Numerical Heat Transfer

(数值传热学)

Chapter 6 Solution Methods for Algebraic Equations

Instructor Tao, Wen-Quan

Key Laboratory of Thermo-Fluid Science & Engineering
Int. Joint Research Laboratory of Thermal Science & Engineering
Xi'an Jiaotong University
Innovative Harbor of West China, Xian

2019-Oct-21

6.1 Introduction to Solution Methods of ABEqs

6.2 Construction of Iteration Methods of Linear Algebraic Equations

6.3 Convergence Conditions and Acceleration Methods for Solving Linear ABEqs.

6.4 Block Correction Method –Promoting Conservation Satisfaction

6.5 Multigrid Techniques –Promoting Simultaneous Attenuation of Different Wave-length Components

6.1 Introduction to Solution Methods of ABEqs

6.1.1 Matrix feature of multi-dimensional discretized equation

6.1.2 Direct method and iteration method for solving ABEqs.

6.1.3 Major idea and key issues of iteration methods

6.1.4 Criteria for terminating iteration

6.1 Introduction to Solution Methods of ABEqs

6.1.1 Matrix feature of multi-dimensional discretized equation

For 2-D, 3-D flow and heat transfer problems, the discretized equations with 2nd order accuracy:

2-D $a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b$

3-D $a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + a_F\phi_F + a_B\phi_B + b$

For a 2D case with L1XM1unknown variables, the general algebraic equation of kth variable is:

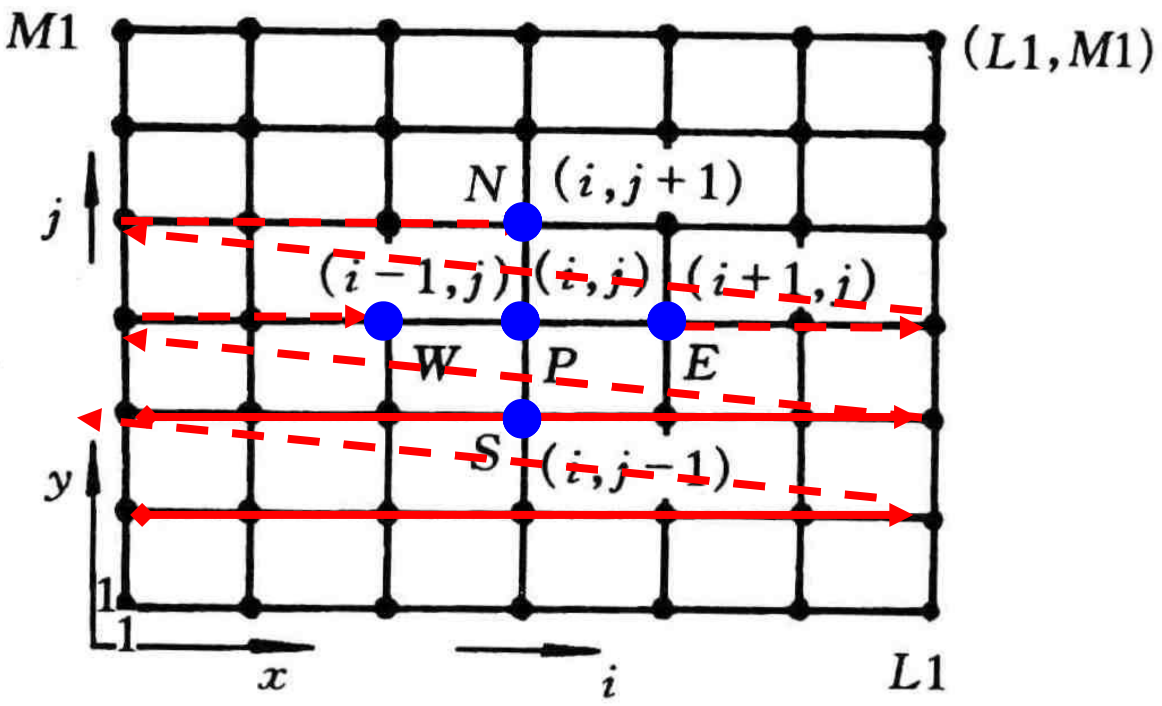
$$a_{k,1}\phi_{,1} + a_{k,2}\phi_2 + ... + a_{k,k-L1}\phi_{k-L1} + a_{k,k-L1+1}\phi_{k-L1+1} + ... + a_{k,k-1}\phi_{k-1}$$

$$+ a_{k,k}\phi_k + a_{k,k+1}\phi_{k+1} + ... + a_{k,k+L1}\phi_{k+L1} + ... + a_{k,L1\bullet M1}\phi_{L1\bullet M1} = b_k$$

For 2-D problem with 2nd order accuracy there are only five coefficients at the left hand side are not equal to zero, and the matrix is of quasi (准)five-diagonal, a large scale sparse matrix (大型稀疏矩阵).

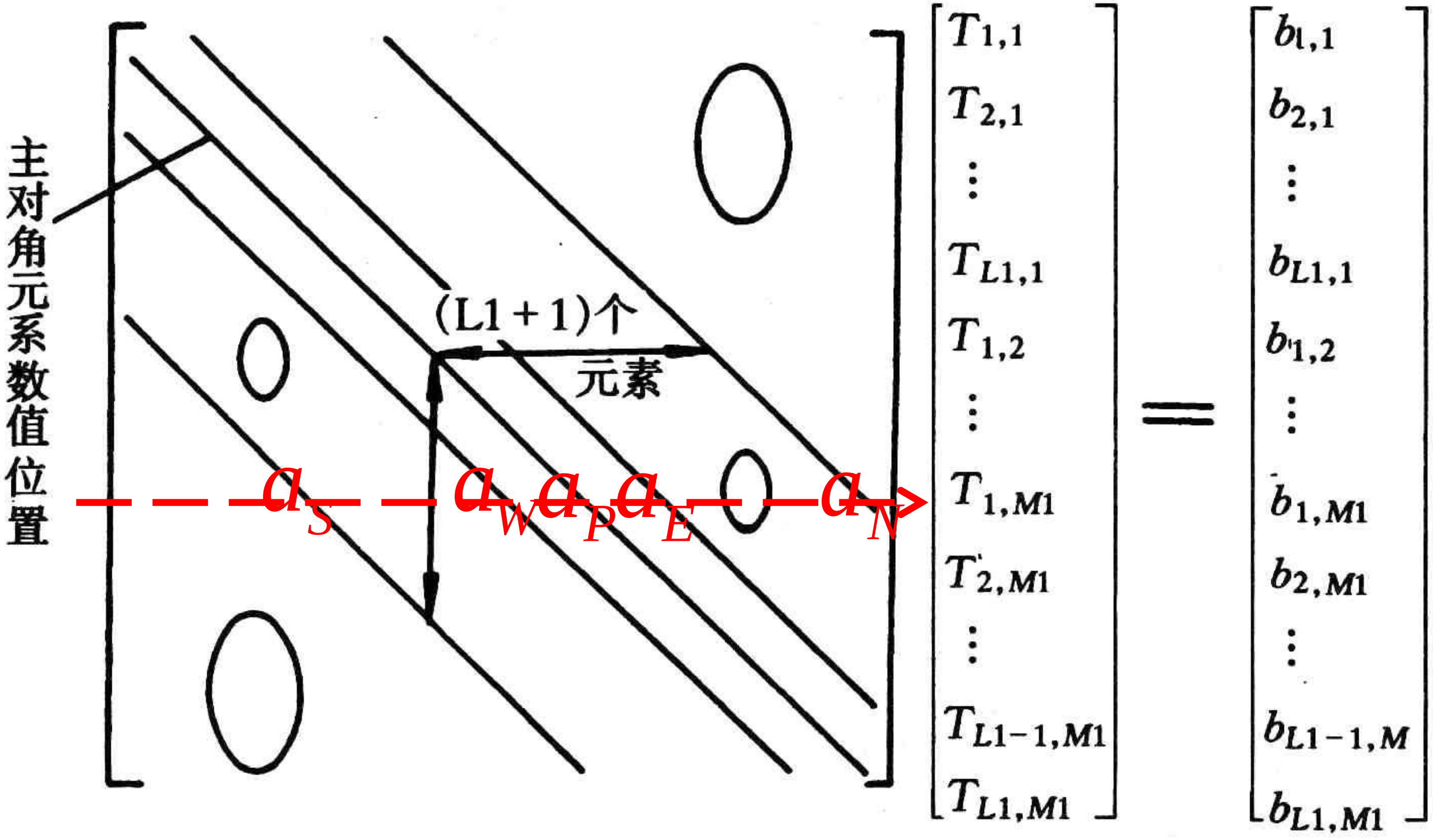
If the 1-D storage of the coefficients is conducted as shown right, then the order of coefficients in one line are:

$$a_S, a_W, a_P, a_E, a_N \longrightarrow$$



$$\cancelto{0}{a_{k,1}\phi_{,1}} + \cancelto{0}{a_{k,2}\phi_2} + \ldots. + \underline{a_{k,k-L1}\phi_{k-L1}} + \cancelto{0}{a_{k,k-L1+1}\phi_{k-L1+1}} + \ldots + \underline{a_{k,k-1}\phi_{k-1}}$$

$$\underline{+a_{k,k}\phi_k} + \underline{a_{k,k+1}\phi_{k+1}} + \ldots + \underline{a_{k,k+L1}\phi_{k+L1}} + \ldots + \cancelto{0}{a_{k,L1\times M1}\phi_{L1\times M1}} = b_k$$



Features of ABEqs. of discretized multi-dimensional flow and heat transfer problems:

1) For conduction of const. properties in uniform grid—matrix is symmetric and positive definite(正定、对称)；

2) For other cases: matrix is neither symmetric nor positive definite.

ABEqs. of large scale sparse matrix are usually solved by iteration methods.

6.1.2 Direct method and iterative method for solving ABEqs.

1.Direct method(直接法)

Accurate solution can be obtained via a finite times of operations if there is no round-off error, such as TDMA，PDMA.

2. Iterative method（迭代法)

From an initial field the solution is progressively improved via the ABEqs. and terminated when a pre-specified criterion is satisfied.

The ABEqs. of fluid flow and heat transfer problems usually are solved by iteration methods:

1)Non-lineairity of the problems， the coefficients need to be updated. There is no need to get the true solution for temporary （临时的） coefficients;

2) The operation times of direct method is proportional to $N^{2.5\sim3}$， where N is the number of unknown variables. When N is very large the operation times becomes very very large, often unmanageable! （2019.10.15）

6.1.3 Major Idea and Key Issues of Iteration Methods

1. Major idea

In matrix form the ABEqs. is ： $\overrightarrow{A}\vec{\phi}=\vec{b}$. Its solution is $\vec{\phi}=(\overrightarrow{A})^{-1}\vec{b}$. Iteration method is to construct a series of $\overrightarrow{\phi^{k}}$ in multi-dimensional space R (the number of dimensions equals the number of unknowns) such that

when $k\rightarrow\infty$ $\vec{\phi}^{(k)}\rightarrow(\overleftarrow{A})^{-1}\vec{b}$

For the kth iteration $\vec{\phi}^{(k)}=f(\overrightarrow{A},\vec{b},\vec{\phi}^{(k-1)})$

2. Key issues of iteration methods

1) How to construct the iteration series?

2) Is the series converged?

3) How to accelerate the convergence speed?

6.1.4 Criteria for terminating (inner) iteration

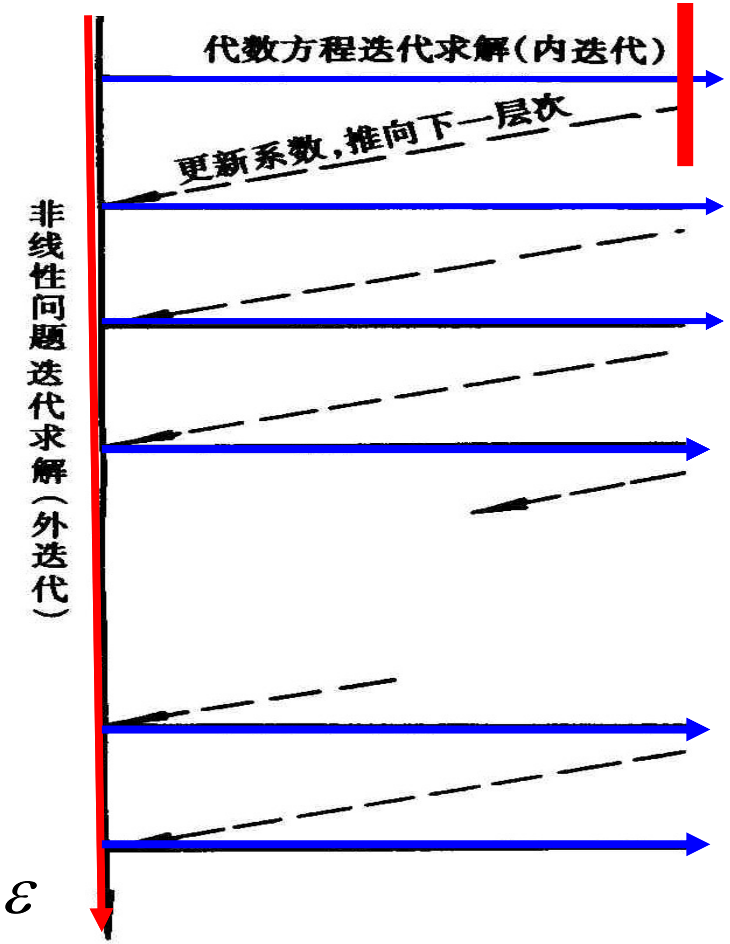
(1) Specifying iteration number;

(2) Specifying the norm of p’eq. residual less than a certain small value;

(3) Specifying the relative norm of p’eq. residual less than a certain small value;

(4) Specifying relative change of variable less than a small value;

$$\left|\frac{\phi^{(k+1)}-\phi^{(k)}}{\phi^{(k+1)}_{\max}}\right|_{\max}\le\varepsilon;\quad \left|\frac{\phi^{(k+1)}-\phi^{(k)}}{\phi^{(k+1)}+\varepsilon_0}\right|_{\max}\le\varepsilon$$



6.2 Construction of Iteration Methods of Linear Algebraic Equations

6.2.1 Point (explicit) iteration

6.2.2 Block (implicit) iteration

6.2.3 Alternative direction iteration－ADI

6.2 Construction of Iteration Methods of Linear Algebraic Equations.

6.2.1 Point (explicit) iteration

The updating is conducted from node to node; After every node has been visited a cycle (轮) of iteration is finished; The updated value at each node is explicitly related to the others (values of previous iteration).

1. Jakob iteration

In the updating of every node the previous cycle values of neighboring nodes are used; The convergence speed is independent of iteration direction.

2. Gauss—Seidel iteration

Present values are used for updating.

3. SOR/SUR iteration

$$\phi^{(k+1)} = \phi^{(k)} + \alpha(\phi^{(k+1)} - \phi^{(k)}) \begin{cases} \alpha < 1 \text{ Under-} \\ (0 \le \alpha \le 2) \\ \alpha > 1 \text{ Over-} \end{cases}$$

Remarks: This relaxation is for solving the linear ABEqs., Not for the non-linearity.

6.2.2 Block (implicit) iteration （块(隐式)）

1. Basic idea

Dividing the solution domain into several regions, within each region direct solution method is used, while from block to block iteration is used, also called implicit iteration.

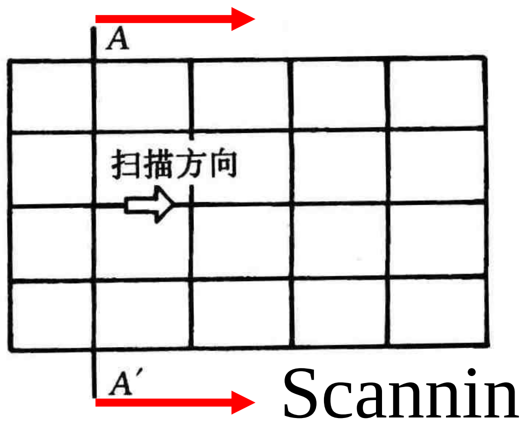
2. **Line iteration** （线迭代)-the most fundamental of block iteration

The smallest block is a line: At the same line TDMA is used for direct solution, from line to line iterative method is used.

Solving in N-S direction and scanning (扫描) in E-W D.:

Jakob: $$a_P\phi_P^{(k+1)} = a_N\phi_N^{(k+1)} + a_S\phi_S^{(k+1)} + [\underline{a_E\phi_E^{(k)} + a_W\phi_W^{(k)} + b}]$$

G-S: $$a_P\phi_P^{(k+1)} = a_N\phi_N^{(k+1)} + a_S\phi_S^{(k+1)} + [\underline{a_E\phi_E^{(k)} + a_W\phi_W^{(k+1)} + b}]$$

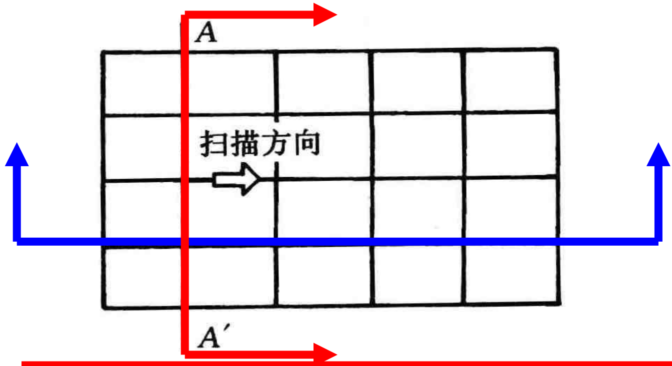


Scanning （扫描） in E-W direction

6.2.3 Alternative direction iteration－ADI

1. Basic idea

First direct solution for each row(行)（or column 列），then direct solution for each column（or row)；The combination of the two updating of the entire domain consists of one cycle iteration:



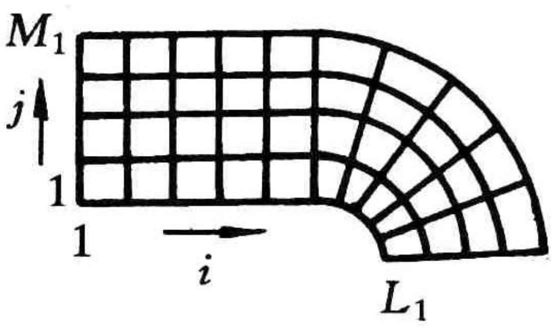
Alternative direction iteration (ADI) vs. alternative direction implicit (ADI)：

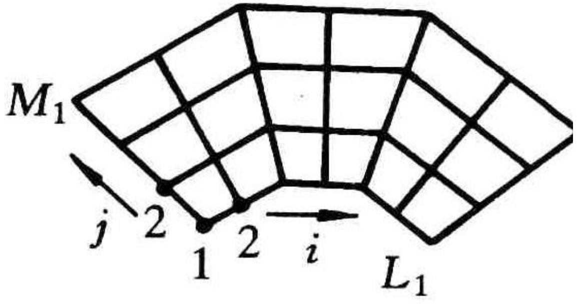
It can be shown that: one-time step forward of transient problem is equivalent to one cycle iteration for steady problem (see appendix).

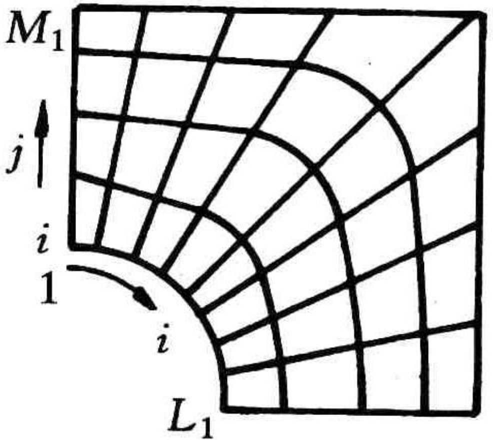
Therefore ADI-iteration of solving multi-dimensional steady problem for one iteration (ADI-iteration) is very similar to the ADI-implicit of solving multidimensional unsteady problem for one time step (ADI-implicit).

2. ADI-line iteration is widely adopted in the numerical solution of flow and heat transfer problem.

ABEqs. generated on structured grid system can be solved by ADI.







6.3 Convergence Conditions and Acceleration Methods for Solving Linear ABEqs.

6.3.1 Sufficient condition for iteration convergence of Jakob and G-S iteration

6.3.2 Analysis of factors influencing iteration convergence speed

6.2.3 Methods for accelerating transferring boundary condition influence into solution domain

6.3 Convergence Conditions and Acceleration Methods for Solving Linear ABEqs.

6.3.1 Sufficient condition for iteration convergence of Jakob and G-S iteration

1. Sufficient condition－Scarborough criterion

Coefficient matrix is non-reducible (不可约), and is diagonally predominant (对角占优)：

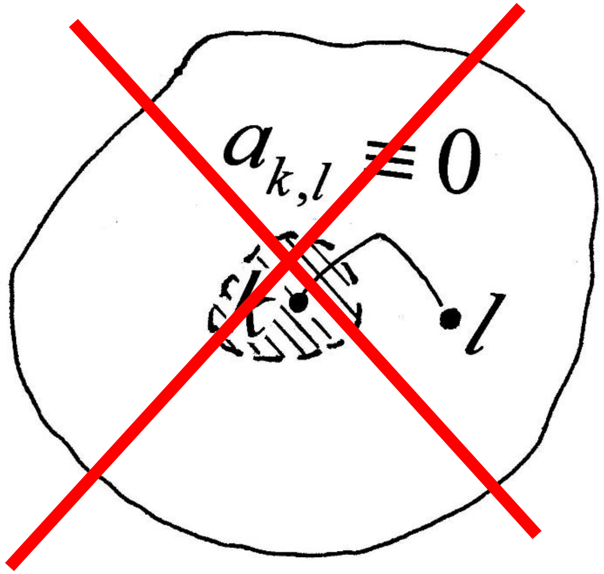
$$\frac{\sum |a_{nb}|}{|a_P|} \le 1 \begin{cases} \le 1 & \text{for all equations} \\ < 1 & \text{at least for one equations} \end{cases}$$

2. Analysis of coefficients of discretized diffusion-convection equation by the recommended method

1) **Matrix is non-reducible**－If matrix is reducible then the set (集合)　of coefficients subscript (矩阵下标) ,W , can be divided into two non-empty (非空) sub-sets, R and S， W＝R＋S， and for any element from R and S, say *k* and *l* respectively， we must have： $a_{k,l} \equiv 0$ ； If such condition does not exist, then the matrix is called non-reducible (不可约)

Analysis： Coefficient of discretized equation represents the influence of neighboring nodes. For nodes in elliptic region any one must has its effects on its neighbors; If matrix is reducible it implies that the computational domain can be divided into two regions which do not affect each other---totally impossible .

Non-reducible matrix is determined by the physical fact that neighboring parts in flow and heat transfer are affected each other.



2) Diagonally predominant—Coefficients constructed in the present course must satisfy this condition:

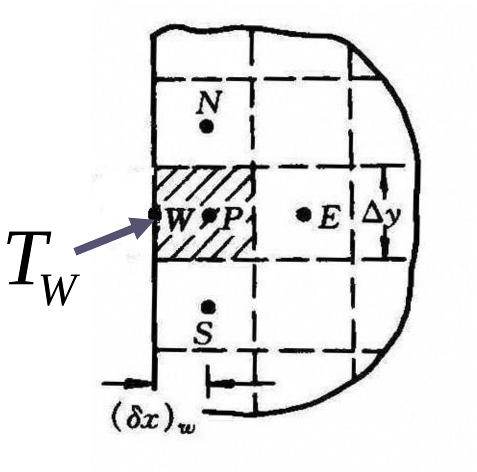
(1) Transient and fully implicit scheme

$$a_P = \sum a_{nb} + a_P^0 - S_P \Delta V \ \ , a_P^0 > 0, - S_P > 0 \ \ , a_P > \sum a_{nb}$$

(2) Steady problem with non-constant source term

$$- S_P > 0 \ \ , a_P > \sum a_{nb}$$

(3) Steady problem without source term



For inner grids: $a_P = \sum a_{nb}$

At least one node in the boundary can be found to satisfy : $a_P > \sum a_{nb}$

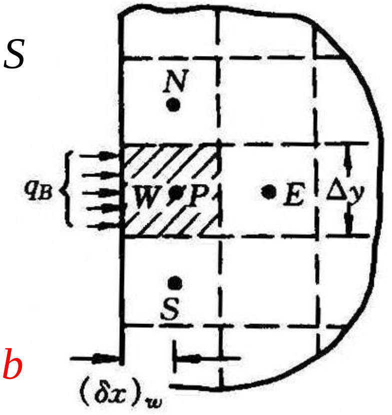
1)Assuming that T_w is known, then when the eq.

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b$$

is solved, it becomes:

$$a_P T_P = a_E T_E + 0 + a_N T_N + a_S T_S + (b + a_W T_W)$$

Hence here: $a_P = \sum a_{nb} > a_E + 0 + a_N + a_S$



2) For 3rd kind boundary condition, additional source term helps

$-S_P > 0$, $a_P = \sum a_{nb} - (-|S_P|) > \sum a_{nb}$

It is impossible that all boundary nodes are of 2nd type, at least one node is of 1st or 3rd type. Otherwise there is no definite solution!

Thus numerical methods recommended by the present course must satisfy this sufficient condition

6.3.2 Analysis of factors influencing iteration convergence speed

1. Transferring effects of B.C. into domain---View P.1

The steady state heat conduction with constant properties are governed by Laplace equation, $\nabla^2 \phi = 0$ for which a uniform field satisfies. However, it is not the solution because B.C. is not satisfied.

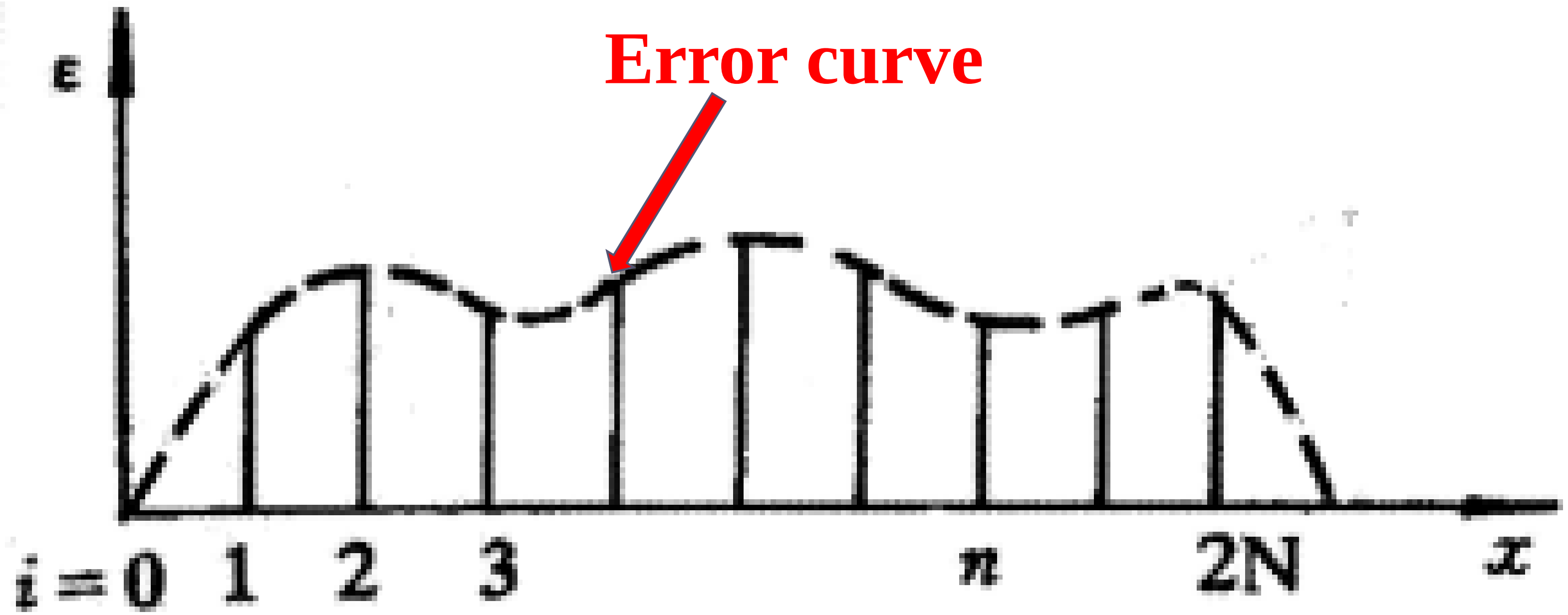
Thus the transferring speed for the effects of boundary condition must affect iteration convergence speed.

2. Satisfaction of conservation condition---View P.2

For a problem with 1st kind boundary condition, it is possible to incorporate all the known boundary values into the initial field, but such an initial field does not satisfy conservation condition. Thus techniques which is in favor of satisfying conservation condition can accelerate convergence speed;

3. Attenuation (衰减) of error vector---View P.3

The error vector is attenuated during iteration. Error vector is composed of components of different frequency. Techniques which can uniformly attenuate different components must can accelerate convergence speed.



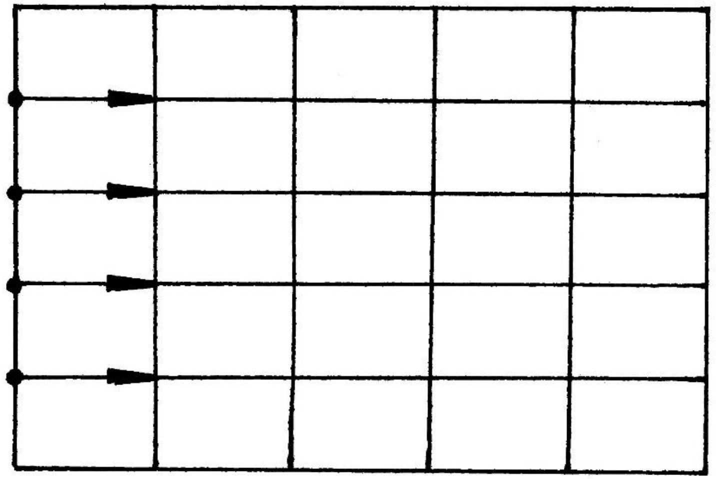
Taking the numerical error of each node as a component of a vector, then all the error components consist a vector, called error vector.

The error curve can be decomposed by a number of sine/cosine components with different frequencies.

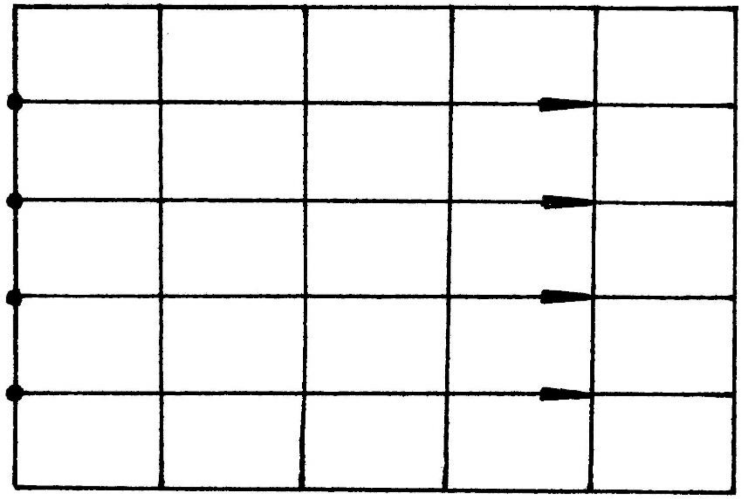
4. Increasing percentage of direct solution---View P.4

Direct solution is the most strong technique that both conservation and boundary condition can be satisfied. Thus appropriately increasing direct solution proportion is in favor of accelerating convergence speed.

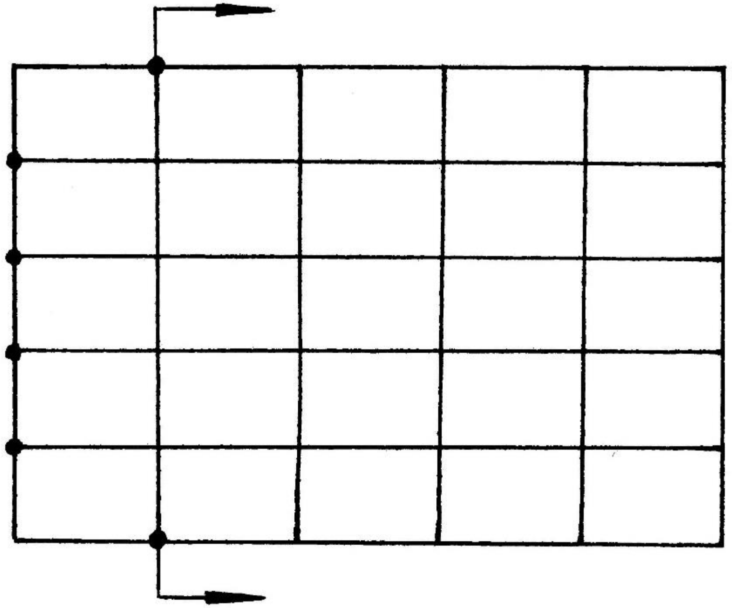
6.3.3 Techniques for accelerating transferring B.C. effects



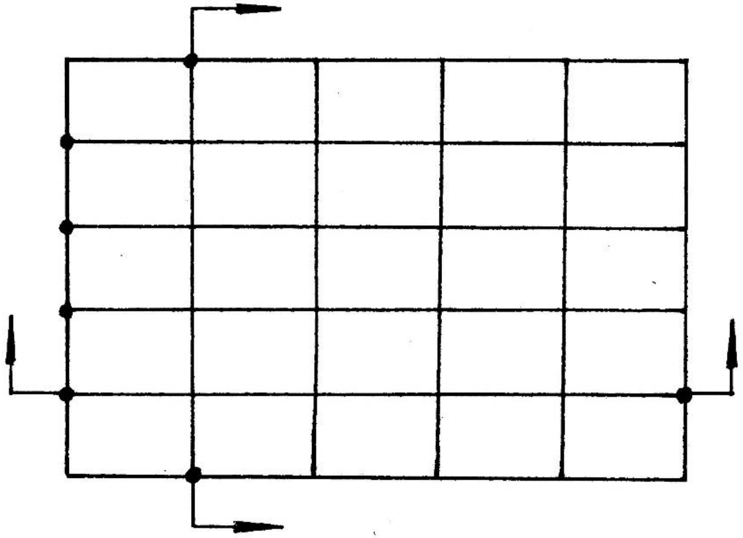
Jakob iteration: In each cycle the effect of B.P. can transfer into inner region by one space step. Very low convergence speed.



G-S iteration: The effects of the iteration starting boundary are transferred into the entire domain; Convergence speed is accelerated.



Line iteration: The effects of iteration starting boundary and the related two end boundaries are all transferred into the entire domain; convergence speed is further accelerated.



ADI line iteration： In every cycle iteration effects of all the boundaries are transferred into the entire domain. The fastest convergence speed.

ADI line iter.>Line iter.>G-S iter.>Jakob iter.

Jakob iteration has the slowest convergence speed. That is the change between two successive iterations is the smallest; This feature is in favor of iteration convergence for highly non-linear problems when iteration cycle number is specified. In the SIMPLEST algorithm, Jakob iteration is used for the convective part of ABEqs.

6.4 Block Correction Method –Promoting Satisfaction of Conservation

6.4.1 Necessity for block correction technique

6.4.2 Basic idea of block correction

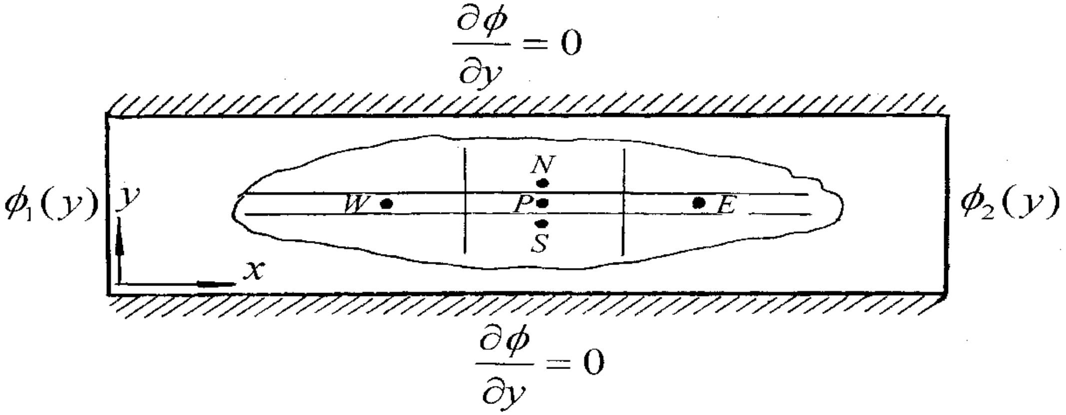
6.4.3 Single block correction and the boundary condition

6.4.4 Remarks of application of B.C. Technique

6.4 Block Correction Method –Promoting Satisfaction of Conservation

6.4.1 Necessity for block correction technique

For 2-D steady heat conduction shown below when ADI is used to solve the ABEqs. convergence speed is very low： EW boundaries have the strongest effect because of 1st kind boundary, but the influencing coefficient is small ； N-S boundary is adiabatic, no definite information can offer, but has larger coefficient－Thus to accelerate convergence of solving ABEqs., a special method is needed



6.4.2 Basic idea of block correction

Physically, iteration is a process for satisfying conservation condition； In one cycle of iteration, a correction, ϕ', is added to previous solution, ϕ^*, which does not satisfy conservation condition, such that ($\phi^* + \phi'$) can satisfy conservation condition better. The process of solving ABEqs. of ϕ' is the process of getting ϕ' .

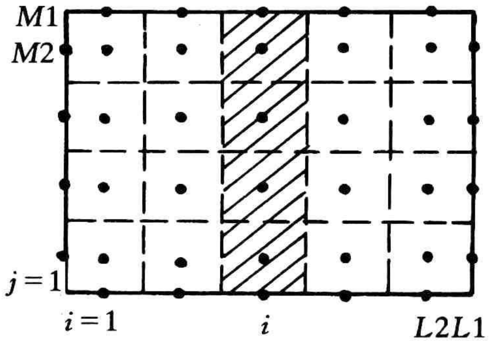
For 2-D problem, corrections are also of 2-D; In order that only 1-D corrections are solved, corrections are somewhat averaged for one block, denoted by $\bar{\phi}'_i$ or $\bar{\phi}'_j$, and it is required that ($\phi^*_{i,j} + \bar{\phi}'_i$) or ($\phi^*_{i,j} + \bar{\phi}'_j$) satisfies the conservation condition.

6.4.3 Single block correction and the boundary condition

1.Equation for correction:

It is required that: $(\phi^*_{i,j} + \overline{\phi}{}'_i)$ satisfy following eq.

$$\sum_j AP(\phi^*_{i,j} + \overline{\phi}_i) = \sum_j AIP(\phi^*_{i+1,j} + \overline{\phi}_{i+1}) + \sum_j AIM(\phi^*_{i-1,j} + \overline{\phi}_{i-1})$$



$$+\sum_j (AJM)(\phi^*_{i,j-1} + \overline{\phi}_i)$$

$$+\sum_j (AJP)(\phi^*_{i,j+1} + \overline{\phi}_i) + \sum_j CON$$

$$(i = IST, \ldots . L2)$$

IST-starting subscript in X-direction; L2-last but one.

Rewrite into ABEqs. of $\bar{\phi}'_{i-1}, \bar{\phi}'_{i}, \bar{\phi}'_{i+1}$:

$$(BL)\underline{\bar{\phi}'_{i}} = (BLP)\underline{\bar{\phi}'_{i+1}} + (BLM)\underline{\bar{\phi}'_{i-1}} + BLC, i = IST, \ldots. L2$$

where

$$BL = \sum_{j=JST}^{M2}(AP) - \sum_{j\neq M2}(AJP) - \sum_{i\neq JST}(AJM)$$

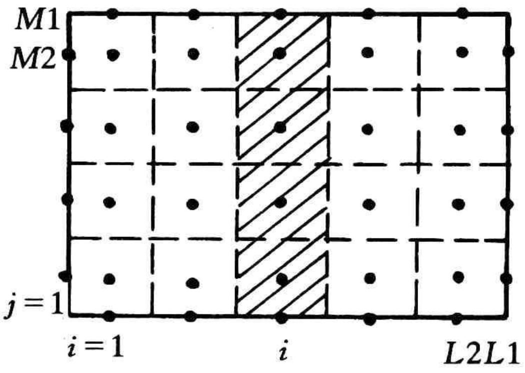
$$BLP = \sum_{j=JST}^{M2}(AIP) \qquad BLM = \sum_{j=JST}^{M2}(AIM)$$

$$BLC = \sum_{j=JST}^{M2} CON + \sum_{j=JST}^{M2}(AJP)\phi^{*}_{i,j+1} + \sum_{j=JST}^{M2}(AJM)\phi^{*}_{i,j-1}$$

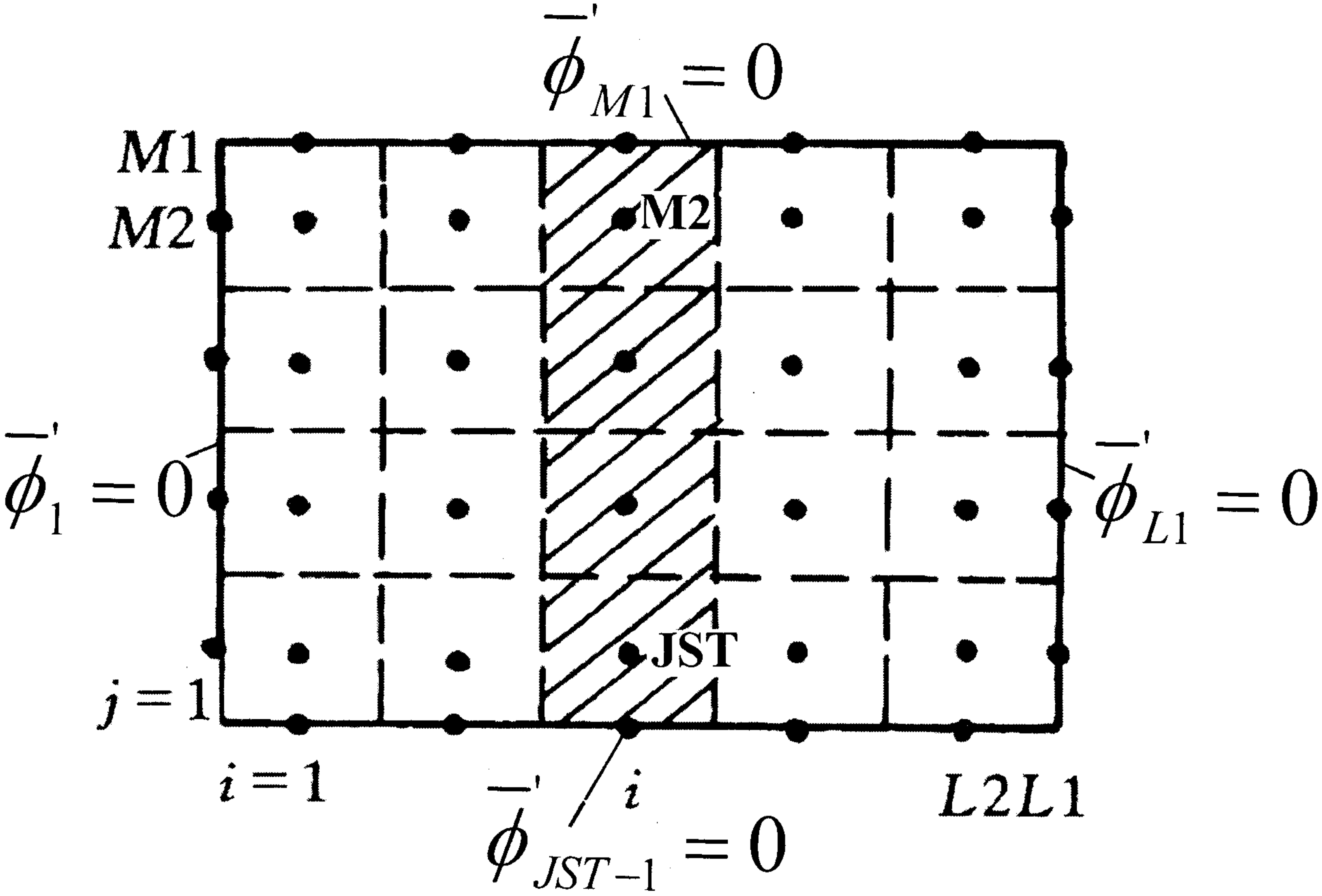
$$+\sum_{j=JST}^{M2}(AIP)\phi^{*}_{i+1,j} + \sum_{j=JST}^{M2}(AIM)\phi^{*}_{i-1,j} - \sum_{j=JST}^{M2}(AP)\phi^{*}_{i,j}$$

$$BL=\sum_{j=JST}^{M2}(AP)-\sum_{j\neq M2}(AJP)-\sum_{i\neq JST}(AJM)$$

ASTM is adopted to deal with 2nd and 3rd kind boundary condition， this is equivalent to that all boundaries are of 1st kind, and the correction for boundary nodes is zero； Thus when summation is conducted in y-direction the 1st term and the last term corrections are zero. Hence, for AJM term JST is not needed， and for AJP M2 is not needed.



2. Boundary condition for the correction ---zero

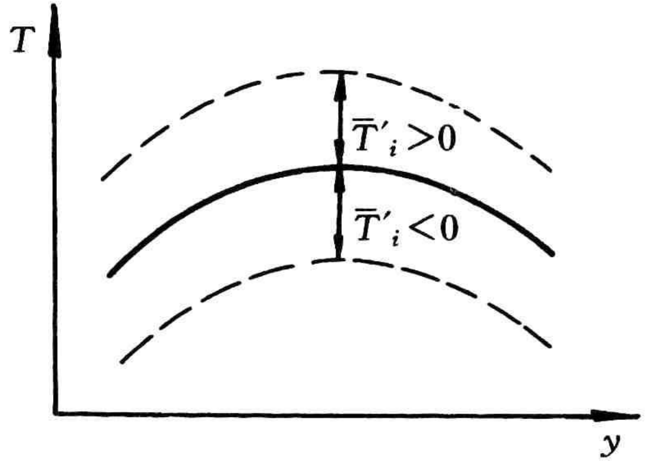


6.4.4 Remarks of application of B.C. technique

1.BCT is not an independent solution method. It should be combined with some other method, such as ADI.;

2. For further accelerating convergence ADI block correction may be used.;

3. For variables of physically larger th zero values the B.C.T. may not be used (such as turbulent kinetic energy, component of a mixed gas). Because B adds or subtracts a constant correction within the entire block, which may lead to minus values.



6.5 Multigrid Techniques –Promoting Simultaneous Attenuation of Different Wave-length Components

6.5.1 Error vector is attenuated(衰减) in the iteration process of solving ABEqs.

6.5.2 Basic idea and key issue of multigrid technique

6.5.3 Transferring solutions between different grid systems

6.5.4 Cycling patterns between different grid systems

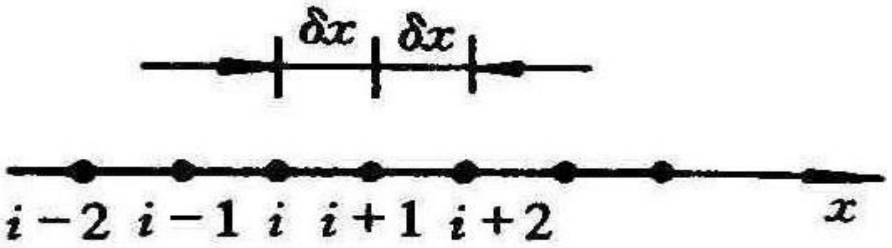
6.5 Multigrid Techniques –Promoting Simultaneous Attenuation of Different Wave-length Components

6.5.1 Error vector is attenuated in the iteration process of solving ABEqs

1. How error vector is attenuated during iteration?

Taking 1-D steady heat conduction problem as an example to analyze how error vector is attenuated:

$$\frac{d^2T}{dx^2}+f(x)=0$$



Discretizing it at a uniform grid system, yielding:

$$T_{i-1}-2T_i+T_{i+1}=-(\delta x)^2 f_i$$

Adopting G-S iteration method from left to right：

$$T_{i-1}^{(k)} - 2T_i^{(k)} + T_{i+1}^{(k-1)} = -(\delta x)^2 f_i$$

In the kth cycle iteration error vector is denoted by $\vec{\varepsilon}^{(k)}$ and its component is denoted by $\varepsilon_i^{(k)}$, then we have:

$$T_i = T_i^{(k)} + \varepsilon_i^{(k)}$$

Substituting this expression to the above equation we can get following variation of error with iteration

$$T_{i-1}^{(k)} - 2T_i^{(k)} + T_{i+1}^{(k-1)} = -(\delta x)^2 f_i$$

$$T_{i\text{-}1}^{(k)} = T_{i\text{-}1} \text{-} \varepsilon_{i\text{-}1}^{(k)}$$

$$T_{i+1}^{(k)} = T_{i+1} \text{-} \varepsilon_{i+1}^{(k)}$$

$$T_i^{(k)} = T_i \text{-} \varepsilon_i^{(k)}$$

Since $$T_{i-1}-2T_i+T_{i+1}=-(\delta x)^2 f_i$$

Then we have:

$$\varepsilon_{i-1}^{(k)}-2\varepsilon_i^{(k)}+\varepsilon_{i+1}^{(k-1)}=0$$

2. **Analysis of attenuation of harmonic components**

It will be shown later that $\varepsilon_i^{(k)}$ can be expressed as: $\psi(k)e^{Ii\theta}$ where $\psi(k)$ is amplitude (振幅) and θ is angle, by substituting this expression to the above eq., yielding

$$\frac{\psi(k)}{\psi(k-1)}=\frac{e^{I\theta}}{2-e^{-I\theta}}=\mu$$

Amplifying factor
(增长因子)

Analyzing amplifying factor for different phase angles：

$$\theta = \pi,$$

$$|\mu| = \frac{|\cos\pi + \cancelto{0}{I\sin\pi}|}{|2-\cos\pi + \cancelto{0}{I\sin\pi}|} = \frac{1}{2+1} = \frac{1}{3}, \quad \boxed{\text{Ite.5 times}} \quad 0.333^5 = 4.09\times10^{-3}$$

$$\theta = \pi/2,$$

$$|\mu| = \frac{\left|\cancelto{0}{\cos\frac{\pi}{2}} + I\sin\frac{\pi}{2}\right|}{\left|2-\cancelto{0}{\cos\frac{\pi}{2}} + I\sin\frac{\pi}{2}\right|} = \frac{1}{\sqrt{2^2+1}} = \frac{1}{\sqrt{5}}, \quad \boxed{\text{Ite5 times}} \quad 0.447^5 = 0.0178$$

$$\theta = \pi/10,$$

$$|\mu| = \frac{\left|\cos\frac{\pi}{10} + I\sin\frac{\pi}{10}\right|}{\left|2-\cos\frac{\pi}{10} + I\sin\frac{\pi}{10}\right|} = \frac{|0.9510+0.3090I|}{|2-(0.9510+0.3090I)|} = \frac{1}{1.094}, \quad \boxed{\text{Ite.5 times}} \quad 0.914^5 = 0.658$$

θ can be expressed by: $\theta = k_x \Delta x = \dfrac{2\pi}{\lambda}\Delta x$

where λ is the wave length. At a fixed space step, short wave has a larger phase angle, and is attenuated (衰减)very fast; while long wave component has small phase angle and attenuated very slowly.

From above calculation phase angle can be an indicator for short/long wave components.

Generally for components with phase angle within following range is regarded as short wave ones:

$$\pi \le \theta \le \pi / 2$$

This phase angle is dependent on space step length $\theta = k_x \Delta x = 2\pi \Delta x / \lambda$. If after several iterations the length step is amplified then originally long wave component may be behaved as a short wave and can be attenuated very fast at that grid system.

In such a way by amplifying space step (放大空间步长） several times during iteration all the error components may be quite uniformly attenuated and the entire ABEqs. may be converged much faster than iteration just at a single grid system.

This is the major idea of multigrid technique for solving ABEqs.

6.5.2 Major idea and key issue of multigrid technique

1. Major idea－Solving ABEqs. is conducted at several grid systems with different space step length such that error components with different frequencies can be attenuated simultaneously.

2. Key issues－

(1) How to transfer solutions at different grid systems?

(2) How to cycle (轮转） the solutions between several grid systems?

6.5.3 Transferring solutions between two gird systems

Basic concept： solution transferred between different grid system － **is the one of the finest grid.**

Taking two grid systems, one coarse and one fine, as an example to show the transferring of solutions.

1.From fine grid to coarse grid

$$\vec{A}^{(k-1)}\phi^{(k-1)}=\vec{b}^{(k-1)}+I_k^{k-1}(\underline{\vec{b}^{(k)}-\vec{A}^{(k)}\phi^{(k)}})$$

Matrix at (k-1)th grid determined from solution of kth grid.

Solution at (k-1)th grid

Source term at (k-1)th grid determined from solution of kth grid

Operator for transferring form kth grid to (k-1)th grid

Residual of fine grid

2. Transferring from coarse grid to fine grid

$$\underline{\phi_{rev}^{(k)}} = \underline{\phi_{old}^{k}} + I_{k-1}^{k}\underline{\underline{(\phi^{(k-1)}} - \underline{I_{k}^{k-1}\phi_{old}^{(k)}})}$$

Improved solution at fine grid

Original solution at fine grid

Operator for transferring correction part of solution at (k-1) th grid to kth grid

New solution of fine grid obtained at (k-1)th grid

Solution of kth grid expressed at (k-1)th grid

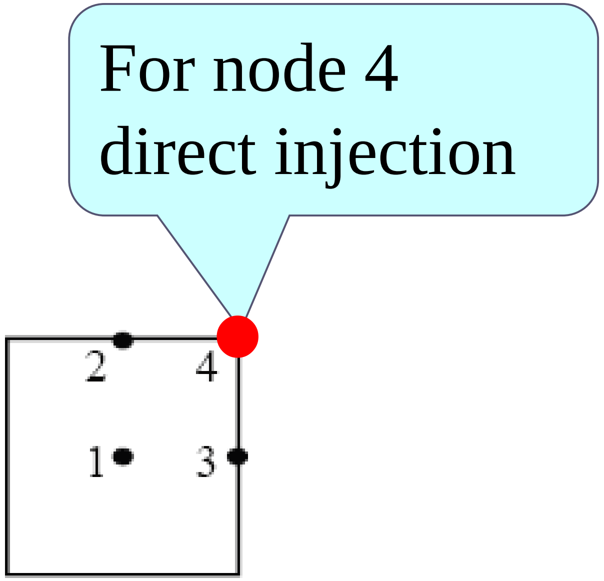
3. Restriction and prolongation operators

1) Restriction operator(限定算子)

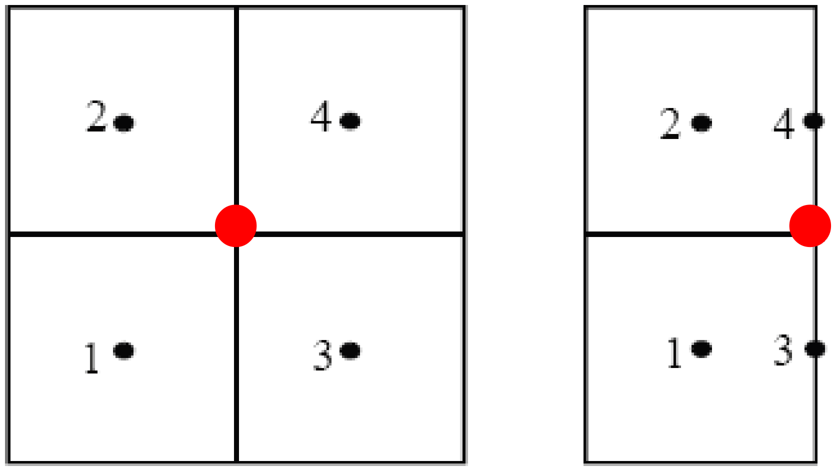
(From fine to course)

$$I_k^{k-1}\begin{cases}\text{Direct injection(直接注入)}\\ \text{Nearby average(就近平均),}\end{cases}$$

Linear interpolation


For node 4
direct injection

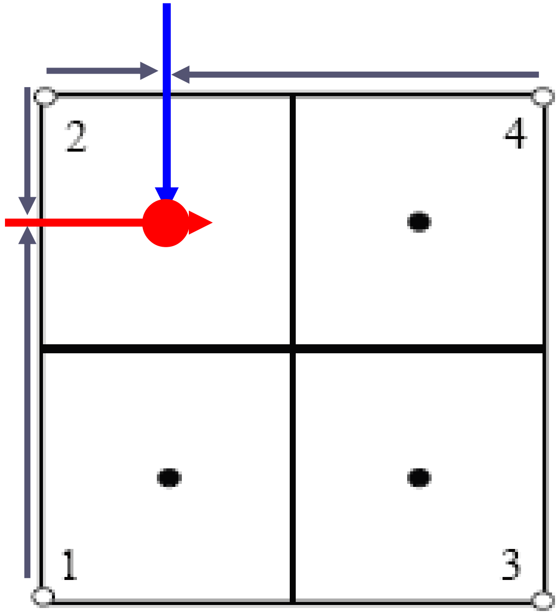
Near average



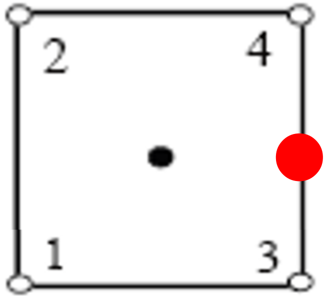
● _fine → ● course

2) Prologation operator (延拓算子) (From course to fine)

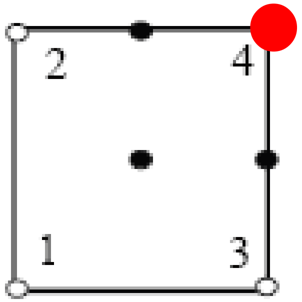
$$I_{k-1}^{k}\begin{cases}\text{Direct injection}\\ \text{linear interpolation}\\ \text{Quadratic interpolation (二次插值)}\end{cases}$$



Quadratic interpolation



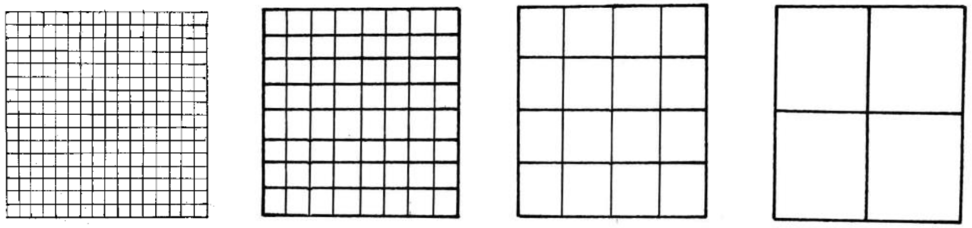
Linear interpolation between nodes 3，4



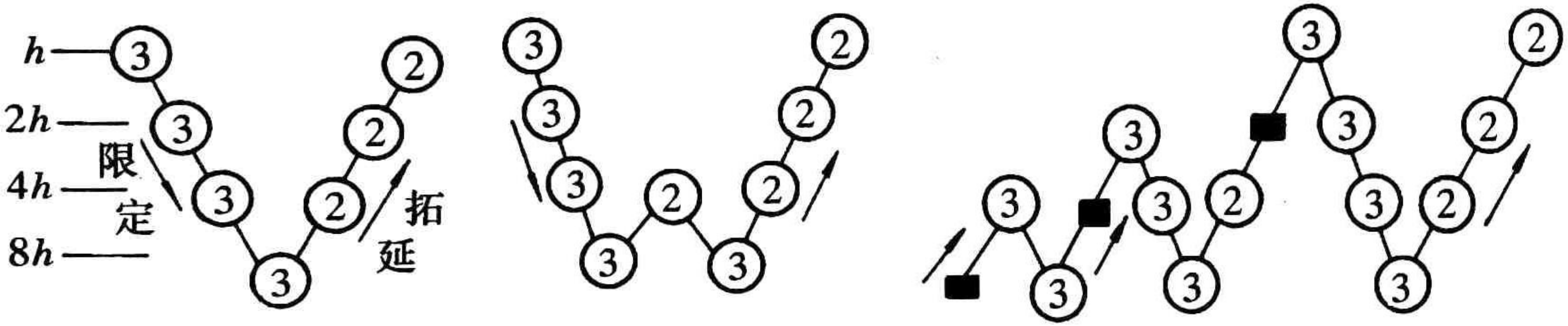
Node 4-Direct injection

● Course → • —Fine

6.5.4 Cycling method between several grids



Three cycling patterns：



(a) V-cycle　　(b) W -cycle　　(c) FMG-cycle

Number in the circle shows times of iteration. Black symbol represents converged solution.　FMG cycle is widely adopted in fluid flow and heat transfer problems.

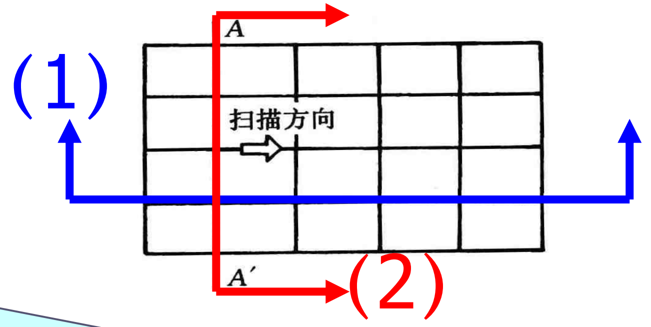
Home work (p.294)

7－1　7－4　7－6　7－8

Due in Oct. 28

Appendix ADI for iteration vs. ADI for marching

ADI-Jakob iteration can be expressed as：：



$b^{(k+1/2)}$

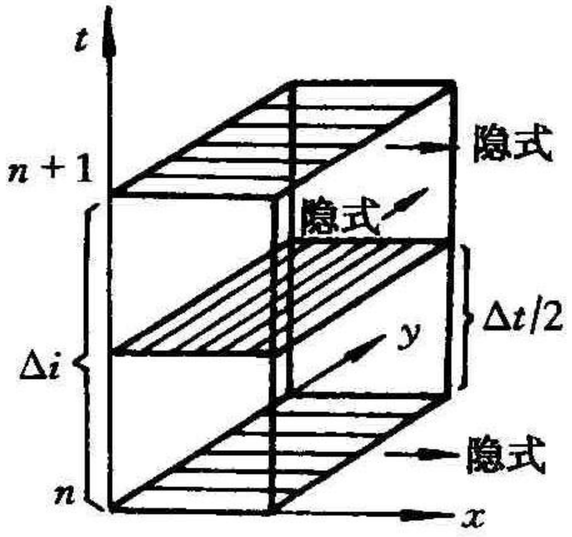
$$a_P\phi_P^{(k+1/2)}=a_E\phi_E^{(k+1/2)}+a_W\phi_W^{(k+1/2)}+[\overline{a_N\phi_N^{(k)}+a_S\phi_S^{(k)}+b}]$$

$$a_P\phi_P^{(k+1)}=a_N\phi_N^{(k+1)}+a_S\phi_S^{(k+1)}+[\underline{a_E\phi_E^{(k+1/2)}+a_W\phi_W^{(k+1/2)}+b}]$$

$b^{(k+1)}$

This expression is very similar to Peaceman-Rachford ADImplicit method for transient problem：

2-D Peaceman-Rachford method



2 – DADImplicit

Dividing Δt into two sub-periods.

In the 1st sub-period $\Delta t / 2$

x-direction is implicit, y-direction is explicit;

In the 2nd $\Delta t / 2$ y-direction is implicit, and x is explicit.

Let $\phi^{(k+1/2)}$ represent temporary values at middle time

$\delta_x^2 \phi_{i,j}^k$ represent CD for 2nd-order x-direction derivative at time level k ; then we have:

1st sub-period：

$$\frac{\phi_{i,j}^{k+1/2}-\phi_{i,j}^{k}}{\Delta t/2}=a(\delta_x^2\phi_{i,j}^{k+1/2}+\delta_y^2\phi_{i,j}^{k})\quad(1)$$

2nd sub-period：

$$\frac{\phi_{i,j}^{k+1}-\phi_{i,j}^{k+1/2}}{\Delta t/2}=a(\delta_x^2\phi_{i,j}^{k+1/2}+\delta_y^2\phi_{i,j}^{k+1})\quad(2)$$

Rewrite Eq.(1):

$$\underbrace{(1+\frac{a\Delta t}{\Delta x^2})}_{a_P}\phi_{i,j}^{k+1/2}=\underbrace{(\frac{a\Delta t}{2\Delta x^2})}_{a_E,a_W}(\phi_{i+1,j}^{k+1/2}+\phi_{i-1,j}^{k+1/2})+\overbrace{\underbrace{(\frac{a\Delta t}{2\Delta x^2})}_{a_S,a_N}(\phi_{i,j+1}^{k}+\phi_{i,j-1}^{k})+(1-\underbrace{\frac{a\Delta t}{\Delta x^2}}_{b})\phi_{i,j}^{k}}^{\boxed{b^{k+1/2}}}$$

Thus one-time step forward of transient problem is equivalent to one cycle iteration for steady problem.

Problem # 7-1

Try to calculate and prove that the following equations are convergent for Jacobi iterative method, whereas are divergent for GS point iterative method.

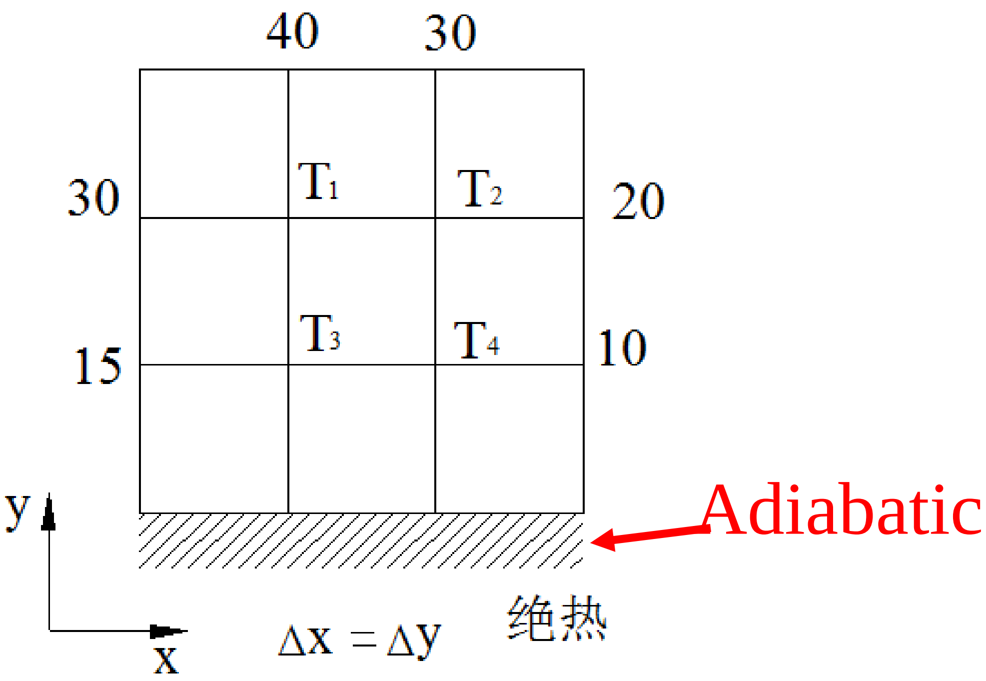
$$x_1 + 2x_2 - 2x_3 = 1$$

$$x_1 + x_2 + x_3 = 3$$

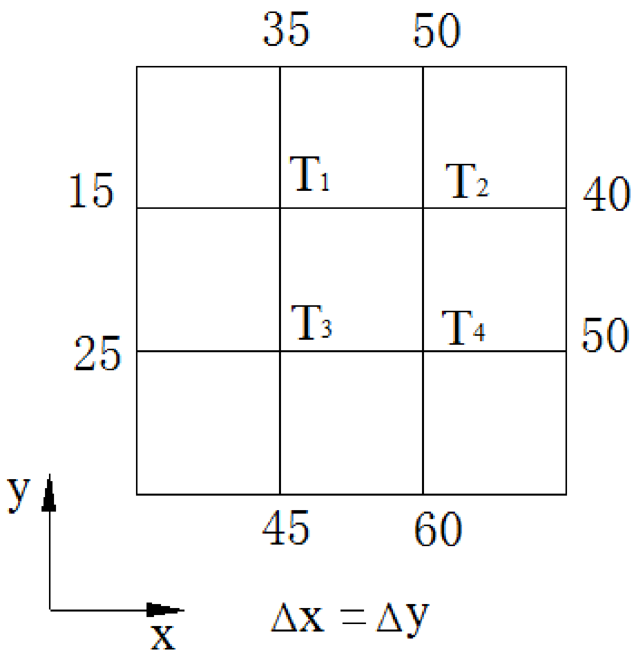
$$2x_1 + 2x_2 + x_3 = 5$$

Problem # 7-4

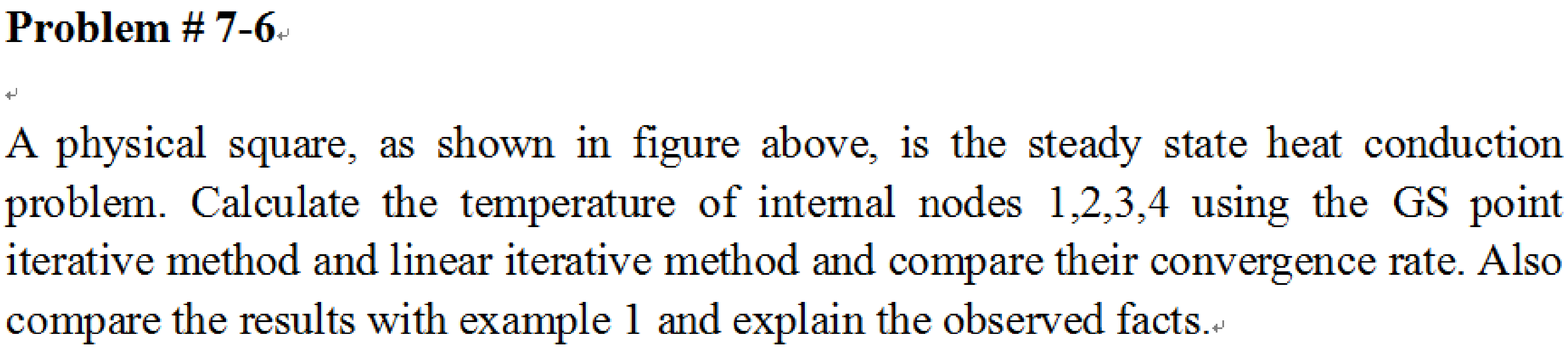
Bottom of a square object is thermally insulated, and the temperature of remaining three sides is shown in figure given below. Determine the temperature inside the square nodes 1,2,3,4. Thermal properties of the object are constants, and without internal heat source.



Problem 7-4



Problem 7-6



Problem # 7-6

A physical square, as shown in figure above, is the steady state heat conduction problem. Calculate the temperature of internal nodes 1,2,3,4 using the GS point iterative method and linear iterative method and compare their convergence rate. Also compare the results with example 1 and explain the observed facts.

Problem # 7-8

A sufficient condition for GS and Jacobi point iteration convergence is that the algebraic equation coefficient matrix must be strictly diagonally dominant, that is the formula (7-21) must be tenable for either rows or columns. Take the following algebraic equations as an example

$4x_1 - x_2 + x_3 = 4$ (Construct the iterative formula for x_1)

$x_1 + 4x_2 + 2x_3 = 9$ (Construct the iterative formula for x_2)

$-x_1 + 2x_2 + 5x_3 = 2$ (Construct the iterative formula for x_3)

Prove that when strictly diagonally dominant is tenable, the error present in one iteration step will be gradually attenuated with the iteration process.

本组网页地址：http://nht.xjtu.edu.cn　欢迎访问！

Teaching PPT will be loaded on ou website

同舟共济
渡彼岸！

People in the same boat help each other to cross to the other bank, where....