scheme.

Problem 7-5 (Problem 3-5 of Textbook)

Judge the conservative property of the following two discretized equations of continuity for incompressible fluid flow:

$$(1)\ \frac{u_{i+1,j}+u_{i+1,j-1}-u_{i,j}-u_{i,j-1}}{2\Delta x}+\frac{v_{i+1,j}-v_{i+1,j-1}}{\Delta y}=0;$$

$$(2)\ \frac{u_{i+1,j}-u_{i,j-1}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}=0;$$

本组网页地址：http://nht.xjtu.edu.cn 欢迎访问！

Teaching PPT will be loaded on ou website

同舟共济

渡彼岸！

People in the same boat help each other to cross to the other bank, where....